Part II — Algebraic Topology
Theorems

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Part IB Analysis II is essential, and Metric and Topological Spaces is highly desirable

The fundamental group
Homotopy of continuous functions and homotopy equivalence between topological spaces. The fundamental group of a space, homomorphisms induced by maps of spaces, change of base point, invariance under homotopy equivalence. [3]

Covering spaces
Covering spaces and covering maps. Path-lifting and homotopy-lifting properties, and their application to the calculation of fundamental groups. The fundamental group of the circle; topological proof of the fundamental theorem of algebra. *Construction of the universal covering of a path-connected, locally simply connected space*. The correspondence between connected coverings of X and conjugacy classes of subgroups of the fundamental group of X. [5]

The Seifert-Van Kampen theorem
Free groups, generators and relations for groups, free products with amalgamation. Statement *and proof* of the Seifert-Van Kampen theorem. Applications to the calculation of fundamental groups. [4]

Simplicial complexes
Finite simplicial complexes and subdivisions; the simplicial approximation theorem. [3]

Homology
Simplicial homology, the homology groups of a simplex and its boundary. Functorial properties for simplicial maps. *Proof of functoriality for continuous maps, and of homotopy invariance*. [4]

Homology calculations
The homology groups of S^n, applications including Brouwer's fixed-point theorem. The Mayer-Vietoris theorem. *Sketch of the classification of closed combinatorical surfaces*; determination of their homology groups. Rational homology groups; the Euler-Poincaré characteristic and the Lefschetz fixed-point theorem. [5]

Contents

0 Introduction

1 Definitions

1.1 Some recollections and conventions

Lemma (Gluing lemma)**.** If $f : X \to Y$ is a function of topological spaces, $X = C \cup K$, C and K are both closed, then f is continuous if and only if the restrictions $f|_C$ and $f|_K$ are continuous.

Lemma. Let (X, d) be a compact metric space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then there is some δ such that for each $x \in X$, there is some $\alpha \in A$ such that $B_\delta(x) \subseteq U_\alpha$. We call δ a *Lebesgue number* of this cover.

1.2 Cell complexes

2 Homotopy and the fundamental group

2.1 Motivation

2.2 Homotopy

Proposition. For spaces X, Y, and $A \subseteq X$, the "homotopic rel A" relation is an equivalence relation. In particular, when $A = \emptyset$, homotopy is an equivalence relation.

Lemma. Consider the spaces and arrows

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

If $f_0 \simeq_H f_1$ and $g_0 \simeq_{H'} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proposition. Homotopy equivalence of spaces is an equivalence relation.

2.3 Paths

Proposition. For any map $f : X \to Y$, there is a well-defined function

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y),$$

defined by

$$\pi_0(f)([x]) = [f(x)].$$

Furthermore,

(i) If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.

(ii) For any maps $A \xrightarrow{h} B \xrightarrow{k} C$, we have $\pi_0(k \circ h) = \pi_0(k) \circ \pi_0(h)$.

(iii) $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $\pi_0(f)$ is a bijection.

Proposition. Let $\gamma_1, \gamma_2 : I \to X$ be paths, $\gamma_1(1) = \gamma_2(0)$. Then if $\gamma_1 \simeq \gamma_1'$ and $\gamma_2 \simeq \gamma_2'$, then $\gamma_1 \cdot \gamma_2 \simeq \gamma_1' \cdot \gamma_2'$.

Proposition. Let $\gamma_0 : x_0 \rightsquigarrow x_1, \gamma_1 : x_1 \rightsquigarrow x_2, \gamma_2 : x_2 \rightsquigarrow x_3$ be paths. Then

(i) $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$

(ii) $\gamma_0 \cdot c_{x_1} \simeq \gamma_0 \simeq c_{x_0} \cdot \gamma_0$.

(iii) $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{x_0}$ and $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{x_1}$.

2.4 The fundamental group

Theorem. The fundamental group is a group.

Proposition. To a based map

$$f : (X, x_0) \to (Y, y_0),$$

there is an associated function

$$f_* = \pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0),$$

defined by $[\gamma] \mapsto [f \circ \gamma]$. Moreover, it satisfies

(i) $\pi_1(f)$ is a homomorphism of groups.

(ii) If $f \simeq f'$, then $\pi_1(f) = \pi_1(f')$.

(iii) For any maps $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$, we have $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$.

(iv) $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X, x_0)}$

Proposition. A path $u : x_0 \rightsquigarrow x_1$ induces a group *isomorphism*

$$u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by

$$[\gamma] \mapsto [u^{-1} \cdot \gamma \cdot u].$$

This satisfies

(i) If $u \simeq u'$, then $u_\# = u'_\#$.

(ii) $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X, x_0)}$

(iii) If $v : x_1 \rightsquigarrow x_2$. Then $(u \cdot v)_\# = v_\# \circ u_\#$.

(iv) If $f : X \to Y$ with $f(x_0) = y_0$, $f(x_1) = y_1$, then

$$(f \circ u)_\# \circ f_* = f_* \circ u_\# : \pi_1(X, x_0) \to \pi_1(Y, y_1).$$

A nicer way of writing this is

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "u_\#"] & \pi_1(Y, y_0) \arrow[d, "(f \circ u)_\#"] \\
\pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

The property says that the composition is the same no matter which way we go from $\pi_1(X, x_0)$ to $\pi_1(Y, y_1)$. We say that the square is a *commutative diagram.* These diagrams will appear all of the time in this course.

(v) If $x_1 = x_0$, then $u_\#$ is an automorphism of $\pi_1(X, x_0)$ given by conjugation by u.

Lemma. The following diagram commutes:

$$\begin{tikzcd}
& \pi_1(Y, f(x_0)) \arrow[dd, "u_\#"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
& \pi_1(Y, g(x_0))
\end{tikzcd}$$

In algebra, we say
$$g_* = u_\# \circ f_*.$$

Theorem. If $f : X \to Y$ is a homotopy equivalence, and $x_0 \in X$, then the induced map
$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$
is an isomorphism.

Lemma. A path-connected space X is simply connected if and only if for any $x_0, x_1 \in X$, there exists a unique homotopy class of paths $x_0 \rightsquigarrow x_1$.

3 Covering spaces

3.1 Covering space

Lemma. Let $p : \tilde{X} \to X$ be a covering map, $f : Y \to X$ be a map, and $\tilde{f}_1, \tilde{f}_2$ be both lifts of f. Then

$$S = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

is both open and closed. In particular, if Y is connected, $\tilde{f}_1$ and $\tilde{f}_2$ agree either everywhere or nowhere.

Lemma (Homotopy lifting lemma)**.** Let $p : \tilde{X} \to X$ be a covering space, $H : Y \times I \to X$ be a homotopy from f_0 to f_1. Let $\tilde{f}_0$ be a lift of f_0. Then there exists a *unique* homotopy $\tilde{H} : Y \times I \to \tilde{X}$ such that

(i) $\tilde{H}(\,\cdot\,, 0) = \tilde{f}_0$; and

(ii) $\tilde{H}$ is a lift of H, i.e. $p \circ \tilde{H} = H$.

Lemma (Path lifting lemma)**.** Let $p : \tilde{X} \to X$ be a covering space, $\gamma : I \to X$ a path, and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0 = \gamma(0)$. Then there exists a *unique* path $\tilde{\gamma} : I \to \tilde{X}$ such that

(i) $\tilde{\gamma}(0) = \tilde{x}_0$; and

(ii) $\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.

Corollary. Suppose $\gamma, \gamma' : I \to X$ are paths $x_0 \rightsquigarrow x_1$ and $\tilde{\gamma}, \tilde{\gamma}' : I \to \tilde{X}$ are lifts of γ and γ' respectively, both starting at $\tilde{x}_0 \in p^{-1}(x_0)$.

If $\gamma \simeq \gamma'$ as *paths*, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ are homotopic as paths. In particular, $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.

Corollary. If X is a path connected space, $x_0, x_1 \in X$, then there is a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$.

Lemma. If $p : \tilde{X} \to X$ is a covering map and $\tilde{x}_0 \in \tilde{X}$, then

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$$

is injective.

Lemma. Suppose X is path connected and $x_0 \in X$.

(i) The action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive if and only if $\tilde{X}$ is path connected. Alternatively, we can say that the orbits of the action correspond to the path components.

(ii) The stabilizer of $\tilde{x}_0 \in p^{-1}(x_0)$ is $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$.

(iii) If $\tilde{X}$ is path connected, then there is a bijection

$$p_*\pi_1(\tilde{X}, \tilde{x}_0)\backslash\pi_1(X, x_0) \to p^{-1}(x_0).$$

Note that $p_*\pi_1(\tilde{X}, \tilde{x}_0)\backslash\pi_1(X, x_0)$ is not a quotient, but simply the set of cosets. We write it the "wrong way round" because we have right cosets instead of left cosets.

Corollary. If $p : \tilde{X} \to X$ is a universal cover, then there is a bijection $\ell : \pi_1(X, x_0) \to p^{-1}(x_0)$.

3.2 The fundamental group of the circle and its applications

Corollary. There is a bijection $\pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$.

Theorem. The map $\ell : \pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$ is a group isomorphism.

Theorem (Brouwer's fixed point theorem)**.** Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ be the unit disk. If $f : D^2 \to D^2$ is continuous, then there is some $x \in D^2$ such that $f(x) = x$.

3.3 Universal covers

Theorem. If X is path connected, locally path connected and semi-locally simply connected, then X has a universal covering.

3.4 The Galois correspondence

Proposition. Let X be a path connected, locally path connected and semi-locally simply connected space. For any subgroup $H \leq \pi_1(X, x_0)$, there is a based covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that $p_*\pi_1(\tilde{X}, \tilde{x}_0) = H$.

Lemma (Lifting criterion)**.** Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map of path-connected based spaces, and (Y, y_0) a path-connected, locally path connected based space. If $f : (Y, y_0) \to (X, x_0)$ is a continuous map, then there is a (unique) lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that the diagram below commutes (i.e. $p \circ \tilde{f} = f$):

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}$$

if and only if the following condition holds:

$$f_*\pi_1(Y, y_0) \leq p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Proposition. Let (X, x_0), $(\tilde{X}_1, \tilde{x}_1)$, $(\tilde{X}_2, \tilde{X}_2)$ be path-connected based spaced, and $p_i : (\tilde{X}_i, \tilde{x}_i) \to (X, x_0)$ be covering maps. Then we have

$$p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$$

if and only if there is some *homeomorphism* h such that the following diagram commutes:

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, "h"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
 & (X, x_0) &
\end{tikzcd}$$

i.e. $p_1 = p_2 \circ h$.

Proposition. Unbased covering spaces correspond to conjugacy classes of subgroups.

4 Some group theory

4.1 Free groups and presentations

Lemma. If G is a group and $\phi : S \to G$ is a set map, then there exists a unique *homomorphism* $f : F(S) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} F(S) \arrow[dr, dashed, "f"] & \\ S \arrow[u] \arrow[r, "\phi"'] & G \end{tikzcd}$$

where the arrow not labeled is the natural inclusion map that sends s_α (as a symbol from the alphabet) to s_α (as a word).

Lemma. If G is a group and $\phi : S \to G$ is a set map such that $f(r) = 1$ for all $r \in R$ (i.e. if $r = s_1^{\pm 1} s_2^{\pm 1} \cdots s_m^{\pm 1}$, then $\phi(r) = \phi(s_1)^{\pm 1}\phi(s_2)^{\pm 1} \cdots \phi(s_m)^{\pm 1} = 1$), then there exists a unique homomorphism $f : \langle S \mid R\rangle \to G$ such that the following triangle commutes:

$$\begin{tikzcd} \langle S \mid R\rangle \arrow[dr, dashed, "f"] & \\ S \arrow[u] \arrow[r, "\phi"'] & G \end{tikzcd}$$

4.2 Another view of free groups

4.3 Free products with amalgamation

Lemma. $G_1 * G_2$ is the group such that for any group K and homomorphisms $\phi_i : G_i \to K$, there exists a unique homomorphism $f : G_1 * G_2 \to K$ such that the following diagram commutes:

$$\begin{tikzcd} & G_2 \arrow[d, "j_2"] \arrow[ddr, bend left, "\phi_2"] & \\ G_1 \arrow[r, "j_1"] \arrow[drr, bend right, "\phi_1"] & G_1 * G_2 \arrow[dr, dashed, "f"] & \\ & & K \end{tikzcd}$$

Corollary. The free product is well-defined.

Lemma. $G_1 \underset{H}{*} G_2$ is the group such that for any group K and homomorphisms $\phi_i : G_i \to K$, there exists a unique homomorphism $G_1 \underset{H}{*} G_2 \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
H \arrow[r, "i_2"] \arrow[d, "i_1"] & G_2 \arrow[d, "j_2"] \arrow[ddr, bend left, "\phi_2"] & \\
G_1 \arrow[r, "j_1"] \arrow[drr, bend right, "\phi_1"] & G_1 \underset{H}{*} G_2 \arrow[dr, dashed, "f"] & \\
& & K
\end{tikzcd}$$

5 Seifert-van Kampen theorem

5.1 Seifert-van Kampen theorem

Theorem (Seifert-van Kampen theorem)**.** Let A, B be open subspaces of X such that $X = A \cup B$, and $A, B, A \cap B$ are path-connected. Then for any $x_0 \in A \cap B$, we have

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0).$$

5.2 The effect on π_1 of attaching cells

Theorem. If $n \geq 3$, then $\pi_1(X \cup_f D^n) \cong \pi_1(X)$. More precisely, the map $\pi_1(X, x_0) \to \pi_1(X \cup_f D^n, x_0)$ induced by inclusion is an isomorphism, where x_0 is a point on the image of f.

Theorem. If $n = 2$, then the natural map $\pi_1(X, X_0) \to \pi_1(X \cup_f D^n, x_0)$ is *surjective*, and the kernel is $\langle\langle [f] \rangle\rangle$. Note that this statement makes sense, since S^{n-1} is a circle, and $f : S^{n-1} \to X$ is a loop in X.

Corollary. For any (finite) group presentation $\langle S \mid R\rangle$, there exists a (finite) cell complex (of dimension 2) X such that $\pi_1(X) \cong \langle S \mid R\rangle$.

5.3 A refinement of the Seifert-van Kampen theorem

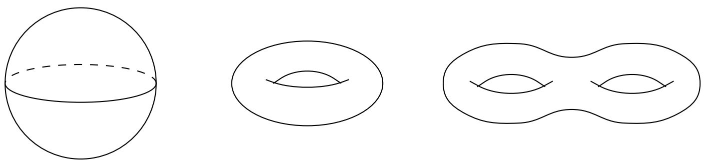
Theorem. Let X be a space, $A, B \subseteq X$ closed subspaces. Suppose that A, B and $A \cap B$ are path connected, and $A \cap B$ is a neighbourhood deformation retract of A and B. Then for any $x_0 \in A \cap B$.

$$\pi_1(X, x_0) = \pi_1(A, x_0) \underset{\pi_1(A\cap B, x_0)}{*} \pi_1(B, x_0).$$

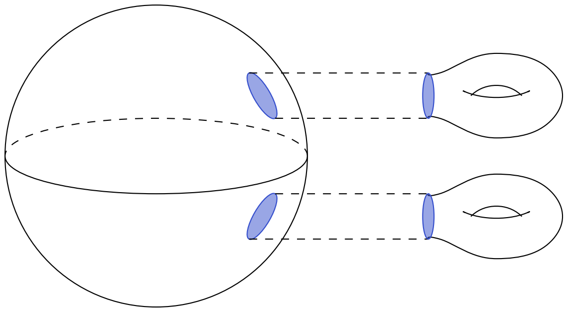
5.4 The fundamental group of all surfaces

Theorem (Classification of compact surfaces)**.** If X is a compact surface, then X is homeomorphic to a space in one of the following two families:

(i) The *orientable surface of genus g*, Σ_g includes the following (please excuse my drawing skills):



A more formal definition of this family is the following: we start with the 2-sphere, and remove a few discs from it to get $S^2 \setminus \cup_{i=1}^g D^2$. Then we take g tori with an open disc removed, and attach them to the circles.



(ii) The *non-orientable surface of genus n*, $E_n = \{\mathbb{RP}^2, K, \cdots\}$ (where K is the Klein bottle). This has a similar construction as above: we start with the sphere S^2, make a few holes, and then glue Möbius strips to them.

6 Simplicial complexes

6.1 Simplicial complexes

Lemma. $a_0, \cdots, a_n \in \mathbb{R}^m$ are affinely independent if and only if $a_1 - a_0, \cdots, a_n - a_0$ are linearly independent.

Lemma. If K is a simplicial complex, then every point $x \in |K|$ lies in the interior of a *unique* simplex.

Lemma. A simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$, and furthermore, we have

$$|f \circ g| = |f| \circ |g|.$$

6.2 Simplicial approximation

Lemma. If $f : |K| \to |L|$ is a map between polyhedra, and $g : V_K \to V_L$ is a simplicial approximation to f, then g is a simplicial map, and $|g| \simeq f$. Furthermore, if f is already simplicial on some subcomplex $M \subseteq K$, then we get $g|_M = f|_M$, and the homotopy can be made rel M.

Proposition. $|K| = |K|'$ and K' really is a simplicial complex.

Theorem (Simplicial approximation theorem)**.** Le K and L be simplicial complexes, and $f : |K| \to |L|$ a continuous map. Then there exists an r and a simplicial map $g : K^{(r)} \to L$ such that g is a simplicial approximation of f. Furthermore, if f is already simplicial on $M \subseteq K$, then we can choose g such that $|g||_M = f|_M$.

Lemma. Let $\dim K \leq n$, then

$$\mu(K^{(r)}) = \left(\frac{n}{n+1}\right)^r \mu(K).$$

7 Simplicial homology

7.1 Simplicial homology

Lemma. $d_{n-1} \circ d_n = 0$.

7.2 Some homological algebra

Lemma. A chain map $f_\bullet : C_\bullet \to D_\bullet$ induces a homomorphism:

$$\begin{aligned} f_* : H_n(C) &\to H_n(D) \\ [c] &\mapsto [f(c)] \end{aligned}$$

Furthermore, if $f_\bullet$ and $g_\bullet$ are chain homotopic, then $f_* = g_*$.

Proposition.

(i) Being chain-homotopic is an equivalence relation of chain maps.

(ii) If $a_\bullet : A_\bullet \to C_\bullet$ is a chain map and $f_\bullet \simeq g_\bullet$, then $f_\bullet \circ a_\bullet \simeq g_\bullet \circ a_\bullet$.

(iii) If $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to A_\bullet$ are chain maps, then
$$g_* \circ f_* = (f_\bullet \circ g_\bullet)_*.$$

(iv) $(\mathrm{id}_{C_\bullet})_* = \mathrm{id}_{H_*(C)}$.

Lemma. Let $f_\bullet : C_\bullet \to D_\bullet$ be a *chain homotopy equivalence*, then $f_* : H_n(C) \to H_n(D)$ is an isomorphism for all n.

7.3 Homology calculations

Lemma. Let $f : K \to L$ be a simplicial map. Then f induces a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$. Hence it also induces $f_* : H_n(K) \to H_n(L)$.

Lemma. If K is a cone with cone point v_0, then inclusion $i : \{v_0\} \to |K|$ induces a chain homotopy equivalence $i_\bullet : C_n(\{v_0\}) \to C_n(K)$. Therefore

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Corollary. If Δ^n is the standard n-simplex, and L consists of Δ^n and all its faces, then

$$H_k(L) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}$$

Corollary. Let K be the standard $(n-1)$-sphere (i.e. the proper faces of L from above). Then for $n \geq 2$, we have

$$H_k(K) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & 0 < k < n-1 \\ \mathbb{Z} & k = n-1 \end{cases}.$$

Lemma (Interpretation of H_0)**.** $H_0(K) \cong \mathbb{Z}^d$, where d is the number of path components of K.

7.4 Mayer-Vietoris sequence

Theorem (Snake lemma)**.** If we have a short exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{i_{\bullet}} B_{\bullet} \xrightarrow{j_{\bullet}} C_{\bullet} \longrightarrow 0$$

then a miracle happens to their homology groups. In particular, there is a long exact sequence (i.e. an exact sequence that is not short)

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial_*"'] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

where i_* and j_* are induced by $i_{\bullet}$ and $j_{\bullet}$, and ∂_* is a map we will define in the proof.

Theorem (Mayer-Vietoris theorem)**.** Let K, L, M, N be simplicial complexes with $K = M \cup N$ and $L = M \cap N$. We have the following inclusion maps:

$$\begin{tikzcd}
L \arrow[r, hook, "i"] \arrow[d, hook, "j"] & M \arrow[d, hook, "k"] \\
N \arrow[r, hook, "\ell"] & K.
\end{tikzcd}$$

Then there exists some natural homomorphism $\partial_* : H_n(K) \to H_{n-1}(L)$ that gives the following long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(L) \arrow[r, "i_*+j_*"] & H_n(M) \oplus H_n(N) \arrow[r, "k_*-\ell_*"] & H_n(K) \arrow[dll, "\partial_*"'] \\
& H_{n-1}(L) \arrow[r, "i_*+j_*"] & H_{n-1}(M) \oplus H_{n-1}(N) \arrow[r, "k_*-\ell_*"] & H_{n-1}(K) \arrow[r] & \cdots \\
& \cdots \arrow[r] & H_0(M) \oplus H_0(N) \arrow[r, "k_*-\ell_*"] & H_0(K) \arrow[r] & 0
\end{tikzcd}$$

Theorem (Snake lemma)**.** If we have a short exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{i_{\bullet}} B_{\bullet} \xrightarrow{j_{\bullet}} C_{\bullet} \longrightarrow 0$$

then there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial_*"'] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

where i_* and j_* are induced by $i_{\bullet}$ and $j_{\bullet}$, and ∂_* is a map we will define in the proof.

7.5 Continuous maps and homotopy invariance

Lemma. If $f, g : K \to L$ are simplicial approximations to the same map F, then f and g are contiguous.

Lemma. If $f, g : K \to L$ are continguous simplicial maps, then

$$f_* = g_* : H_n(K) \to H_n(L)$$

for all n.

Lemma. Each vertex $\hat{\sigma} \in K'$ is a barycenter of some $\sigma \in K$. Then we choose $a(\hat{\sigma})$ to be an arbitrary vertex of σ. This defines a function $a : V_{K'} \to V_K$. This a is a simplicial approximation to the identity. Moreover, *every* simplicial approximation to the identity is of this form.

Proposition. Let K' be the barycentric subdivision of K, and $a : K' \to K$ a simplicial approximation to the identity map. Then the induced map $a_* : H_n(K') \to H_n(K)$ is an isomorphism for all n.

Lemma (Five lemma)**.** Consider the following commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "a"] & B_1 \arrow[r] \arrow[d, "b"] & C_1 \arrow[r] \arrow[d, "c"] & D_1 \arrow[r] \arrow[d, "d"] & E_1 \arrow[d, "e"] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2
\end{tikzcd}$$

If the top and bottom rows are exact, and a, b, d, e are isomorphisms, then c is also an isomorphism.

Proposition. To each continuous map $f : |K| \to |L|$, there is an associated map $f_* : H_n(K) \to H_n(L)$ (for all n) given by

$$f_* = s_* \circ \nu_{K,r}^{-1},$$

where $s : K^{(r)} \to L$ is a simplicial approximation to f, and $\nu_{K,r} : H_n(K^{(r)}) \to H_n(K)$ is the isomorphism given by composing maps $H_n(K^{(i)}) \to H_n(K^{(i-1)})$ induced by simplical approximations to the identity.

Furthermore:

(i) f_* does not depend on the choice of r or s.

(ii) If $g : |M| \to |K|$ is another continuous map, then

$$(f \circ g)_* = f_* \circ g_*.$$

Corollary. If $f : |K| \to |L|$ is a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism for all n.

Lemma. Let L be a simplicial complex (with $|L| \subseteq \mathbb{R}^n$). Then there is an $\varepsilon = \varepsilon(L) > 0$ such that if $f, g : |K| \to |L|$ satisfy $\|f(x) - g(x)\| < \varepsilon$, then $f_* = g_* : H_n(K) \to H_n(L)$ for all n.

Theorem. Let $f \simeq g : |K| \to |L|$. Then $f_* = g_*$.

Lemma. $H_n(X)$ is well-defined, i.e. it does not depend on the choice of K.

7.6 Homology of spheres and applications

Lemma. The sphere S^{n-1} is triangulable, and

$$H_k(S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition. $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $m \neq n$.

Theorem (Brouwer's fixed point theorem (in all dimensions))**.** There is no retraction D^n onto $\partial D^n \cong S^{n-1}$. So every continuous map $f : D^n \to D^n$ has a fixed point.

Lemma. In the triangulation of S^n given by vertices $V_K = \{\pm \mathbf{e}_0, \pm \mathbf{e}_1, \cdots, \pm \mathbf{e}_n\}$, the element

$$x = \sum_{\boldsymbol{\varepsilon} \in \{\pm 1\}^{n+1}} \varepsilon_0 \cdots \varepsilon_n (\varepsilon_0 \mathbf{e}_0, \cdots, \varepsilon_n \mathbf{e}_n)$$

is a cycle and generates $H_n(S^n)$.

Proposition. If n is even, then the antipodal map $a \not\simeq \mathrm{id}$.

7.7 Homology of surfaces

7.8 Rational homology, Euler and Lefschetz numbers

Lemma. If $H_n(K) \cong \mathbb{Z}^k \oplus F$ for F a finite group, then $H_n(K; \mathbb{Q}) \cong \mathbb{Q}^k$.

Lemma. Let V be a finite-dimensional vector space and $W \leq V$ a subspace. Let $A : V \to V$ be a linear map such that $A(W) \subseteq W$. Let $B = A|_W : W \to W$ and $C : V/W \to V/W$ the induced map on the quotient. Then

$$\mathrm{tr}(A) = \mathrm{tr}(B) + \mathrm{tr}(C).$$

Corollary. Let $f_\bullet : C_\bullet(K; \mathbb{Q}) \to C_\bullet(K; \mathbb{Q})$ be a chain map. Then

$$\sum_{i \geq 0} (-1)^i \,\mathrm{tr}(f_i : C_i(K) \to C_i(K)) = \sum_{i \geq 0} (-1)^i \,\mathrm{tr}(f_* : H_i(K) \to H_i(K)),$$

with homology groups understood to be over $\mathbb{Q}$.

Theorem (Lefschetz fixed point theorem)**.** Let $f : X \to X$ be a continuous map from a triangulable space to itself. If $L(f) \neq 0$, then f has a fixed point.