Algebraic Topology, Examples 1

Michaelmas 2015

1. Let $a : S^n \to S^n$ be the antipodal map, $a(x) = -x$. Show that a is homotopic to the identity map when n is odd. [Try $n = 1$ first.]

2. Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that there exists an $x \in S^1$ such that $f(x) = x$, and a $y \in S^1$ so that $f(y) = -y$.

3. Suppose that $f : X \to Y$ is a map for which there exist maps $g, h : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ h \simeq \mathrm{Id}_Y$. Show that f, g, and h are homotopy equivalences.

4. Show that a retract of a contractible space is contractible.

5. Show that if a space X deformation retracts to a point $x_0 \in X$, then for every open neighbourhood $x_0 \in U$ there exists a smaller open neighbourhood $x_0 \in V \subset U$ such that the inclusion $(V, x_0) \hookrightarrow (U, x_0)$ is based homotopic to the constant map.

6. Construct a space which contains both the annulus $S^1 \times I$ and the Möbius band as deformation retracts.

7. For $m < n$, consider S^m as a subspace of S^n given by

$$\{(x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0) \mid \sum x_i^2 = 1\}.$$

Show that the complement $S^n - S^m$ is homotopy equivalent to S^{n-m-1}.

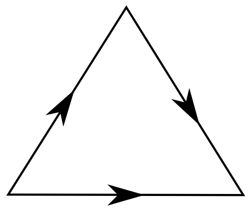
8. A space is called *locally path connected* if for every point $x \in X$ and every neighbourhood $U \ni x$, there exists a smaller neighbourhood V of x (i.e. $x \in V \subset U$) which is path connected. Show that a locally path connected space which is connected is also path connected.

9. For a map $f : S^{n-1} \to X$ we define the *space obtained by attaching an n-cell to X along f* to be the quotient space

$$X \cup_f D^n := (X \amalg D^n)/\sim$$

where $\sim$ is the smallest equivalence relation containing $b \sim f(b)$ for every $b \in S^{n-1} \subset D^n$. Show that if $f, f' : S^{n-1} \to X$ are homotopic maps then $X \cup_f D^n \simeq X \cup_{f'} D^n$.

10. The *dunce cap* is the space obtained from a solid triangle by gluing the edges together as shown.



Show that this space is contractible. [Use the previous question.]

11. Show that the Möbius band does not retract onto its boundary.

12. For based spaces (X, x_0) and (Y, y_0) show there is an isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

13. Construct a covering map $\pi : \mathbb{R}^2 \to K$ of the Klein bottle, and hence show that $\pi_1(K, k_0)$ is isomorphic to the group G with elements $(m, n) \in \mathbb{Z}^2$ and group operation

$$(m, n) * (p, q) = (m + (-1)^n \cdot p, n + q).$$

Show that K has a covering space homeomorphic to the torus $S^1 \times S^1$, but that the torus does not have a covering space homeomorphic to K.

14.* Let G be a path-connected, locally path connected topological group, and $p : \hat{G} \to G$ be a path connected covering space. Let $\epsilon \in p^{-1}(e)$ be a point in the fibre over the identity $e \in G$.

1. Show that $\hat{G}$ has a unique structure of a topological group with unit ϵ so that p is a homomorphism.
2. Show that $\mathrm{Ker}(p) \subset \hat{G}$ lies in the centre of $\hat{G}$.
3. Show that $SO(3)$, the group of rotations of $\mathbb{R}^3$ (or equivalently of orthogonal 3×3 matrices of determinant 1), is homeomorphic to the projective space $\mathbb{RP}^3$.
4. Together, 1. and 3. give a group $\widehat{SO(3)}$ homeomorphic to S^3. Identify this group with a well-known matrix group.

Algebraic Topology, Examples 2

Michaelmas 2015

1. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = e$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \,|\, g(U) \cap U \neq \emptyset\}$ is finite).

1. Show that the quotient map $X \to X/G$ is a covering map.
2. Deduce that if X is simply-connected and locally path-connected then for any point $[x] \in X/G$ we have an isomorphism of groups $\pi_1(X/G, [x]) \cong G$.
3. Hence show that for $n \geq 2$ odd and any $m \geq 2$ there is a space X with fundamental group $\mathbb{Z}/m$ and universal cover S^n. [Consider S^n as the unit sphere in $\mathbb{C}^k$.]

2. Show that the Klein bottle has a cell structure with a single 0-cell, two 1-cells, and a single 2-cell. Deduce that its fundamental group has a presentation $\langle a, b \,|\, baba^{-1} \rangle$, and show this is isomorphic to the group in Q13 of Sheet 1.

3. Show that the inclusion $i : (S^1 \times \{1\}) \cup (\{1\} \times S^1) \hookrightarrow S^1 \times S^1$ does not admit a retraction. [Where $S^1 \subset \mathbb{C}$ is the elements of unit modulus, containing 1.]

4. A *graph* G is a space obtained by starting with a set $E(G)$ of copies of the interval I and an equivalence relation $\sim$ on $E(G) \times \{0, 1\}$, and forming the quotient space of $E(G) \times I$ by the minimal equivalence relation containing $\sim$. (More practically, it is a space obtained from a collection of copies of I by gluing their ends together.) The *vertices* are the equivalence classes represented by the ends of the intervals.

A *tree* is a graph which is contractible. A tree T inside a graph G is *maximal* if no strictly larger subgraph is a tree.

(a) If $T \subset G$ is a tree, show that the quotient map $G \to G/T$ is a homotopy equivalence, and that G/T is again a graph. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex.

[Hint: You may wish to apply Proposition 0.17 from Hatcher].

(b) Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. [We have *not* required that a graph have finitely many edges.]

(c) Show that a covering space of a graph is again a graph, and deduce that a subgroup of a free group is again a free group.

5. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the two characteristic loops. Describe covering spaces associated to

1. $\langle\langle a \rangle\rangle$, the normal subgroup generated by a,
2. $\langle a \rangle$, the subgroup generated by a,
3. the kernel of the homomorphism $\phi : \langle a, b \rangle \to \mathbb{Z}/4$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

6. Consider the 2-dimensional cell complex Y obtained from X in the previous question by attaching 2-cells along loops in the homotopy classes a^2 and b^2, so that

$$\pi_1(Y, x_0) \cong \langle a, b \,|\, a^2, b^2 \rangle.$$

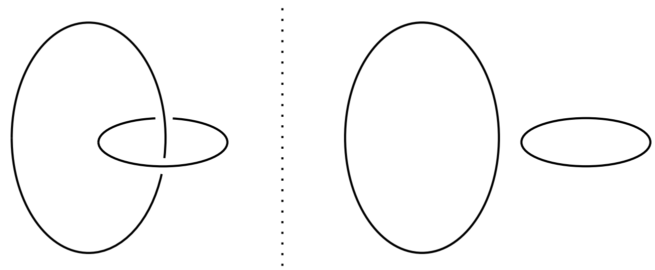
1. Construct (in pictures) the covering space of Y corresponding to the subgroup $\langle a \,|\, a^2 \rangle$.
2. Construct (in pictures) the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \,|\, a^2, b^2 \rangle \to \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\mathrm{Ker}(\phi)$ is isomorphic to $\langle a, b \,|\, a^2, b^2 \rangle$.

7. Show that the groups

$$G = \langle a, b \,|\, a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \,|\, xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. Show that this group is non-abelian and infinite. [Construct surjective homomorphisms to S_3 and $\mathbb{Z}$.]

8. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity).

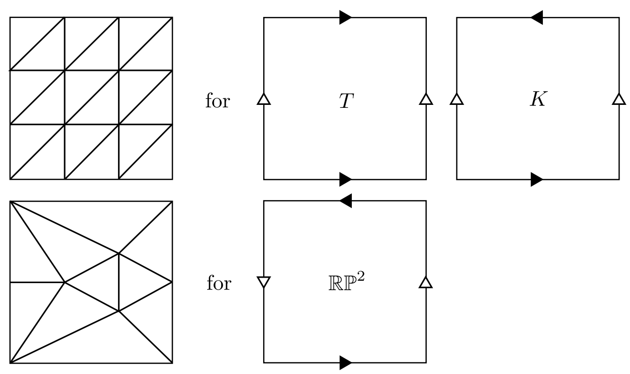


By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

Algebraic Topology, Examples 3

Michaelmas 2015

1. Show that there are triangulations of the torus, Klein bottle, and projective plane as follows:



How many vertices, edges and faces does each triangulation have? What is the number χ = vertices − edges + faces for each triangulation?

2. Use the simplicial approximation theorem to show that:

(i) if K and L are simplicial complexes, there are at most countably many homotopy classes of continuous maps $f : |K| \to |L|$;

(ii) if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map;

(iii) for any vertex v of a simplicial complex K the based map $(|K_{(2)}|, v) \to (|K|, v)$ (i.e. the inclusion of the 2-skeleton) induces an isomorphism on fundamental groups.

3. An *abstract simplicial complex* consists of a finite set V_X (called the *vertices*) and a collection X (called the *simplices*) of subsets of V_X such that if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$. A map $f : (V_X, X) \to (V_Y, Y)$ of abstract simplicial complexes is a function $f : V_X \to V_Y$ such that $f(\sigma) \in Y$ for all $\sigma \in X$.

(i) For a simplicial complex K in $\mathbb{R}^m$, show that the *abstraction* of K,

$$V_X = \{0\text{-simplices of } K\} \qquad X = \{\{a_0, \dots, a_n\} \subset V_X \mid \langle a_0, \dots, a_n \rangle \in K\}$$

is an abstract simplicial complex. Show that if simplicial complexes K and L have isomorphic abstractions, then $|K|$ and $|L|$ are homeomorphic.

(ii) Show that there exists an infinite sequence of points $(x_1, x_2, \ldots) \in \mathbb{R}^m$ such that any $(m+1)$ of them are affinely independent. Hence show that if (V_X, X) is an abstract simplicial complex with all simplices of dimension $\leq n$, then there is a simplicial complex K in $\mathbb{R}^{2n+1}$ with abstraction isomorphic to (V_X, X).

4. Let K be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex. Show that $L = K \setminus \{\sigma\}$ is again a simplicial complex, and that the inclusion $V_L \to V_K$ defines a simplicial map $i : L \to K$.

If σ has dimension n, note that $d_n(\sigma)$ is an $(n-1)$-cycle and consists of simplices of L, so represents a class $[d_n(\sigma)] \in H_{n-1}(L)$; this defines a homomorphism $\varphi : \mathbb{Z} \to H_{n-1}(L)$ by $1 \mapsto [d_n(\sigma)]$. Construct a homomorphism $\phi : H_n(K) \to \mathbb{Z}$ such that

$$0 \longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\varphi} H_{n-1}(L) \xrightarrow{i_*} H_{n-1}(K) \longrightarrow 0$$

is exact (i.e. the image of one map is *precisely* the kernel of the next), and show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for $j \neq n-1, n$.

5. Let K be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex, and that $\tau \leq \sigma$ is a face of one dimension lower which is not a face of any other simplex. Show that $L = K \setminus \{\sigma, \tau\}$ is again a simplicial complex, and that the inclusion $V_L \to V_K$ defines a simplicial map $i : L \to K$.

(i) By constructing a chain homotopy inverse to $i_\bullet : C_\bullet(L) \to C_\bullet(K)$, show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for all j.

(ii) * Prove the same thing using the previous question (twice) instead.

6. Using the two previous questions, compute the homology groups of the simplicial complexes described in Q1, and describe generators for each homology group.

7. * Let K be an n-dimensional simplicial complex such that

(i) every $(n-1)$-simplex is a face of precisely two n-simplices, and

(ii) every pair of n-simplices can be connected by a sequence of n-simplices such that adjacent terms share an $(n-1)$-dimensional face.

Show that $H_n(K)$ is either $\mathbb{Z}$ or trivial. In the first case show $H_n(K)$ is generated by a cycle which is a sum of all n-simplices with suitable orientations.

8. * For simplicial complexes K and L inside $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, show that $|K| \times |L| \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is the polyhedron of a simplicial complex. [Prove it first in the case where both K and L consist of a single simplex (plus all its faces).]

Algebraic Topology, Examples 4

Michaelmas 2015

1. Show that if $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

2. For each of the following exact sequences of abelian groups and homomorphisms say as much as possible about the unknown group G and homomorphism α.

(i) $0 \longrightarrow \mathbb{Z}/2 \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 0$,

(ii) $0 \longrightarrow \mathbb{Z}/2 \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

(iii) $0 \longrightarrow G \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$,

(iv) $0 \longrightarrow \mathbb{Z}/3 \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$.

3. Consider a commutative diagram

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

in which the rows are exact and each square commutes. If h_1, h_2, h_4, and h_5 are isomorphisms, show that h_3 is too.

4. For triangulated surfaces X and Y, let $X\#Y$ be the surface obtained by cutting out a 2-simplex from both X and Y and then gluing together the two copies of $\partial\Delta^2$ formed.

(i) Use the Mayer–Vietoris sequence to compute the homology of $\Sigma_g\#\Sigma_h$, and deduce that it is homeomorphic to Σ_{g+h}.

(ii) Use the Mayer–Vietoris sequence to compute the homology of $\Sigma_g\#S_n$, and hence deduce that it is homeomorphic to S_{n+2g}.

[Recall that Σ_g denotes the orientable surface described in Example 5.4.1 of the printed notes, and that S_n denotes the non-orientable surface described in Example 5.4.3 of the printed notes.]

5. Let $p : \widetilde{X} \to X$ be a finite-sheeted covering space, and $h : |K| \to X$ a triangulation. Show that there is an $r \geq 1$ and triangulation $g : |L| \to \widetilde{X}$ so that the composition $h^{-1} \circ p \circ g : |L| \to |K^{(r)}|$ is a simplicial map. If p has n sheets, show that $\chi(\widetilde{X}) = n \cdot \chi(X)$. Hence show that Σ_g is a covering space of Σ_h if and only if $\frac{1-g}{1-h}$ is an integer.

[Hint: If $g = 1 + k \cdot (h-1)$, show that $\mathbb{Z}/k$ acts freely and properly discontinuously on a particular orientable surface of genus g, and identify the quotient.]

6. Let $p : S^{2k} \to X$ be a covering map, $G = \pi_1(X, [x_0])$, and recall that G then acts freely on S^{2k}. Show that for any $g \in G$ the map $g_* : H_{2k}(S^{2k}) \to H_{2k}(S^{2k})$ is multiplication by -1. Deduce that G is either trivial or $\mathbb{Z}/2$, and that $\mathbb{RP}^{2k}$ is not a proper covering space of any other space.

7. * If $f : K \to K$ is a simplicial isomorphism, let $X \subset |K|$ be the fixed set of $|f|$ i.e. $\{x \in |K| \text{ s.t. } |f|(x) = x\}$. Show that the Lefschetz number $L(f)$ is equal to $\chi(X)$.

[Hint: Barycentrically subdivide K so that X is the polyhedron of a sub simplicial complex.]

8. * Let K be a simplicial complex in $\mathbb{R}^m$, and consider this as lying inside $\mathbb{R}^{m+1}$ as the vectors of the form $(x_1, \ldots, x_n, 0)$. Let $e_+ = (0, \ldots, 0, 1) \in \mathbb{R}^{m+1}$ and $e_- = (0, \ldots, 0, -1) \in \mathbb{R}^{m+1}$. The *suspension* of K is the simplicial complex in $\mathbb{R}^{m+1}$

$$SK := K \cup \{\langle v_0, \ldots, v_n, e_+\rangle, \langle v_0, \ldots, v_n, e_-\rangle \mid \langle v_0, \ldots, v_n\rangle \in K\}.$$

(i) Show that SK is a simplicial complex, and that if $|K| \cong S^n$ then $|SK| \cong S^{n+1}$.

(ii) Using the Mayer–Vietoris sequence, show that if K is connected then $H_0(SK) \cong \mathbb{Z}$, $H_1(SK) = 0$, and $H_i(SK) \cong H_{i-1}(K)$ if $i \geq 2$.

(iii) If $f : K \to K$ is a simplicial map, let $Sf : SK \to SK$ be the unique simplicial map which agrees with f on the subcomplex K and fixes the points e_+ and e_-. Show that under the isomorphism in (ii), the maps f_* and Sf_* agree. [It may help to describe the isomorphism in (ii) at the level of chains.]

(iv) Deduce that for every $n \geq 1$ and $d \in \mathbb{Z}$ there is a map $f : S^n \to S^n$ so that f_* induces multiplication by d on $H_n(S^n) \cong \mathbb{Z}$.