Part II — Algebraic Topology
Theorems with proof

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Part IB Analysis II is essential, and Metric and Topological Spaces is highly desirable

The fundamental group
Homotopy of continuous functions and homotopy equivalence between topological spaces. The fundamental group of a space, homomorphisms induced by maps of spaces, change of base point, invariance under homotopy equivalence. [3]

Covering spaces
Covering spaces and covering maps. Path-lifting and homotopy-lifting properties, and their application to the calculation of fundamental groups. The fundamental group of the circle; topological proof of the fundamental theorem of algebra. *Construction of the universal covering of a path-connected, locally simply connected space*. The correspondence between connected coverings of X and conjugacy classes of subgroups of the fundamental group of X. [5]

The Seifert-Van Kampen theorem
Free groups, generators and relations for groups, free products with amalgamation. Statement *and proof* of the Seifert-Van Kampen theorem. Applications to the calculation of fundamental groups. [4]

Simplicial complexes
Finite simplicial complexes and subdivisions; the simplicial approximation theorem. [3]

Homology
Simplicial homology, the homology groups of a simplex and its boundary. Functorial properties for simplicial maps. *Proof of functoriality for continuous maps, and of homotopy invariance*. [4]

Homology calculations
The homology groups of S^n, applications including Brouwer's fixed-point theorem. The Mayer-Vietoris theorem. *Sketch of the classification of closed combinatorical surfaces*; determination of their homology groups. Rational homology groups; the Euler-Poincaré characteristic and the Lefschetz fixed-point theorem. [5]

Contents

0 Introduction

1 Definitions

1.1 Some recollections and conventions

Lemma (Gluing lemma)**.** If $f : X \to Y$ is a function of topological spaces, $X = C \cup K$, C and K are both closed, then f is continuous if and only if the restrictions $f|_C$ and $f|_K$ are continuous.

Proof. Suppose f is continuous. Then for any closed $A \subseteq Y$, we have

$$f|_C^{-1}(A) = f^{-1}(A) \cap C,$$

which is closed. So $f|_C$ is continuous. Similarly, $f|_K$ is continuous.

If $f|_C$ and $f|_K$ are continuous, then for any closed $A \subseteq Y$, we have

$$f^{-1}(A) = f|_C^{-1}(A) \cup f|_K^{-1}(A),$$

which is closed. So f is continuous. □

Lemma. Let (X, d) be a compact metric space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then there is some δ such that for each $x \in X$, there is some $\alpha \in A$ such that $B_\delta(x) \subseteq U_\alpha$. We call δ a *Lebesgue number* of this cover.

Proof. Suppose not. Then for each $n \in \mathbb{N}$, there is some $x_n \in X$ such that $B_{1/n}(x_n)$ is not contained in any U_α. Since X is compact, the sequence (x_n) has a convergent subsequence. Suppose this subsequence converges to y.

Since $\mathcal{U}$ is an open cover, there is some $\alpha \in A$ such that $y \in U_\alpha$. Since U_α is open, there is some $r > 0$ such that $B_r(y) \subseteq U_\alpha$. But then we can find a sufficiently large n such that $\frac{1}{n} < \frac{r}{2}$ and $d(x_n, y) < \frac{r}{2}$. But then

$$B_{1/n}(x_n) \subseteq B_r(y) \subseteq U_\alpha.$$

Contradiction. □

1.2 Cell complexes

2 Homotopy and the fundamental group

2.1 Motivation

2.2 Homotopy

Proposition. For spaces X, Y, and $A \subseteq X$, the "homotopic $\mathrm{rel}\, A$" relation is an equivalence relation. In particular, when $A = \emptyset$, homotopy is an equivalence relation.

Proof.

(i) Reflexivity: $f \simeq f$ since $H(x, t) = f(x)$ is a homotopy.

(ii) Symmetry: if $H(x, t)$ is a homotopy from f to g, then $H(x, 1 - t)$ is a homotopy from g to f.

(iii) Transitivity: Suppose $f, g, h : X \to Y$ and $f \simeq_H g \,\mathrm{rel}\, A$, $g \simeq_{H'} h \,\mathrm{rel}\, A$. We want to show that $f \simeq h \,\mathrm{rel}\, A$. The idea is to "glue" the two maps together.

We know how to continuously deform f to g, and from g to h. So we just do these one after another. We define $H'' : X \times I \to Y$ by

$$H''(x, t) = \begin{cases} H(x, 2t) & 0 \le t \le \frac{1}{2} \\ H'(x, 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

This is well-defined since $H(x, 1) = g(x) = H'(x, 0)$. This is also continuous by the gluing lemma. It is easy to check that H'' is a homotopy $\mathrm{rel}\, A$. □

Lemma. Consider the spaces and arrows

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

If $f_0 \simeq_H f_1$ and $g_0 \simeq_{H'} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof. We will show that $g_0 \circ f_0 \simeq g_0 \circ f_1 \simeq g_1 \circ f_1$. Then we are done since homotopy between maps is an equivalence relation. So we need to write down two homotopies.

(i) Consider the following composition:

$$X \times I \xrightarrow{H} Y \xrightarrow{g_0} Z$$

It is easy to check that this is the first homotopy we need to show $g_0 \circ f_0 \simeq g_0 \circ f_1$.

(ii) The following composition is a homotopy from $g_0 \circ f_1$ to $g_1 \circ f_1$:

$$X \times I \xrightarrow{f_1 \times \mathrm{id}_I} Y \times I \xrightarrow{H'} Z$$ □

Proposition. Homotopy equivalence of spaces is an equivalence relation.

Proof. Symmetry and reflexivity are trivial. To show transitivity, let $f : X \to Y$ and $h : Y \to Z$ be homotopy equivalences, and $g : Y \to X$ and $k : Z \to Y$ be their homotopy inverses. We will show that $h \circ f : X \to Z$ is a homotopy equivalence with homotopy inverse $g \circ k$. We have

$$(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ \mathrm{id}_Y \circ k = h \circ k \simeq \mathrm{id}_Z .$$

Similarly,

$$(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X .$$

So done. □

2.3 Paths

Proposition. For any map $f : X \to Y$, there is a well-defined function

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y),$$

defined by

$$\pi_0(f)([x]) = [f(x)].$$

Furthermore,

(i) If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.

(ii) For any maps $A \xrightarrow{h} B \xrightarrow{k} C$, we have $\pi_0(k \circ h) = \pi_0(k) \circ \pi_0(h)$.

(iii) $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$

Proof. To show this is well-defined, suppose $[x] = [y]$. Then let $\gamma : I \to X$ be a path from x to y. Then $f \circ \gamma$ is a path from $f(x)$ to $f(y)$. So $[f(x)] = [f(y)]$.

(i) If $f \simeq g$, let $H : X \times I \to Y$ be a homotopy from f to g. Let $x \in X$. Then $H(x, \cdot\,)$ is a path from $f(x)$ to $g(x)$. So $[f(x)] = [g(x)]$, i.e. $\pi_0(f)([x]) = \pi_0(g)([x])$. So $\pi_0(f) = \pi_0(g)$.

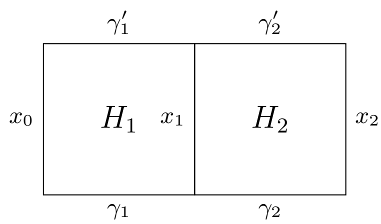
(ii) $\pi_0(k \circ h)([x]) = \pi_0(k) \circ \pi_0(h)([x]) = [k(h(x))]$.

(iii) $\pi_0(\mathrm{id}_X)([x]) = [\mathrm{id}_X(x)] = [x]$. So $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$. □

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $\pi_0(f)$ is a bijection.

Proposition. Let $\gamma_1, \gamma_2 : I \to X$ be paths, $\gamma_1(1) = \gamma_2(0)$. Then if $\gamma_1 \simeq \gamma_1'$ and $\gamma_2 \simeq \gamma_2'$, then $\gamma_1 \cdot \gamma_2 \simeq \gamma_1' \cdot \gamma_2'$.

Proof. Suppose that $\gamma_1 \simeq_{H_1} \gamma_1'$ and $\gamma_2 \simeq_{H_2} \gamma_2'$. Then we have the diagram



We can thus construct a homotopy by

$$H(s,t) = \begin{cases} H_1(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(s, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}. \qquad \square$$

Proposition. Let $\gamma_0 : x_0 \rightsquigarrow x_1, \gamma_1 : x_1 \rightsquigarrow x_2, \gamma_2 : x_2 \rightsquigarrow x_3$ be paths. Then

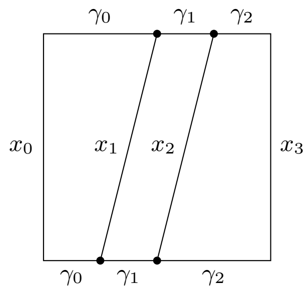
(i) $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$

(ii) $\gamma_0 \cdot c_{x_1} \simeq \gamma_0 \simeq c_{x_0} \cdot \gamma_0$.

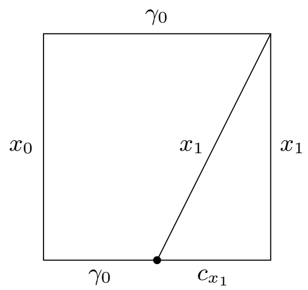
(iii) $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{x_0}$ and $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{x_1}$.

Proof.

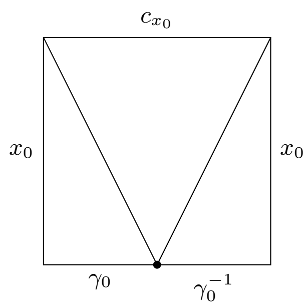
(i) Consider the following diagram:



(ii) Consider the following diagram:



(iii) Consider the following diagram:



Turning these into proper proofs is left as an exercise for the reader. $\square$

2.4 The fundamental group

Theorem. The fundamental group is a group.

Proof. Immediate from our previous lemmas. □

Proposition. To a based map

$$f : (X, x_0) \to (Y, y_0),$$

there is an associated function

$$f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0),$$

defined by $[\gamma] \mapsto [f \circ \gamma]$. Moreover, it satisfies

(i) $\pi_1(f)$ is a homomorphism of groups.

(ii) If $f \simeq f'$, then $\pi_1(f) = \pi_1(f')$.

(iii) For any maps $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$, we have $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$.

(iv) $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}$

Proof. Exercise. □

Proposition. A path $u : x_0 \rightsquigarrow x_1$ induces a group *isomorphism*

$$u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by

$$[\gamma] \mapsto [u^{-1} \cdot \gamma \cdot u].$$

This satisfies

(i) If $u \simeq u'$, then $u_\# = u'_\#$.

(ii) $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X, x_0)}$

(iii) If $v : x_1 \rightsquigarrow x_2$. Then $(u \cdot v)_\# = v_\# \circ u_\#$.

(iv) If $f : X \to Y$ with $f(x_0) = y_0$, $f(x_1) = y_1$, then

$$(f \circ u)_\# \circ f_* = f_* \circ u_\# : \pi_1(X, x_0) \to \pi_1(Y, y_1).$$

A nicer way of writing this is

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "u_\#"] & \pi_1(Y, y_0) \arrow[d, "(f \circ u)_\#"] \\ \pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

The property says that the composition is the same no matter which way we go from $\pi_1(X, x_0)$ to $\pi_1(Y, y_1)$. We say that the square is a *commutative diagram*. These diagrams will appear all of the time in this course.

(v) If $x_1 = x_0$, then $u_\#$ is an automorphism of $\pi_1(X, x_0)$ given by conjugation by u.

Proof. Yet another exercise. Note that $(u^{-1})_\# = (u_\#)^{-1}$, which is why we have an isomorphism. ☐

Lemma. The following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "u_\#"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

In algebra, we say

$$g_* = u_\# \circ f_*.$$

Proof. Suppose we have a loop $\gamma : I \to X$ based at x_0.

We need to check that

$$g_*([\gamma]) = u_\# \circ f_*([\gamma]).$$

In other words, we want to show that

$$g \circ \gamma \simeq u^{-1} \cdot (f \circ \gamma) \cdot u.$$

To prove this result, we want to build a homotopy.

Consider the composition:

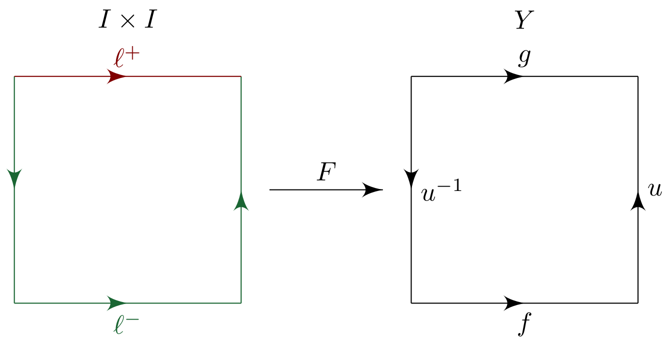
$$F : \; I \times I \xrightarrow{\gamma \times \mathrm{id}_I} X \times I \xrightarrow{H} Y.$$

Our plan is to exhibit two homotopic paths ℓ^+ and ℓ^- in $I \times I$ such that

$$F \circ \ell^+ = g \circ \gamma, \quad F \circ \ell^- = u^{-1} \cdot (f \circ \gamma) \cdot u.$$

This is in general a good strategy — X is a complicated and horrible space we don't understand. So to construct a homotopy, we make ourselves work in a much nicer space $I \times I$.

Our ℓ^+ and ℓ^- are defined in a rather simple way.



More precisely, ℓ^+ is the path $s \mapsto (s, 1)$, and ℓ^- is the concatenation of the paths $s \mapsto (0, 1-s)$, $s \mapsto (s, 0)$ and $s \mapsto (1, s)$.

Note that ℓ^+ and ℓ^- are homotopic as paths. If this is not obvious, we can manually check the homotopy

$$L(s, t) = t\ell^+(s) + (1-t)\ell^-(s).$$

This works because $I \times I$ is convex. Hence $F \circ \ell^+ \simeq_{F \circ L} F \circ \ell^-$ as paths.

Now we check that the compositions $F \circ \ell^\pm$ are indeed what we want. We have

$$F \circ \ell^+(s) = H(\gamma(s), 1) = g \circ \gamma(s).$$

Similarly, we can show that

$$F \circ \ell^-(s) = u^{-1} \cdot (f \circ \gamma) \cdot u(s).$$

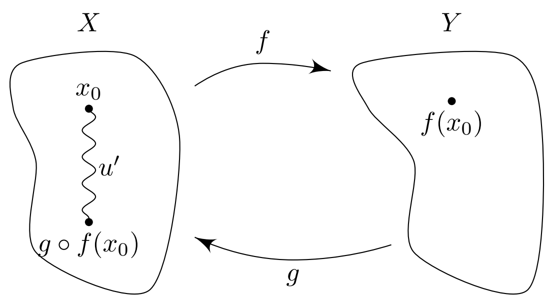
So done. □

Theorem. If $f : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then the induced map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse. So $f \circ g \simeq_H \mathrm{id}_Y$ and $g \circ f \simeq_{H'} \mathrm{id}_X$.



We have no guarantee that $g \circ f(x_0) = x_0$, but we know that our homotopy H' gives us $u' = H'(x_0, \cdot\,) : x_0 \rightsquigarrow g \circ f(x_0)$.

Applying our previous lemma with id_X for "f" and $g \circ f$ for "g", we get

$$u'_\# \circ (\mathrm{id}_X)_* = (g \circ f)_*$$

Using the properties of the $_*$ operation, we get that

$$g_* \circ f_* = u'_\#.$$

However, we know that $u'_\#$ is an isomorphism. So f_* is injective and g_* is surjective.

Doing it the other way round with $f \circ g$ instead of $g \circ f$, we know that g_* is injective and f_* is surjective. So both of them are isomorphisms. □

Lemma. A path-connected space X is simply connected if and only if for any $x_0, x_1 \in X$, there exists a unique homotopy class of paths $x_0 \rightsquigarrow x_1$.

Proof. Suppose X is simply connected, and let $u, v : x_0 \rightsquigarrow x_1$ be paths. Now note that $u \cdot v^{-1}$ is a loop based at x_0, it is homotopic to the constant path, and $v^{-1} \cdot v$ is trivially homotopic to the constant path. So we have

$$u \simeq u \cdot v^{-1} \cdot v \simeq v.$$

On the other hand, suppose there is a unique homotopy class of paths $x_0 \rightsquigarrow x_1$ for all $x_0, x_1 \in X$. Then in particular there is a unique homotopy class of loops based at x_0. So $\pi_1(X, x_0)$ is trivial. $\square$

3 Covering spaces

3.1 Covering space

Lemma. Let $p : \tilde{X} \to X$ be a covering map, $f : Y \to X$ be a map, and $\tilde{f}_1, \tilde{f}_2$ be both lifts of f. Then

$$S = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

is both open and closed. In particular, if Y is connected, $\tilde{f}_1$ and $\tilde{f}_2$ agree either everywhere or nowhere.

Proof. First we show it is open. Let y be such that $\tilde{f}_1(y) = \tilde{f}_2(y)$. Then there is an evenly covered open neighbourhood $U \subseteq X$ of $f(y)$. Let $\tilde{U}$ be such that $\tilde{f}_1(y) \in \tilde{U}$, $p(\tilde{U}) = U$ and $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Let $V = \tilde{f}_1^{-1}(\tilde{U}) \cap \tilde{f}_2^{-1}(\tilde{U})$. We will show that $\tilde{f}_1 = \tilde{f}_2$ on V.

Indeed, by construction

$$p|_{\tilde{U}} \circ \tilde{f}_1|_V = p|_{\tilde{U}} \circ \tilde{f}_2|_V.$$

Since $p|_{\tilde{U}}$ is a homeomorphism, it follows that

$$\tilde{f}_1|_V = \tilde{f}_2|_V.$$

Now we show S is closed. Suppose not. Then there is some $y \in \bar{S} \setminus S$. So $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Let U be an evenly covered neighbourhood of $f(y)$. Let $p^{-1}(U) = \coprod U_\alpha$. Let $\tilde{f}_1(y) \in U_\beta$ and $\tilde{f}_2(y) \in U_\gamma$, where $\beta \neq \gamma$. Then $V = \tilde{f}_1^{-1}(U_\beta) \cap \tilde{f}_2^{-1}(U_\gamma)$ is an open neighbourhood of y, and hence intersects S by definition of closure. So there is some $x \in V$ such that $\tilde{f}_1(x) = \tilde{f}_2(x)$. But $\tilde{f}_1(x) \in U_\beta$ and $\tilde{f}_2(x) \in U_\gamma$, and hence U_β and U_γ have a non-trivial intersection. This is a contradiction. So S is closed. □

Lemma (Homotopy lifting lemma)**.** Let $p : \tilde{X} \to X$ be a covering space, $H : Y \times I \to X$ be a homotopy from f_0 to f_1. Let $\tilde{f}_0$ be a lift of f_0. Then there exists a *unique* homotopy $\tilde{H} : Y \times I \to \tilde{X}$ such that

(i) $\tilde{H}(\,\cdot\,, 0) = \tilde{f}_0$; and

(ii) $\tilde{H}$ is a lift of H, i.e. $p \circ \tilde{H} = H$.

Lemma (Path lifting lemma)**.** Let $p : \tilde{X} \to X$ be a covering space, $\gamma : I \to X$ a path, and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0 = \gamma(0)$. Then there exists a *unique* path $\tilde{\gamma} : I \to \tilde{X}$ such that

(i) $\tilde{\gamma}(0) = \tilde{x}_0$; and

(ii) $\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.

Proof. Let

$$S = \{s \in I : \tilde{\gamma} \text{ exists on } [0, s] \subseteq I\}.$$

Observe that

(i) $0 \in S$.

(ii) S is open. If $s \in S$ and $\tilde{\gamma}(s) \in V_\beta \subseteq p^{-1}(U)$, we can define $\tilde{\gamma}$ on some small neighbourhood of s by

$$\tilde{\gamma}(t) = (p|_{V_\beta})^{-1} \circ \gamma(t)$$

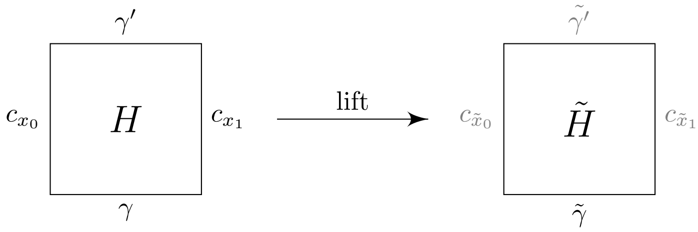
(iii) S is closed. If $s \notin S$, then pick an evenly covered neighbourhood U of $\gamma(s)$. Suppose $\gamma((s - \varepsilon, s)) \subseteq U$. So $s - \frac{\varepsilon}{2} \notin S$. So $(s - \frac{\varepsilon}{2}, 1] \cap S = \emptyset$. So S is closed.

Since S is both open and closed, and is non-empty, we have $S = I$. So $\tilde{\gamma}$ exists. □

Corollary. Suppose $\gamma, \gamma' : I \to X$ are paths $x_0 \rightsquigarrow x_1$ and $\tilde{\gamma}, \tilde{\gamma}' : I \to \tilde{X}$ are lifts of γ and γ' respectively, both starting at $\tilde{x}_0 \in p^{-1}(x_0)$.

If $\gamma \simeq \gamma'$ as *paths*, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ are homotopic as paths. In particular, $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

Proof. The homotopy lifting lemma gives us an $\tilde{H}$, a lift of H with $\tilde{H}(\,\cdot\,, 0) = \tilde{\gamma}$.



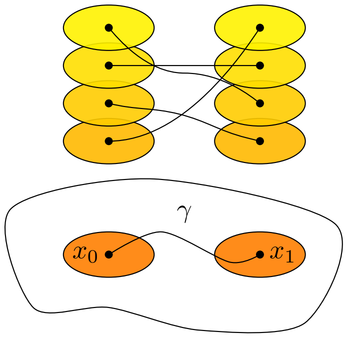
In this diagram, we by assumption know the bottom of the $\tilde{H}$ square is $\tilde{\gamma}$. To show that this is a path homotopy from $\tilde{\gamma}$ to $\tilde{\gamma}'$, we need to show that the other edges are $c_{\tilde{x}_0}$, $c_{\tilde{x}_1}$ and $\tilde{\gamma}'$ respectively.

Now $\tilde{H}(\,\cdot\,, 1)$ is a lift of $H(\,\cdot\,, 1) = \gamma'$, starting at $\tilde{x}_0$. Since lifts are unique, we must have $\tilde{H}(\,\cdot\,, 1) = \tilde{\gamma}'$. So this is indeed a homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$. Now we need to check that this is a homotopy of paths.

We know that $\tilde{H}(0, \,\cdot\,)$ is a lift of $H(0, \,\cdot\,) = c_{x_0}$. We are aware of one lift of c_{x_0}, namely $c_{\tilde{x}_0}$. By uniqueness of lifts, we must have $\tilde{H}(0, \,\cdot\,) = c_{\tilde{x}_0}$. Similarly, $\tilde{H}(1, \,\cdot\,) = c_{\tilde{x}_1}$. So this is a homotopy of paths. □

Corollary. If X is a path connected space, $x_0, x_1 \in X$, then there is a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$.

Proof. Let $\gamma : x_0 \rightsquigarrow x_1$ be a path. We want to use this to construct a bijection between each preimage of x_0 and each preimage of x_1. The obvious thing to do is to use lifts of the path γ.



Define a map $f_\gamma : p^{-1}(x_0) \to p^{-1}(x_1)$ that sends $\tilde{x}_0$ to the end point of the unique lift of γ at $\tilde{x}_0$.

The inverse map is obtained by replacing γ with γ^{-1}, i.e. $f_{\gamma^{-1}}$. To show this is an inverse, suppose we have some lift $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{x}_1$, so that $f_\gamma(\tilde{x}_0) = \tilde{x}_1$. Now notice that $\tilde{\gamma}^{-1}$ is a lift of γ^{-1} starting at $\tilde{x}_1$ and ending at $\tilde{x}_0$. So $f_{\gamma^{-1}}(\tilde{x}_1) = \tilde{x}_0$. So $f_{\gamma^{-1}}$ is an inverse to f_γ, and hence f_γ is bijective. $\square$

Lemma. If $p : \tilde{X} \to X$ is a covering map and $\tilde{x}_0 \in \tilde{X}$, then

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective.

Proof. To show that a group homomorphism p_* is injective, we have to show that if $p_*(x)$ is trivial, then x must be trivial.

Consider a based loop $\tilde{\gamma}$ in $\tilde{X}$. We let $\gamma = p \circ \tilde{\gamma}$. If γ is trivial, i.e. $\gamma \simeq c_{x_0}$ as paths, the homotopy lifting lemma then gives us a homotopy upstairs between $\tilde{\gamma}$ and $c_{\tilde{x}_0}$. So $\tilde{\gamma}$ is trivial. $\square$

Lemma. Suppose X is path connected and $x_0 \in X$.

(i) The action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive if and only if $\tilde{X}$ is path connected. Alternatively, we can say that the orbits of the action correspond to the path components.

(ii) The stabilizer of $\tilde{x}_0 \in p^{-1}(x_0)$ is $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$.

(iii) If $\tilde{X}$ is path connected, then there is a bijection

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \backslash \pi_1(X, x_0) \to p^{-1}(x_0).$$

Note that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \backslash \pi_1(X, x_0)$ is not a quotient, but simply the set of cosets. We write it the "wrong way round" because we have right cosets instead of left cosets.

Proof.

(i) If $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$, then since $\tilde{X}$ is path connected, we know that there is some $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{x}_0'$. Then we can project this to $\gamma = p \circ \tilde{\gamma}$. Then γ is a path from $x_0 \rightsquigarrow x_0$, i.e. a loop. Then by the definition of the action, $\tilde{x}_0 \cdot [\gamma] = \tilde{\gamma}(1) = \tilde{x}_0'$.

(ii) Suppose $[\gamma] \in \mathrm{stab}(\tilde{x}_0)$. Then $\tilde{\gamma}$ is a loop based at $\tilde{x_0}$. So $\tilde{\gamma}$ defines $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ and $\gamma = p \circ \tilde{\gamma}$.

(iii) This follows directly from the orbit-stabilizer theorem. □

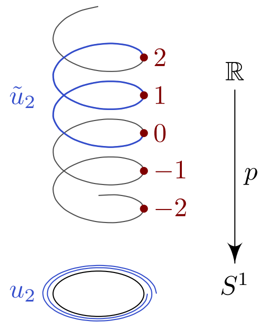
Corollary. If $p : \tilde{X} \to X$ is a universal cover, then there is a bijection $\ell : \pi_1(X, x_0) \to p^{-1}(x_0)$.

3.2 The fundamental group of the circle and its applications

Corollary. There is a bijection $\pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$.

Theorem. The map $\ell : \pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$ is a group isomorphism.

Proof. We know it is a bijection. So we need to check it is a group homomorphism. The idea is to write down representatives for what we think the elements should be.



Let $\tilde{u}_n : I \to \mathbb{R}$ be defined by $t \mapsto nt$, and let $u_n = p \circ \tilde{u}_n$. Since $\mathbb{R}$ is simply connected, there is a unique homotopy class between any two points. So for any $[\gamma] \in \pi_1(S^1, 1)$, if $\tilde{\gamma}$ is the lift to $\mathbb{R}$ at 0 and $\tilde{\gamma}(1) = n$, then $\tilde{\gamma} \simeq \tilde{u_n}$ as paths. So $[\gamma] = [u_n]$.

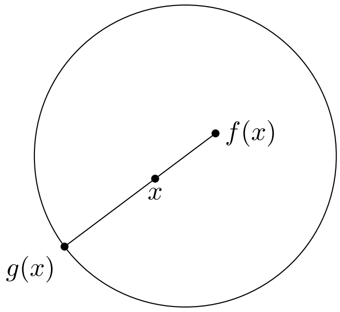
To show that this has the right group operation, we can easily see that $\widetilde{u_m \cdot u_n} = \tilde{u}_{m+n}$, since we are just moving by $n + m$ in both cases. Therefore

$$\ell([u_m][u_n]) = \ell([u_m \cdot u_m]) = m + n = \ell([u_{m+n}]).$$

So ℓ is a group isomorphism. □

Theorem (Brouwer's fixed point theorem)**.** Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ be the unit disk. If $f : D^2 \to D^2$ is continuous, then there is some $x \in D^2$ such that $f(x) = x$.

Proof. Suppose not. So $x \neq f(x)$ for all $x \in D^2$.



We define $g : D^2 \to S^1$ as in the picture above. Then we know that g is continuous and g is a retraction from D^2 onto S^1. In other words, the following composition is the identity:

$$\begin{tikzcd} S^1 \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\mathrm{id}_{S^1}"'] & D^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

Then this induces a homomorphism of groups whose composition is the identity:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "\iota_*"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{Z}}"'] & \{0\} \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

But this is clearly nonsense! So we must have had a fixed point. □

3.3 Universal covers

Theorem. If X is path connected, locally path connected and semi-locally simply connected, then X has a universal covering.

Proof. (idea) We pick a basepoint $x_0 \in X$ for ourselves. Suppose we have a universal covering $\tilde{X}$. Then this lifts to some $\tilde{x}_0$ in $\tilde{X}$. If we have any other point $\tilde{x} \in \tilde{X}$, since $\tilde{X}$ should be path connected, there is a path $\tilde{\alpha} : \tilde{x}_0 \rightsquigarrow \tilde{x}$. If we have another path, then since $\tilde{X}$ is simply connected, the paths are homotopic. Hence, we can identify each point in $\tilde{X}$ with a path from $\tilde{x}_0$, i.e.

$$\{\text{points of } \tilde{X}\} \longleftrightarrow \{\text{paths } \tilde{\alpha} \text{ from } \tilde{x}_0 \in \tilde{X}\}/\simeq.$$

This is not too helpful though, since we are defining $\tilde{X}$ in terms of things in $\tilde{X}$. However, by path lifting, we know that paths $\tilde{\alpha}$ from $\tilde{x}_0$ in $\tilde{X}$ biject with paths α from x_0 in X. Also, by homotopy lifting, homotopies of paths in X can be lifted to homotopies of paths in $\tilde{X}$. So we have

$$\{\text{points of } \tilde{X}\} \longleftrightarrow \{\text{paths } \alpha \text{ from } x_0 \in X\}/\simeq.$$

So we can produce our $\tilde{X}$ by picking a basepoint $x_0 \in X$, and defining

$$\tilde{X} = \{\text{paths } \alpha : I \to X \text{ such that } \alpha(0) = x_0\}/\simeq.$$

The covering map $p : \tilde{X} \to X$ is given by $[\alpha] \mapsto \alpha(1)$.

One then has to work hard to define the topology, and then show this is simply connected. □

3.4 The Galois correspondence

Proposition. Let X be a path connected, locally path connected and semi-locally simply connected space. For any subgroup $H \leq \pi_1(X, x_0)$, there is a based covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = H$.

Proof. Since X is a path connected, locally path connected and semi-locally simply connected space, let $\bar{X}$ be a universal covering. We have an intermediate group H such that $\pi_1(\tilde{X}, \tilde{x}_0) = 1 \leq H \leq \pi_1(X, x_0)$. How can we obtain a corresponding covering space?

Note that if we have $\bar{X}$ and we want to recover X, we can quotient $\bar{X}$ by the action of $\pi_1(X, x_0)$. Since $\pi_1(X, x_0)$ acts on $\bar{X}$, so does $H \leq \pi_1(X, x_0)$. Now we can define our covering space by taking quotients. We define $\sim_H$ on $\bar{X}$ to be the orbit relation for the action of H, i.e. $\tilde{x} \sim_H \tilde{y}$ if there is some $h \in H$ such that $\tilde{y} = h\tilde{x}$. We then let $\tilde{X}$ be the quotient space $\bar{X}/{\sim_H}$.

We can now do the messy algebra to show that this is the covering space we want. ☐

Lemma (Lifting criterion)**.** Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map of path-connected based spaces, and (Y, y_0) a path-connected, locally path connected based space. If $f : (Y, y_0) \to (X, x_0)$ is a continuous map, then there is a (unique) lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that the diagram below commutes (i.e. $p \circ \tilde{f} = f$):

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

if and only if the following condition holds:

$$f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0).$$

Proof. One direction is easy: if $\tilde{f}$ exists, then $f = p \circ \tilde{f}$. So $f_* = p_* \circ \tilde{f}_*$. So we know that $\operatorname{im} f_* \subseteq \operatorname{im} p_*$. So done.

In the other direction, uniqueness follows from the uniqueness of lifts. So we only need to prove existence. We define $\tilde{f}$ as follows:

Given a $y \in Y$, there is some path $\alpha_y : y_0 \rightsquigarrow y$. Then f maps this to $\beta_y : x_0 \rightsquigarrow f(y)$ in X. By path lifting, this path lifts uniquely to $\tilde{\beta}_y$ in $\tilde{X}$. Then we set $\tilde{f}(y) = \tilde{\beta}_y(1)$. Note that if $\tilde{f}$ exists, then this *must* be what $\tilde{f}$ sends y to. What we need to show is that this is well-defined.

Suppose we picked a different path $\alpha'_y : y_0 \rightsquigarrow y$. Then this α'_y would have differed from α_y by a loop γ in Y.

Our condition that $f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0)$ says $f \circ \gamma$ is the image of a loop in $\tilde{X}$. So $\tilde{\beta}_y$ and $\tilde{\beta}'_y$ also differ by a loop in $\tilde{X}$, and hence have the same end point. So this shows that $\tilde{f}$ is well-defined.

Finally, we show that $\tilde{f}$ is continuous. First, observe that any open set $U \subseteq \tilde{X}$ can be written as a union of $\tilde{V}$ such that $p|_{\tilde{V}} : \tilde{V} \to p(\tilde{V})$ is a homeomorphism. Thus, it suffices to show that if $p|_{\tilde{V}} : \tilde{V} \to p(\tilde{V}) = V$ is a homeomorphism, then $\tilde{f}^{-1}(\tilde{V})$ is open.

Let $y \in \tilde{f}^{-1}(\tilde{V})$, and let $x = f(y)$. Since $f^{-1}(V)$ is open and Y is locally path-connected, we can pick an open $W \subseteq f^{-1}(V)$ such that $y \in W$ and W is path connected. We claim that $W \subseteq \tilde{f}^{-1}(\tilde{V})$.

Indeed, if $z \in W$, then we can pick a path γ from y to z. Then f sends this to a path from x to $f(z)$. The lift of this path to $\tilde{X}$ is given by $p|_{\tilde{V}}^{-1}(f(\gamma))$, whose end point is $p|_{\tilde{V}}^{-1}(f(z)) \in \tilde{V}$. So it follows that $\tilde{f}(z) = p|_{\tilde{V}}^{-1}(f(z)) \in \tilde{V}$. $\square$

Proposition. Let (X, x_0), $(\tilde{X}_1, \tilde{x}_1)$, $(\tilde{X}_2, \tilde{x}_2)$ be path-connected based spaced, and $p_i : (\tilde{X}_i, \tilde{x}_i) \to (X, x_0)$ be covering maps. Then we have

$$p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$$

if and only if there is some *homeomorphism* h such that the following diagram commutes:

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, "h"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0)
\end{tikzcd}$$

i.e. $p_1 = p_2 \circ h$.

Proof. If such a homeomorphism exists, then clearly the subgroups are equal. If the subgroups are equal, we rotate our diagram a bit:

$$\begin{tikzcd}
& (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\
(\tilde{X}_1, \tilde{x}_1) \arrow[ur, dashed, "h = \tilde{p}_1"] \arrow[r, "p_1"'] & (X, x_0)
\end{tikzcd}$$

Then $h = \tilde{p}_1$ exists by the lifting criterion. By symmetry, we can get $h^{-1} = \tilde{p}_2$. To show $\tilde{p}_2$ is indeed the inverse of $\tilde{p}_1$, note that $\tilde{p}_2 \circ \tilde{p}_1$ is a lift of $p_2 \circ \tilde{p}_1 = p_1$. Since $\mathrm{id}_{\tilde{X}_1}$ is also a lift, by the uniqueness of lifts, we know $\tilde{p}_2 \circ \tilde{p}_1$ is the identity map. Similarly, $\tilde{p}_1 \circ \tilde{p}_2$ is also the identity.

$$\begin{tikzcd}
& & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\
(\tilde{X}_1, \tilde{x}_1) \arrow[r, "\tilde{p}_1"'] \arrow[rr, bend right, "p_1"'] & (\tilde{X}_2, \tilde{x}_2) \arrow[r, "p_2"'] \arrow[ur, "\tilde{p}_2"] & (X, x_0)
\end{tikzcd}$$

$\square$

Proposition. Unbased covering spaces correspond to conjugacy classes of subgroups.

4 Some group theory

4.1 Free groups and presentations

Lemma. If G is a group and $\phi : S \to G$ is a set map, then there exists a unique *homomorphism* $f : F(S) \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
F(S) \arrow[dr, dashed, "f"] \\
S \arrow[u] \arrow[r, "\phi"'] & G
\end{tikzcd}$$

where the arrow not labeled is the natural inclusion map that sends s_α (as a symbol from the alphabet) to s_α (as a word).

Proof. Clearly if f exists, then f must send each s_α to $\phi(s_\alpha)$ and s_α^{-1} to $\phi(s_\alpha)^{-1}$. Then the values of f on all other elements must be determined by

$$f(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$$

since f is a homomorphism. So if f exists, it must be unique. So it suffices to show that this f is a well-defined homomorphism.

This is well-defined if we define $F(S)$ to be the set of all reduced words, since each reduced word has a unique representation (since it is *defined* to be the representation itself).

To show this is a homomorphism, suppose

$$x = x_1 \cdots x_n a_1 \cdots a_k, \quad y = a_k^{-1} \cdots a_1^{-1} y_1 \cdots y_m,$$

where $y_1 \neq x_n^{-1}$. Then

$$xy = x_1 \cdots x_n y_1 \cdots y_m.$$

Then we can compute

$$\begin{aligned}
f(x)f(y) &= \big(\phi(x_1) \cdots \phi(x_n)\phi(a_1) \cdots \phi(a_k)\big)\big(\phi(a_k)^{-1} \cdots \phi(a_1)^{-1}\phi(y_1) \cdots \phi(y_m)\big)\\
&= \phi(x_1) \cdots \phi(x_n) \cdots \phi(y_1) \cdots \phi(y_m)\\
&= f(xy).
\end{aligned}$$

So f is a homomorphism. □

Lemma. If G is a group and $\phi : S \to G$ is a set map such that $f(r) = 1$ for all $r \in R$ (i.e. if $r = s_1^{\pm 1} s_2^{\pm 1} \cdots s_m^{\pm 1}$, then $\phi(r) = \phi(s_1)^{\pm 1}\phi(s_2)^{\pm 1} \cdots \phi(s_m)^{\pm 1} = 1$), then there exists a unique homomorphism $f : \langle S \mid R\rangle \to G$ such that the following triangle commutes:

$$\begin{tikzcd}
\langle S \mid R \rangle \arrow[dr, dashed, "f"] \\
S \arrow[u] \arrow[r, "\phi"'] & G
\end{tikzcd}$$

4.2 Another view of free groups

4.3 Free products with amalgamation

Lemma. $G_1 * G_2$ is the group such that for any group K and homomorphisms $\phi_i : G_i \to K$, there exists a unique homomorphism $f : G_1 * G_2 \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
 & G_2 \arrow[d, "j_2"] \arrow[ddr, bend left, "\phi_2"] & \\
G_1 \arrow[r, "j_1"] \arrow[drr, bend right, "\phi_1"] & G_1 * G_2 \arrow[dr, dashed, "f"] & \\
 & & K
\end{tikzcd}$$

Proof. It is immediate from the universal property of the definition of presentations. □

Corollary. The free product is well-defined.

Proof. The conclusion of the universal property can be seen to characterize $G_1 * G_2$ up to isomorphism. □

Lemma. $G_1 \underset{H}{*} G_2$ is the group such that for any group K and homomorphisms $\phi_i : G_i \to K$, there exists a unique homomorphism $G_1 \underset{H}{*} G_2 \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
H \arrow[r, "i_2"] \arrow[d, "i_1"] & G_2 \arrow[d, "j_2"] \arrow[ddr, bend left, "\phi_2"] & \\
G_1 \arrow[r, "j_1"] \arrow[drr, bend right, "\phi_1"] & G_1 \underset{H}{*} G_2 \arrow[dr, dashed, "f"] & \\
 & & K
\end{tikzcd}$$

5 Seifert-van Kampen theorem

5.1 Seifert-van Kampen theorem

Theorem (Seifert-van Kampen theorem)**.** Let A, B be open subspaces of X such that $X = A \cup B$, and $A, B, A \cap B$ are path-connected. Then for any $x_0 \in A \cap B$, we have

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A \cap B, x_0)}{*} \pi_1(B, x_0).$$

5.2 The effect on π_1 of attaching cells

Theorem. If $n \geq 3$, then $\pi_1(X \cup_f D^n) \cong \pi_1(X)$. More precisely, the map $\pi_1(X, x_0) \to \pi_1(X \cup_f D^n, x_0)$ induced by inclusion is an isomorphism, where x_0 is a point on the image of f.

Proof. Again, the difficulty of applying Seifert-van Kampen theorem is that we need to work with open sets.

Let $0 \in D^n$ be any point in the interior of D^n. We let $A = X \cup_f (D^n \setminus \{0\})$. Note that $D^n \setminus \{0\}$ deformation retracts to the boundary S^{n-1}. So A deformation retracts to X. Let $B = \mathring{D}$, the interior of D^n. Then

$$A \cap B = \mathring{D}^n \setminus 0 \cong S^{n-1} \times (-1, 1)$$

We cannot use y_0 as our basepoint, since this point is not in $A \cap B$. Instead, pick an arbitrary $y_1 \in A \cap B$. Since D^n is path connected, we have a path $\gamma : y_1 \rightsquigarrow y_0$, and we can use this to recover the fundamental groups based at y_0.

Now Seifert-van Kampen theorem says

$$\pi_1(X \cup_f D^n, y_1) \cong \pi_1(A, y_1) \underset{\pi_1(A \cap B, y_1)}{*} \pi_1(B, y_1).$$

Since B is just a disk, and $A \cap B$ is simply connected ($n \geq 3$ implies S^{n-1} is simply connected), their fundamental groups are trivial. So we get

$$\pi_1(X \cup_f D^n, y_1) \cong \pi_1(A, y_1).$$

We can now use γ to change base points from y_1 to y_0. So

$$\pi_1(X \cup_f D^n, y_0) \cong \pi_1(A, y_0) \cong \pi_1(X, y_0). \qquad \square$$

Theorem. If $n = 2$, then the natural map $\pi_1(X, x_0) \to \pi_1(X \cup_f D^n, x_0)$ is *surjective*, and the kernel is $\langle\langle [f] \rangle\rangle$. Note that this statement makes sense, since S^{n-1} is a circle, and $f : S^{n-1} \to X$ is a loop in X.

Proof. As before, we get

$$\pi_1(X \cup_f D^n, y_1) \cong \pi_1(A, y_1) \underset{\pi_1(A \cap B, y_1)}{*} \pi_1(B, y_1).$$

Again, B is contractible, and $\pi_1(B, y_1) \cong 1$. However, $\pi_1(A \cap B, y_1) \cong \mathbb{Z}$. Since $\pi_1(A \cap B, y_1)$ is just (homotopic to) the loop induced by f, it follows that

$$\pi_1(A, y_1) \underset{\pi_1(A \cap B, y_1)}{*} 1 = (\pi_1(A, y_1) * 1)/\langle\langle \pi_1(A \cap B, y_1) \rangle\rangle \cong \pi_1(X, x_0)/\langle\langle f \rangle\rangle. \quad \square$$

Corollary. For any (finite) group presentation $\langle S \mid R\rangle$, there exists a (finite) cell complex (of dimension 2) X such that $\pi_1(X) \cong \langle S \mid R\rangle$.

Proof. Let $S = \{a_1, \cdots, a_m\}$ and $R = \{r_1, \cdots, r_n\}$. We start with a single point, and get our $X^{(1)}$ by adding a loop about the point for each $a_i \in S$. We then get our 2-cells e_j^2 for $j = 1, \cdots, n$, and attaching them to $X^{(1)}$ by $f_i : S^1 \to X^{(1)}$ given by a based loop representing $r_i \in F(S)$. □

5.3 A refinement of the Seifert-van Kampen theorem

Theorem. Let X be a space, $A, B \subseteq X$ closed subspaces. Suppose that A, B and $A \cap B$ are path connected, and $A \cap B$ is a neighbourhood deformation retract of A and B. Then for any $x_0 \in A \cap B$.

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0).$$

Proof. Pick open neighbourhoods $A \cap B \subseteq U \subseteq A$ and $A \cap B \subseteq V \subseteq B$ that strongly deformation retract to $A \cap B$. Let U be such that U retracts to $A \cap B$. Since U retracts to A, it follows that U is path connected since path-connectedness is preserved by homotopies.

Let $A' = A \cup V$ and $B' = B \cup U$. Since $A' = (X \setminus B) \cup V$, and $B' = (X \setminus A) \cup U$, it follows that A' and B' are open.

Since U and V retract to $A \cap B$, we know $A' \simeq A$ and $B' \simeq B$. Also, $A' \cap B' = (A \cup V) \cap (B \cup U) = U \cup V \simeq A \cap B$. In particular, it is path connected. So by Seifert van-Kampen, we get

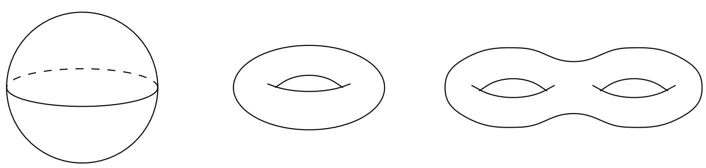
$$\pi_1(A \cup B) = \pi_1(A', x_0) \underset{\pi_1(A'\cap B', x_0)}{*} \pi_1(B', x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0).$$

□

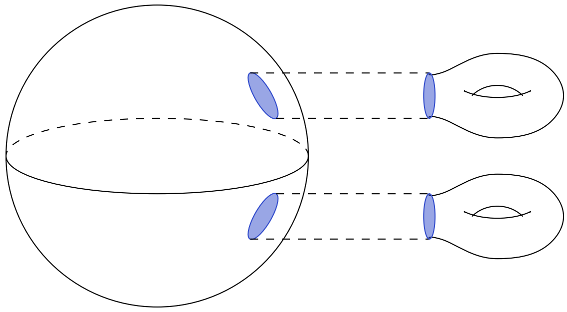
5.4 The fundamental group of all surfaces

Theorem (Classification of compact surfaces)**.** If X is a compact surface, then X is homeomorphic to a space in one of the following two families:

(i) The *orientable surface of genus g*, Σ_g includes the following (please excuse my drawing skills):



A more formal definition of this family is the following: we start with the 2-sphere, and remove a few discs from it to get $S^2 \setminus \cup_{i=1}^g D^2$. Then we take g tori with an open disc removed, and attach them to the circles.



(ii) The *non-orientable surface of genus n*, $E_n = \{\mathbb{RP}^2, K, \cdots\}$ (where K is the Klein bottle). This has a similar construction as above: we start with the sphere S^2, make a few holes, and then glue Möbius strips to them.

6 Simplicial complexes

6.1 Simplicial complexes

Lemma. $a_0, \cdots, a_n \in \mathbb{R}^m$ are affinely independent if and only if $a_1 - a_0, \cdots, a_n - a_0$ are linearly independent.

Proof. Suppose $a_0, \cdots, a_n$ are affinely independent. Suppose

$$\sum_{i=1}^{n} \lambda_i (a_i - a_0) = 0.$$

Then we can rewrite this as

$$\left(-\sum_{i=1}^{n} \lambda_i\right) a_0 + \lambda_1 a_1 + \cdots + \lambda_n a_n = 0.$$

Now the sum of the coefficients is 0. So affine independence implies that all coefficients are 0. So $a_1 - a_0, \cdots, a_n - a_0$ are linearly independent.

On the other hand, suppose $a_1 - a_0, \cdots, a_n - a_0$ are linearly independent. Now suppose

$$\sum_{i=0}^{n} t_i a_i = 0, \quad \sum_{i=0}^{n} t_i = 0.$$

Then we can write

$$t_0 = -\sum_{i=1}^{n} t_i.$$

Then the first equation reads

$$0 = \left(-\sum_{i=1}^{n} t_i\right) a_0 + t_1 a_1 + \cdots + t_n a_n = \sum_{i=1}^{n} t_i (a_i - a_0).$$

So linear independence implies all $t_i = 0$. □

Lemma. If K is a simplicial complex, then every point $x \in |K|$ lies in the interior of a *unique* simplex.

Lemma. A simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$, and furthermore, we have

$$|f \circ g| = |f| \circ |g|.$$

Proof. For any point in a simplex $\sigma = \langle a_0, \cdots, a_n \rangle$, we define

$$|f| \left(\sum_{i=0}^{n} t_i a_i\right) = \sum_{i=0}^{n} t_i f(a_i).$$

The result is in L because $\{f(a_i)\}$ spans a simplex. It is not difficult to see this is well-defined when the point lies on the boundary of a simplex. This is clearly continuous on σ, and is hence continuous on $|K|$ by the gluing lemma.

The final property is obviously true by definition. □

6.2 Simplicial approximation

Lemma. If $f : |K| \to |L|$ is a map between polyhedra, and $g : V_K \to V_L$ is a simplicial approximation to f, then g is a simplicial map, and $|g| \simeq f$. Furthermore, if f is already simplicial on some subcomplex $M \subseteq K$, then we get $g|_M = f|_M$, and the homotopy can be made rel M.

Proof. First we want to check g is really a simplicial map if it satisfies $(*)$. Let $\sigma = \langle a_0, \cdots, a_n\rangle$ be a simplex in K. We want to show that $\{g(a_0), \cdots, g(a_n)\}$ spans a simplex in L.

Pick an arbitrary $x \in \mathring{\sigma}$. Since σ contains each a_i, we know that $x \in \mathrm{St}_K(a_i)$ for all i. Hence we know that

$$f(x) \in \bigcap_{i=0}^n f(\mathrm{St}_K(a_i)) \subseteq \bigcap_{i=0}^n \mathrm{St}_L(g(a_i)).$$

Hence we know that there is one simplex, say, τ that contains all $g(a_i)$ whose interior contains $f(x)$. Since each $g(a_i)$ is a vertex in L, each $g(a_i)$ must be a vertex of τ. So they span a face of τ, as required.

We now want to prove that $|g| \simeq f$. We let $H : |K| \times I \to |L| \subseteq \mathbb{R}^m$ be defined by

$$(x, t) \mapsto t|g|(x) + (1 - t)f(x).$$

This is clearly continuous. So we need to check that $\operatorname{im} H \subseteq |L|$. But we know that both $|g|(x)$ and $f(x)$ live in τ and τ is convex. It thus follows that $H(x \times I) \subseteq \tau \subseteq |L|$.

To prove the last part, it suffices to show that every simplicial approximation to a simplicial map must be the map itself. Then the homotopy is rel M by the construction above. This is easily seen to be true — if g is a simplicial approximation to f, then $f(v) \in f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(g(v))$. Since $f(v)$ is a vertex and $g(v)$ is the only vertex in $\mathrm{St}_L(g(v))$, we must have $f(v) = g(v)$. So done. □

Proposition. $|K| = |K'|$ and K' really is a simplicial complex.

Proof. Too boring to be included in lectures. □

Theorem (Simplicial approximation theorem)**.** Let K and L be simplicial complexes, and $f : |K| \to |L|$ a continuous map. Then there exists an r and a simplicial map $g : K^{(r)} \to L$ such that g is a simplicial approximation of f. Furthermore, if f is already simplicial on $M \subseteq K$, then we can choose g such that $|g||_M = f|_M$.

Lemma. Let $\dim K = n$, then

$$\mu(K^{(r)}) \leq \left(\frac{n}{n+1}\right)^r \mu(K).$$

Proof of simplicial approximation theorem. Suppose we are given the map $f : |K| \to |L|$. We have a natural cover of $|L|$, namely the open stars of all vertices. We can use f to pull these back to $|K|$ to obtain a cover of $|K|$:

$$\{f^{-1}(\mathrm{St}_L(w)) : w \in V_L\}.$$

The idea is to barycentrically subdivide our K such that each open star of K is contained in one of these things.

By the Lebesgue number lemma, there exists some δ, the *Lebesgue number* of the cover, such that for each $x \in |K|$, $B_\delta(x)$ is contained in *some* element of the cover. By the previous lemma, there is an r such that $\mu(K^{(r)}) < \delta$.

Now since the mesh $\mu(K^{(r)})$ is the smallest distance between any two vertices, the radius of every open star $\mathrm{St}_{K^{(r)}}(x)$ is at most $\mu(K^{(r)})$. Hence it follows that $\mathrm{St}_{K^{(r)}}(x) \subseteq B_\delta(x)$ for all vertices $x \in V_{K^{(r)}}$. Therefore, for all $x \in V_{K^{(r)}}$, there is some $w \in V_L$ such that

$$\mathrm{St}_{K^{(r)}}(x) \subseteq B_\delta(x) \subseteq f^{-1}(\mathrm{St}_L(w)).$$

Therefore defining $g(x) = w$, we get

$$f(\mathrm{St}_{K^{(r)}}(x)) \subseteq \mathrm{St}_L(g(x)).$$

So g is a simplicial approximation of f.

The last part follows from the observation that if f is a simplicial map, then it maps vertices to vertices. So we can pick $g(v) = f(v)$. $\square$

7 Simplicial homology

7.1 Simplicial homology

Lemma. $d_{n-1} \circ d_n = 0$.

Proof. This just involves expanding the definition and working through the mess. □

7.2 Some homological algebra

Lemma. A chain map $f_\bullet : C_\bullet \to D_\bullet$ induces a homomorphism:

$$\begin{aligned} f_* : H_n(C) &\to H_n(D) \\ [c] &\mapsto [f(c)] \end{aligned}$$

Furthermore, if $f_\bullet$ and $g_\bullet$ are chain homotopic, then $f_* = g_*$.

Proof. Since the homology groups are defined as the cycles quotiented by the boundaries, to show that f_* defines a homomorphism, we need to show f sends cycles to cycles and boundaries to boundaries. This is an easy check. If $d_n(\sigma) = 0$, then

$$d_n(f_n(\sigma)) = f_n(d_n(\sigma)) = f_n(0) = 0.$$

So $f_n(\sigma) \in Z_n(D)$.

Similarly, if σ is a boundary, say $\sigma = d_n(\tau)$, then

$$f_n(\sigma) = f_n(d_n(\tau)) = d_n(f_n(\tau)).$$

So $f_n(\sigma)$ is a boundary. It thus follows that f_* is well-defined.

Now suppose h_n is a chain homotopy between f and g. For any $c \in Z_n(C)$, we have

$$g_n(c) - f_n(c) = d_{n+1} \circ h_n(c) + h_{n-1} \circ d_n(c).$$

Since $c \in Z_n(C)$, we know that $d_n(c) = 0$. So

$$g_n(c) - f_n(c) = d_{n+1} \circ h_n(c) \in B_n(D).$$

Hence $g_n(c)$ and $f_n(c)$ differ by a boundary. So $[g_n(c)] - [f_n(c)] = 0$ in $H_n(D)$, i.e. $f_*(c) = g_*(c)$. □

Proposition.

(i) Being chain-homotopic is an equivalence relation of chain maps.

(ii) If $a_\bullet : A_\bullet \to C_\bullet$ is a chain map and $f_\bullet \simeq g_\bullet$, then $f_\bullet \circ a_\bullet \simeq g_\bullet \circ a_\bullet$.

(iii) If $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to A_\bullet$ are chain maps, then

$$g_* \circ f_* = (g_\bullet \circ f_\bullet)_*.$$

(iv) $(\mathrm{id}_{C_\bullet})_* = \mathrm{id}_{H_*(C)}$.

Lemma. Let $f_\bullet : C_\bullet \to D_\bullet$ be a *chain homotopy equivalence*, then $f_* : H_n(C) \to H_n(D)$ is an isomorphism for all n.

Proof. Let $g_\bullet$ be the homotopy inverse. Since $f_\bullet \circ g_\bullet \simeq \mathrm{id}_{D_\bullet}$, we know $f_* \circ g_* = \mathrm{id}_{H_*(D)}$. Similarly, $g_* \circ f_* = \mathrm{id}_{H_*(C)}$. So we get isomorphisms between $H_n(C)$ and $H_n(D)$. □

7.3 Homology calculations

Lemma. Let $f : K \to L$ be a simplicial map. Then f induces a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$. Hence it also induces $f_* : H_n(K) \to H_n(L)$.

Proof. This is fairly obvious, except that simplicial maps are allowed to "squash" simplices, so f might send an n-simplex to an $(n-1)$-simplex, which is not in $D_n(L)$. We solve this problem by just killing these troublesome simplices.

Let σ be an oriented n-simplex in K, corresponding to a basis element of $C_n(K)$. Then we define

$$f_n(\sigma) = \begin{cases} f(\sigma) & f(\sigma) \text{ is an } n\text{-simplex} \\ 0 & f(\sigma) \text{ is a } k\text{-simplex for } k < n \end{cases}.$$

More precisely, if $\sigma = (a_0, \cdots, a_n)$, then

$$f_n(\sigma) = \begin{cases} (f(a_0), \cdots, f(a_n)) & f(a_0), \cdots, f(a_n) \text{ spans an } n\text{-simplex} \\ 0 & \text{otherwise} \end{cases}.$$

We then extend f_n linearly to obtain $f_n : C_n(K) \to C_n(L)$.

It is immediate from this that this satisfies the chain map condition, i.e. $f_\bullet$ commutes with the boundary operators. □

Lemma. If K is a cone with cone point v_0, then inclusion $i : \{v_0\} \to |K|$ induces a chain homotopy equivalence $i_\bullet : C_n(\{v_0\}) \to C_n(K)$. Therefore

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Corollary. If Δ^n is the standard n-simplex, and L consists of Δ^n and all its faces, then

$$H_k(L) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}$$

Proof. K is a cone (on any vertex). □

Corollary. Let K be the standard $(n-1)$-sphere (i.e. the proper faces of L from above). Then for $n \geq 2$, we have

$$H_k(K) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & 0 < k < n-1 \\ \mathbb{Z} & k = n-1 \end{cases}.$$

Proof. We write down the chain groups for K and L.

$$\begin{tikzcd}
0 & C_0(L) \arrow[l] \arrow[d, equal] & C_1(L) \arrow[l, "d_1^L"'] \arrow[d, equal] & \cdots \arrow[l] & C_{n-1}(L) \arrow[l, "d_{n-1}^L"'] \arrow[d, equal] & C_n(L) \arrow[l, "d_n^L"'] \\
0 & C_0(K) \arrow[l] & C_1(K) \arrow[l, "d_1^K"] & \cdots \arrow[l] & C_{n-1}(K) \arrow[l, "d_{n-1}^K"] & C_n(K) = 0 \arrow[l]
\end{tikzcd}$$

For $k < n-1$, we have $C_k(K) = C_k(L)$ and $C_{k+1}(K) = C_{k+1}(L)$. Also, the boundary maps are equal. So

$$H_k(K) = H_k(L) = 0.$$

We now need to compute

$$H_{n-1}(K) = \ker d_{n-1}^K = \ker d_{n-1}^L = \operatorname{im} d_n^L.$$

We get the last equality since

$$\frac{\ker d_{n-1}^L}{\operatorname{im} d_n^L} = H_{n-1}(L) = 0.$$

We also know that $C_n(L)$ is generated by just one simplex $(e_0, \cdots, e_n)$. So $C_n(L) \cong \mathbb{Z}$. Also d_n^L is injective since it does not kill the generator $(e_0, \cdots, e_n)$. So

$$H_{n-1}(K) \cong \operatorname{im} d_n^L \cong \mathbb{Z}. \qquad \square$$

Lemma (Interpretation of H_0)**.** $H_0(K) \cong \mathbb{Z}^d$, where d is the number of path components of K.

Proof. Let K be our simplicial complex and $v, w \in V_k$. We note that by definition, v, w represent the same homology class in $H_0(K)$ if and only if there is some c such that $d_1 c = w - v$. The requirement that $d_1 c = w - v$ is equivalent to saying c is a path from v to w. So $[v] = [w]$ if and only if v and w are in the same path component of K. $\square$

7.4 Mayer-Vietoris sequence

Theorem (Snake lemma)**.** If we have a short exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0$$

then a miracle happens to their homology groups. In particular, there is a long exact sequence (i.e. an exact sequence that is not short)

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial_*"'] \\ & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

where i_* and j_* are induced by $i_\bullet$ and $j_\bullet$, and ∂_* is a map we will define in the proof.

Theorem (Mayer-Vietoris theorem)**.** Let K, L, M, N be simplicial complexes with $K = M \cup N$ and $L = M \cap N$. We have the following inclusion maps:

$$\begin{tikzcd} L \arrow[r, hook, "i"] \arrow[d, hook, "j"] & M \arrow[d, hook, "k"] \\ N \arrow[r, hook, "\ell"] & K. \end{tikzcd}$$

Then there exists some natural homomorphism $\partial_* : H_n(K) \to H_{n-1}(L)$ that gives the following long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(L) \arrow[r, "i_* + j_*"] & H_n(M) \oplus H_n(N) \arrow[r, "k_* - \ell_*"] & H_n(K) \arrow[dlll, "\partial_*"] \\
H_{n-1}(L) \arrow[r, "i_* + j_*"] & H_{n-1}(M) \oplus H_{n-1}(N) \arrow[r, "k_* - \ell_*"] & H_{n-1}(K) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \longrightarrow H_0(M) \oplus H_0(N) \xrightarrow{k_* - \ell_*} H_0(K) \longrightarrow 0$$

Proof. All we have to do is to produce a short exact sequence of complexes. We have

$$0 \longrightarrow C_n(L) \xrightarrow{i_n + j_n} C_n(M) \oplus C_n(N) \xrightarrow{k_n - \ell_n} C_n(K) \longrightarrow 0$$

Here $i_n + j_n : C_n(L) \to C_n(M) \oplus C_n(N)$ is the map $x \mapsto (x, x)$, while $k_n - \ell_n : C_n(M) \oplus C_n(N) \to C_n(K)$ is the map $(a, b) \mapsto a - b$ (after applying the appropriate inclusion maps).

It is easy to see that this is a short exact sequence of chain complexes. The image of $i_n + j_n$ is the set of all elements of the form (x, x), and the kernel of $k_n - \ell_n$ is also these. It is also easy to see that $i_n + j_n$ is injective and $k_n - \ell_n$ is surjective. $\square$

Theorem (Snake lemma)**.** If we have a short exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{i_{\bullet}} B_{\bullet} \xrightarrow{j_{\bullet}} C_{\bullet} \longrightarrow 0$$

then there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dlll, "\partial_*"] \\
H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

where i_* and j_* are induced by $i_{\bullet}$ and $j_{\bullet}$, and ∂_* is a map we will define in the proof.

Proof. The proof of this is in general not hard. It just involves a lot of checking of the details, such as making sure the homomorphisms are well-defined, are actually homomorphisms, are exact at all the places etc. The only important and non-trivial part is just the construction of the map ∂_*.

First we look at the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "j_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

To construct $\partial_* : H_n(C) \to H_{n-1}(A)$, let $[x] \in H_n(C)$ be a class represented by $x \in Z_n(C)$. We need to find a cycle $z \in A_{n-1}$. By exactness, we know the map $j_n : B_n \to C_n$ is surjective. So there is a $y \in B_n$ such that $j_n(y) = x$. Since our target is A_{n-1}, we want to move down to the next level. So consider $d_n(y) \in B_{n-1}$. We would be done if $d_n(y)$ is in the image of i_{n-1}. By exactness, this is equivalent to saying $d_n(y)$ is in the kernel of j_{n-1}. Since the diagram is commutative, we know

$$j_{n-1} \circ d_n(y) = d_n \circ j_n(y) = d_n(x) = 0,$$

using the fact that x is a cycle. So $d_n(y) \in \ker j_{n-1} = \operatorname{im} i_{n-1}$. Moreover, by exactness again, i_{n-1} is injective. So there is a unique $z \in A_{n-1}$ such that $i_{n-1}(z) = d_n(y)$. We have now produced our z.

We are not done. We have $\partial_*[x] = [z]$ as our candidate definition, but we need to check many things:

(i) We need to make sure ∂_* is indeed a homomorphism.

(ii) We need $d_{n-1}(z) = 0$ so that $[z] \in H_{n-1}(A)$;

(iii) We need to check $[z]$ is well-defined, i.e. it does not depend on our choice of y and x for the homology class $[x]$.

(iv) We need to check the exactness of the resulting sequence.

We now check them one by one:

(i) Since everything involved in defining ∂_* are homomorphisms, it follows that ∂_* is also a homomorphism.

(ii) We check $d_{n-1}(z) = 0$. To do so, we need to add an additional layer.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "j_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "d_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, "d_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i_{n-2}"] & B_{n-2} \arrow[r, "j_{n-2}"] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

We want to check that $d_{n-1}(z) = 0$. We will use the commutativity of the diagram. In particular, we know

$$i_{n-2} \circ d_{n-1}(z) = d_{n-1} \circ i_{n-1}(z) = d_{n-1} \circ d_n(y) = 0.$$

By exactness at A_{n-2}, we know i_{n-2} is injective. So we must have $d_{n-1}(z) = 0$.

(iii) (a) First, in the proof, suppose we picked a different y' such that $j_n(y') = j_n(y) = x$. Then $j_n(y' - y) = 0$. So $y' - y \in \ker j_n = \operatorname{im} i_n$. Let $a \in A_n$ be such that $i_n(a) = y' - y$. Then

$$\begin{aligned} d_n(y') &= d_n(y' - y) + d_n(y) \\ &= d_n \circ i_n(a) + d_n(y) \\ &= i_{n-1} \circ d_n(a) + d_n(y). \end{aligned}$$

Hence when we pull back $d_n(y')$ and $d_n(y)$ to A_{n-1}, the results differ by the boundary $d_n(a)$, and hence produce the same homology class.

(b) Suppose $[x'] = [x]$. We want to show that $\partial_*[x] = \partial_*[x']$. This time, we add a layer above.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d_{n+1}"] & B_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "d_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "j_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

By definition, since $[x'] = [x]$, there is some $c \in C_{n+1}$ such that

$$x' = x + d_{n+1}(c).$$

By surjectivity of j_{n+1}, we can write $c = j_{n+1}(b)$ for some $b \in B_{n+1}$. By commutativity of the squares, we know

$$x' = x + j_n \circ d_{n+1}(b).$$

The next step of the proof is to find some y such that $j_n(y) = x$. Then

$$j_n(y + d_{n+1}(b)) = x'.$$

So the corresponding y' is $y' = y + d_{n+1}(b)$. So $d_n(y) = d_n(y')$, and hence $\partial_*[x] = \partial_*[x']$.

(iv) This is yet another standard diagram chasing argument. When reading this, it is helpful to look at a diagram and see how the elements are chased along. It is even more beneficial to attempt to prove this yourself.

(a) $\operatorname{im} i_* \subseteq \ker j_*$: This follows from the assumption that $i_n \circ j_n = 0$.

(b) $\ker j_* \subseteq \operatorname{im} i_*$: Let $[b] \in H_n(B)$. Suppose $j_*([b]) = 0$. Then there is some $c \in C_{n+1}$ such that $j_n(b) = d_{n+1}(c)$. By surjectivity of j_{n+1}, there is some $b' \in B_{n+1}$ such that $j_{n+1}(b') = c$. By commutativity, we know $j_n(b) = j_n \circ d_{n+1}(b')$, i.e.

$$j_n(b - d_{n+1}(b')) = 0.$$

By exactness of the sequence, we know there is some $a \in A_n$ such that

$$i_n(a) = b - d_{n+1}(b').$$

Moreover,

$$i_{n-1} \circ d_n(a) = d_n \circ i_n(a) = d_n(b - d_{n+1}(b')) = 0,$$

using the fact that b is a cycle. Since i_{n-1} is injective, it follows that $d_n(a) = 0$. So $[a] \in H_n(A)$. Then

$$i_*([a]) = [b] - [d_{n+1}(b')] = [b].$$

So $[b] \in \operatorname{im} i_*$.

(c) $\operatorname{im} j_* \subseteq \ker \partial_*$: Let $[b] \in H_n(B)$. To compute $\partial_*(j_*([b]))$, we first pull back $j_n(b)$ to $b \in B_n$. Then we compute $d_n(b)$ and then pull it back to A_{n+1}. However, we know $d_n(b) = 0$ since b is a cycle. So $\partial_*(j_*([b])) = 0$, i.e. $\partial_* \circ j_* = 0$.

(d) $\ker \partial_* \subseteq \operatorname{im} j_*$: Let $[c] \in H_n(C)$ and suppose $\partial_*([c]) = 0$. Let $b \in B_n$ be such that $j_n(b) = c$, and $a \in A_{n-1}$ such that $i_{n-1}(a) = d_n(b)$. By assumption, $\partial_*([c]) = [a] = 0$. So we know a is a boundary, say $a = d_n(a')$ for some $a' \in A_n$. Then by commutativity we know $d_n(b) = d_n \circ i_n(a')$. In other words,

$$d_n(b - i_n(a')) = 0.$$

So $[b - i_n(a')] \in H_n(B)$. Moreover,

$$j_*([b - i_n(a')]) = [j_n(b) - j_n \circ i_n(a')] = [c].$$

So $[c] \in \operatorname{im} j_*$.

(e) $\operatorname{im} \partial_* \subseteq \ker i_*$: Let $[c] \in H_n(C)$. Let $b \in B_n$ be such that $j_n(b) = c$, and $a \in A_{n-1}$ be such that $i_n(a) = d_n(b)$. Then $\partial_*([c]) = [a]$. Then

$$i_*([a]) = [i_n(a)] = [d_n(b)] = 0.$$

So $i_* \circ \partial_* = 0$.

(f) $\ker i_* \subseteq \operatorname{im} \partial_*$: Let $[a] \in H_n(A)$ and suppose $i_*([a]) = 0$. So we can find some $b \in B_{n+1}$ such that $i_n(a) = d_{n+1}(b)$. Let $c = j_{n+1}(b)$. Then

$$d_{n+1}(c) = d_{n+1} \circ j_{n+1}(b) = j_n \circ d_{n+1}(b) = j_n \circ i_n(a) = 0.$$

So $[c] \in H_n(C)$. Then $[a] = \partial_*([c])$ by definition of ∂_*. So $[a] \in \operatorname{im} \partial_*$. □

7.5 Continuous maps and homotopy invariance

Lemma. If $f, g : K \to L$ are simplicial approximations to the same map F, then f and g are contiguous.

Proof. Let $\sigma \in K$, and pick some $s \in \mathring{\sigma}$. Then $F(s) \in \mathring{\tau}$ for some $\tau \in L$. Then the definition of simplicial approximation implies that for any simplicial approximation f to F, $f(\sigma)$ spans a face of τ. □

Lemma. If $f, g : K \to L$ are continguous simplicial maps, then

$$f_* = g_* : H_n(K) \to H_n(L)$$

for all n.

Proof. We will write down a chain homotopy between f and g:

$$h_n((a_0, \cdots, a_n)) = \sum_{i=0}^{n} (-1)^i [f(a_0), \cdots, f(a_i), g(a_i), \cdots, g(a_n)],$$

where the square brackets means the corresponding oriented simplex if there are no repeats, 0 otherwise.

We can now check by direct computation that this is indeed a chain homotopy. □

Lemma. Each vertex $\hat{\sigma} \in K'$ is a barycenter of some $\sigma \in K$. Then we choose $a(\hat{\sigma})$ to be an arbitrary vertex of σ. This defines a function $a : V_{K'} \to V_K$. This a is a simplicial approximation to the identity. Moreover, *every* simplicial approximation to the identity is of this form.

Proof. Omitted. □

Proposition. Let K' be the barycentric subdivision of K, and $a : K' \to K$ a simplicial approximation to the identity map. Then the induced map $a_* : H_n(K') \to H_n(K)$ is an isomorphism for all n.

Proof. We first deal with K being a simplex σ and its faces. Now that K is just a cone (with any vertex as the cone vertex), and K' is also a cone (with the barycenter as the cone vertex). Therefore

$$H_n(K) \cong H_n(K') \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

So only $n = 0$ is (a little) interesting, but it is easy to check that a_* is an isomorphism in this case, since it just maps a vertex to a vertex, and all vertices in each simplex are in the same homology class.

To finish the proof, note that K is built up by gluing up simplices, and K' is built by gluing up subdivided simplices. So to understand their homology groups, we use the Mayer-Vietoris sequence.

Given a complicated simplicial complex K, let σ be a maximal dimensional simplex of K. We let $L = K \setminus \{\sigma\}$ (note that L includes the boundary of σ). We let $S = \{\sigma \text{ and all its faces}\} \subseteq K$ and $T = L \cap S$.

We can do similarly for K', and let L', S', T' be the corresponding barycentric subdivisions. We have $K = L \cup S$ and $K' = L' \cup S'$ (and $L' \cap S' = T'$). By the previous lemma, we see our construction of a gives $a(L') \subseteq L$, $a(S') \subseteq S$ and $a(T') \subseteq T$. So these induce maps of the corresponding homology groups

$$\begin{tikzcd}
H_n(T') \arrow[r] \arrow[d, "a_*"] & H_n(S') \oplus H_n(L') \arrow[r] \arrow[d, "a_* \oplus a_*"] & H_n(K') \arrow[r] \arrow[d, "a_*"] & H_{n-1}(T') \arrow[r] \arrow[d, "a_*"] & H_{n-1}(S') \oplus H_{n-1}(L') \arrow[d, "a_* \oplus a_*"] \\
H_n(T) \arrow[r] & H_n(S) \oplus H_n(L) \arrow[r] & H_n(K) \arrow[r] & H_{n-1}(T) \arrow[r] & H_{n-1}(S) \oplus H_{n-1}(L)
\end{tikzcd}$$

By induction, we can assume all but the middle maps are isomorphisms. By the five lemma, this implies the middle map is an isomorphism, where the five lemma is as follows: □

Lemma (Five lemma)**.** Consider the following commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "a"] & B_1 \arrow[r] \arrow[d, "b"] & C_1 \arrow[r] \arrow[d, "c"] & D_1 \arrow[r] \arrow[d, "d"] & E_1 \arrow[d, "e"] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2
\end{tikzcd}$$

If the top and bottom rows are exact, and a, b, d, e are isomorphisms, then c is also an isomorphism.

Proof. Exercise in example sheet. □

Proposition. To each continuous map $f : |K| \to |L|$, there is an associated map $f_* : H_n(K) \to H_n(L)$ (for all n) given by

$$f_* = s_* \circ \nu_{K,r}^{-1},$$

where $s : K^{(r)} \to L$ is a simplicial approximation to f, and $\nu_{K,r} : H_n(K^{(r)}) \to H_n(K)$ is the isomorphism given by composing maps $H_n(K^{(i)}) \to H_n(K^{(i-1)})$ induced by simplical approximations to the identity.

Furthermore:

(i) f_* does not depend on the choice of r or s.

(ii) If $g : |M| \to |K|$ is another continuous map, then

$$(f \circ g)_* = f_* \circ g_*.$$

Proof. Omitted. □

Corollary. If $f : |K| \to |L|$ is a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism for all n.

Proof. Immediate from (ii) of previous proposition. □

Lemma. Let L be a simplicial complex (with $|L| \subseteq \mathbb{R}^n$). Then there is an $\varepsilon = \varepsilon(L) > 0$ such that if $f, g : |K| \to |L|$ satisfy $\|f(x) - g(x)\| < \varepsilon$, then $f_* = g_* : H_n(K) \to H_n(L)$ for all n.

Proof. By the Lebesgue number lemma, there is an $\varepsilon > 0$ such that each ball of radius 2ε in $|L|$ lies in some star $\mathrm{St}_L(w)$.

Now apply the Lebesgue number lemma again to $\{f^{-1}(B_\varepsilon(y))\}_{y \in |L|}$, an open cover of $|K|$, and get $\delta > 0$ such that for all $x \in |K|$, we have

$$f(B_\delta(x)) \subseteq B_\varepsilon(y) \subseteq B_{2\varepsilon}(y) \subseteq \mathrm{St}_L(w)$$

for some $y \in |L|$ and $\mathrm{St}_L(w)$. Now since g and f differ by at most ε, we know

$$g(B_\delta(x)) \subseteq B_{2\varepsilon}(y) \subseteq \mathrm{St}_L(w).$$

Now subdivide r times so that $\mu(K^{(r)}) < \frac{1}{2}\delta$. So for all $v \in V_{K^{(r)}}$, we know

$$\mathrm{St}_{K^{(r)}}(v) \subseteq B_\delta(v).$$

This gets mapped by *both* f and g to $\mathrm{St}_L(w)$ for the same $w \in V_L$. We define $s : V_{K^{(r)}} \to V_L$ sending $v \mapsto w$. □

Theorem. Let $f \simeq g : |K| \to |L|$. Then $f_* = g_*$.

Proof. Let $H : |K| \times I \to |L|$. Since $|K| \times I$ is compact, we know H is uniformly continuous. Pick $\varepsilon = \varepsilon(L)$ as in the previous lemma. Then there is some δ such that $|s - t| < \delta$ implies $|H(x, s) - H(x, t)| < \varepsilon$ for all $x \in |K|$.

Now choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_i - t_{i-1} < \delta$ for all i. Define $f_i : |K| \to |L|$ by $f_i(x) = H(x, t_i)$. Then we know $\|f_i - f_{i-1}\| < \varepsilon$ for all i. Hence $(f_i)_* = (f_{i-1})_*$. Therefore $(f_0)_* = (f_n)_*$, i.e. $f_* = g_*$. □

Lemma. $H_n(X)$ is well-defined, i.e. it does not depend on the choice of K.

Proof. Clear from previous theorem. □

7.6 Homology of spheres and applications

Lemma. The sphere S^{n-1} is triangulable, and

$$H_k(S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We already did this computation for the standard $(n-1)$-sphere $\partial\Delta^n$, where Δ^n is the standard n-simplex. □

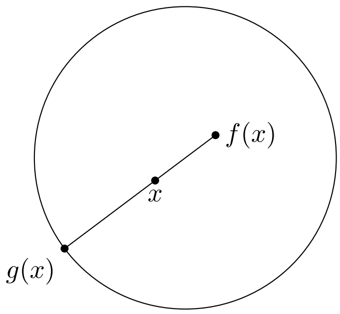
Proposition. $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $m \neq n$.

Proof. See example sheet 4. □

Theorem (Brouwer's fixed point theorem (in all dimensions))**.** There is no retraction D^n onto $\partial D^n \cong S^{n-1}$. So every continuous map $f : D^n \to D^n$ has a fixed point.

Proof. The proof is exactly the same as the two-dimensional case, with homology groups instead of the fundamental group.

We first show the second part from the first. Suppose $f : D^n \to D^n$ has no fixed point. Then the following $g : D^n \to \partial D^n$ is a continuous retraction.



So we now show no such continuous retraction can exist. Suppose $r : D^n \to \partial D^n$ is a retraction, i.e. $r \circ i \simeq \text{id} : \partial D^n \to \partial D^n$.

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$$

We now take the $(n-1)$th homology groups to obtain

$$H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1}).$$

Since $r \circ i$ is homotopic to the identity, this composition must also be the identity, but this is clearly nonsense, since $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ while $H_{n-1}(D^n) \cong 0$. So such a continuous retraction cannot exist. □

Lemma. In the triangulation of S^n given by vertices $V_K = \{\pm\mathbf{e}_0, \pm\mathbf{e}_1, \cdots, \pm\mathbf{e}_n\}$, the element

$$x = \sum_{\boldsymbol{\varepsilon} \in \{\pm 1\}^{n+1}} \varepsilon_0 \cdots \varepsilon_n (\varepsilon_0 \mathbf{e}_0, \cdots, \varepsilon_n \mathbf{e}_n)$$

is a cycle and generates $H_n(S^n)$.

Proof. By direct computation, we see that $dx = 0$. So x is a cycle. To show it generates $H_n(S^n)$, we note that everything in $H_n(S^n) \cong \mathbb{Z}$ is a multiple of the generator, and since x has coefficients ± 1, it cannot be a multiple of anything else (apart from $-x$). So x is indeed a generator. □

Proposition. If n is even, then the antipodal map $a \not\simeq \mathrm{id}$.

Proof. We can directly compute that $a_* x = (-1)^{n+1} x$. If n is even, then $a_* = -1$, but $\mathrm{id}_* = 1$. So $a \not\simeq \mathrm{id}$. □

7.7 Homology of surfaces

7.8 Rational homology, Euler and Lefschetz numbers

Lemma. If $H_n(K) \cong \mathbb{Z}^k \oplus F$ for F a finite group, then $H_n(K; \mathbb{Q}) \cong \mathbb{Q}^k$.

Proof. Exercise. □

Lemma. Let V be a finite-dimensional vector space and $W \leq V$ a subspace. Let $A : V \to V$ be a linear map such that $A(W) \subseteq W$. Let $B = A|_W : W \to W$ and $C : V/W \to V/W$ the induced map on the quotient. Then

$$\mathrm{tr}(A) = \mathrm{tr}(B) + \mathrm{tr}(C).$$

Proof. In the right basis,

$$A = \begin{pmatrix} B & A' \\ 0 & C \end{pmatrix}.$$

□

Corollary. Let $f_\bullet : C_\bullet(K; \mathbb{Q}) \to C_\bullet(K; \mathbb{Q})$ be a chain map. Then

$$\sum_{i \geq 0} (-1)^i \,\mathrm{tr}(f_i : C_i(K) \to C_i(K)) = \sum_{i \geq 0} (-1)^i \,\mathrm{tr}(f_* : H_i(K) \to H_i(K)),$$

with homology groups understood to be over $\mathbb{Q}$.

Proof. There is an exact sequence

$$0 \longrightarrow B_i(K; \mathbb{Q}) \longrightarrow Z_i(K; \mathbb{Q}) \longrightarrow H_i(K; \mathbb{Q}) \longrightarrow 0$$

This is since $H_i(K, \mathbb{Q})$ is defined as the quotient of Z_i over B_i. We also have the exact sequence

$$0 \longrightarrow Z_i(K; \mathbb{Q}) \longrightarrow C_i(K; \mathbb{Q}) \xrightarrow{d_i} B_{i-1}(K; \mathbb{Q}) \longrightarrow 0$$

This is true by definition of B_{i-1} and Z_i. Let $f_i^H, f_i^B, f_i^Z, f_i^C$ be the various maps induced by f on the corresponding groups. Then we have

$$\begin{aligned} L(|f|) &= \sum_{i \geq 0} (-1)^i \,\mathrm{tr}(f_i^H) \\ &= \sum_{i \geq 0} (-1)^i (\mathrm{tr}(f_i^Z) - \mathrm{tr}(f_i^B)) \\ &= \sum_{i \geq 0} (-1)^i (\mathrm{tr}(f_i^C) - \mathrm{tr}(f_{i-1}^B) - \mathrm{tr}(f_i^B)). \end{aligned}$$

Because of the alternating signs in dimension, each f_i^B appears twice in the sum with opposite signs. So all f_i^B cancel out, and we are left with

$$L(|f|) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(f_i^C). \qquad \square$$

Theorem (Lefschetz fixed point theorem)**.** Let $f : X \to X$ be a continuous map from a triangulable space to itself. If $L(f) \neq 0$, then f has a fixed point.

Proof. We prove the contrapositive. Suppose f has no fixed point. We will show that $L(f) = 0$. Let

$$\delta = \inf\{|x - f(x)| : x \in X\}$$

thinking of X as a subset of $\mathbb{R}^n$. We know this is non-zero, since f has no fixed point, and X is compact (and hence the infimum point is achieved by some x).

Choose a triangulation $L : |K| \to X$ such that $\mu(K) < \frac{\delta}{2}$. We now let

$$g : K^{(r)} \to K$$

be a simplicial approximation to f. Since we picked our triangulation to be so fine, for $x \in \sigma \in K$, we have

$$|f(x) - g(x)| < \frac{\delta}{2}$$

since the mesh is already less than $\frac{\delta}{2}$. Also, we know

$$|f(x) - x| \geq \delta.$$

So we have

$$|g(x) - x| > \frac{\delta}{2}.$$

So we must have $g(x) \not\in \sigma$. The conclusion is that for any $\sigma \in K$, we must have

$$g(\sigma) \cap \sigma = \emptyset.$$

Now we compute $L(f) = L(|g|)$. The only complication here is that g is a map from $K^{(r)}$ to K, and the domains and codomains are different. So we need to compose it with $s_i : C_i(K; \mathbb{Q}) \to C_i(K^{(r)}; \mathbb{Q})$ induced by inverses of simplicial approximations to the identity map. Then we have

$$\begin{aligned} L(|g|) &= \sum_{i \geq 0} (-1)^i \operatorname{tr}(g_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})) \\ &= \sum_{i \geq 0} (-1)^i \operatorname{tr}(g_i \circ s_i : C_i(K; \mathbb{Q}) \to C_i(K; \mathbb{Q})) \end{aligned}$$

Now note that s_i takes simplices of σ to sums of subsimplices of σ. So $g_i \circ s_i$ takes every simplex off itself. So each diagonal terms of the matrix of $g_i \circ s_i$ is 0! Hence the trace is

$$L(|g|) = 0. \qquad \square$$