Part II — Algebraic Topology
Definitions

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Part IB Analysis II is essential, and Metric and Topological Spaces is highly desirable

The fundamental group
Homotopy of continuous functions and homotopy equivalence between topological spaces. The fundamental group of a space, homomorphisms induced by maps of spaces, change of base point, invariance under homotopy equivalence. [3]

Covering spaces
Covering spaces and covering maps. Path-lifting and homotopy-lifting properties, and their application to the calculation of fundamental groups. The fundamental group of the circle; topological proof of the fundamental theorem of algebra. *Construction of the universal covering of a path-connected, locally simply connected space*. The correspondence between connected coverings of X and conjugacy classes of subgroups of the fundamental group of X. [5]

The Seifert-Van Kampen theorem
Free groups, generators and relations for groups, free products with amalgamation. Statement *and proof* of the Seifert-Van Kampen theorem. Applications to the calculation of fundamental groups. [4]

Simplicial complexes
Finite simplicial complexes and subdivisions; the simplicial approximation theorem. [3]

Homology
Simplicial homology, the homology groups of a simplex and its boundary. Functorial properties for simplicial maps. *Proof of functoriality for continuous maps, and of homotopy invariance*. [4]

Homology calculations
The homology groups of S^n, applications including Brouwer's fixed-point theorem. The Mayer-Vietoris theorem. *Sketch of the classification of closed combinatorical surfaces*; determination of their homology groups. Rational homology groups; the Euler-Poincaré characteristic and the Lefschetz fixed-point theorem. [5]

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0 Introduction

1 Definitions

1.1 Some recollections and conventions

Definition (Map)**.** In this course, the word *map* will always refer to continuous maps. We are doing topology, and never care about non-continuous functions.

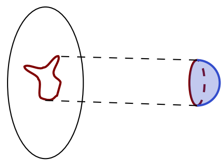
1.2 Cell complexes

Definition (Cell attachment)**.** For a space X, and a map $f : S^{n-1} \to X$, the space obtained by *attaching an n-cell* to X along f is

$$X \cup_f D^n = (X \amalg D^n)/{\sim},$$

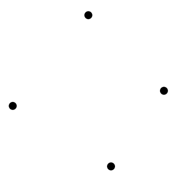
where the equivalence relation $\sim$ is the equivalence relation generated by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$ (and $\amalg$ is the disjoint union).

Intuitively, a map $f : S^{n-1} \to X$ just picks out a subset of X that looks like the sphere. So we are just sticking a disk onto X by attaching the boundary of the disk onto a sphere within X.



Definition (Cell complex)**.** A (finite) *cell complex* is a space X obtained by

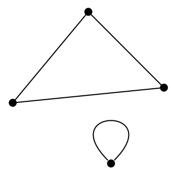
(i) Start with a discrete finite set $X^{(0)}$.



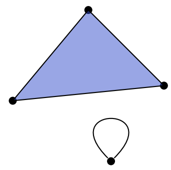
(ii) Given $X^{(n-1)}$, form $X^{(n)}$ by taking a finite set of maps $\{f_\alpha : S^{n-1} \to X^{(n-1)}\}$ and attaching n-cells along the f_α:

$$X^{(n)} = \left(X^{(n-1)} \amalg \coprod_\alpha D_\alpha^n \right) / \{x \sim f_\alpha(x)\}.$$

For example, given the $X^{(0)}$ above, we can attach some loops and lines to obtain the following $X^{(1)}$



We can add surfaces to obtain the following $X^{(2)}$



(iii) Stop at some $X = X^{(k)}$. The minimum such k is the *dimension* of X.

To define non-finite cell complexes, we just have to remove the words "finite" in the definition and remove the final stopping condition.

2 Homotopy and the fundamental group

2.1 Motivation

2.2 Homotopy

Notation.
$$I = [0, 1] \subseteq \mathbb{R}.$$

Definition (Homotopy)**.** Let $f, g : X \to Y$ be maps. A *homotopy* from f to g is a map
$$H : X \times I \to Y$$
such that
$$H(x, 0) = f(x), \quad H(x, 1) = g(x).$$
We think of the interval I as time. For each time t, $H(\,\cdot\,, t)$ defines a map $X \to Y$. So we want to start from f, move with time, and eventually reach g.

If such an H exists, we say f is *homotopic* to g, and write $f \simeq g$. If we want to make it explicit that the homotopy is H, we write $f \simeq_H g$.

Definition (Homotopy rel A)**.** We say f is *homotopic to* g $\operatorname{rel} A$, written $f \simeq g \operatorname{rel} A$, if for all $a \in A \subseteq X$, we have
$$H(a, t) = f(a) = g(a).$$

Definition (Homotopy equivalence)**.** A map $f : X \to Y$ is a *homotopy equivalence* if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We call g a *homotopy inverse* for f.

If a homotopy equivalence $f : X \to Y$ exists, we say that X and Y are homotopy equivalent and write $X \simeq Y$.

Notation. $*$ denotes the one-point space $\{0\}$.

Definition (Contractible space)**.** If $X \simeq *$, then X is *contractible.*

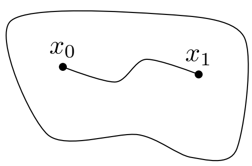
Definition (Retraction)**.** Let $A \subseteq X$ be a subspace. A *retraction* $r : X \to A$ is a map such that $r \circ i = \mathrm{id}_A$, where $i : A \hookrightarrow X$ is the inclusion. If such an r exists, we call A a *retract* of X.

Definition (Deformation retraction)**.** The retraction r is a *deformation retraction* if $i \circ r \simeq \mathrm{id}_X$. A deformation retraction is *strong* if we require this homotopy to be a homotopy rel A.

2.3 Paths

Definition (Path)**.** A *path* in a space X is a map $\gamma : I \to X$. If $\gamma(0) = x_0$ and $\gamma(1) = x_1$, we say γ is a path from x_0 to x_1, and write $\gamma : x_0 \rightsquigarrow x_1$.

If $\gamma(0) = \gamma(1)$, then γ is called a *loop* (based at x_0).



Definition (Concatenation of paths)**.** If we have two paths γ_1 from x_0 to x_1; and γ_2 from x_1 to x_2, we define the *concatenation* to be

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is continuous by the gluing lemma.

Definition (Inverse of path)**.** The *inverse* of a path $\gamma : I \to X$ is defined by

$$\gamma^{-1}(t) = \gamma(1-t).$$

This is exactly the same path but going in the opposite direction.

Definition (Constant path)**.** The *constant path* at a point $x \in X$ is given by $c_x(t) = x$.

Definition (Path components)**.** We can define a relation on X: $x_1 \sim x_2$ if there exists a path from x_1 to x_2. By the concatenation, inverse and constant paths, $\sim$ is an equivalence relation. The equivalence classes $[x]$ are called *path components*. We denote the quotient $X/{\sim}$ by $\pi_0(X)$.

Definition (Homotopy of paths)**.** Paths $\gamma, \gamma' : I \to X$ are *homotopic as paths* if they are homotopic rel $\{0, 1\} \subseteq I$, i.e. the end points are fixed. We write $\gamma \simeq \gamma'$.

2.4 The fundamental group

Definition (Fundamental group)**.** Let X be a space and $x_0 \in X$. The *fundamental group* of X (based at x_0), denoted $\pi_1(X, x_0)$, is the set of homotopy classes of loops in X based at x_0 (i.e. $\gamma(0) = \gamma(1) = x_0$). The group operations are defined as follows:

We define an operation by $[\gamma_0][\gamma_1] = [\gamma_0 \cdot \gamma_1]$; inverses by $[\gamma]^{-1} = [\gamma^{-1}]$; and the identity as the constant path $e = [c_{x_0}]$.

Definition (Based space)**.** A *based space* is a pair (X, x_0) of a space X and a point $x_0 \in X$, the *basepoint*. A *map of based spaces*

$$f : (X, x_0) \to (Y, y_0)$$

is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. A *based homotopy* is a homotopy rel $\{x_0\}$.

Definition (Simply connected space)**.** A space X is *simply connected* if it is path connected *and* $\pi_1(X, x_0) \cong 1$ for some (any) choice of $x_0 \in X$.

3 Covering spaces

3.1 Covering space

Definition (Covering space)**.** A *covering space* of X is a pair $(\tilde{X}, p : \tilde{X} \to X)$, such that each $x \in X$ has a neighbourhood U which is *evenly covered.*

Definition (Evenly covered)**.** $U \subseteq X$ is *evenly covered* by $p : \tilde{X} \to X$ if

$$p^{-1}(U) \cong \coprod_{\alpha \in \Lambda} V_\alpha,$$

where $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism, and each of the $V_\alpha \subseteq \tilde{X}$ is open.

Definition (Lifting)**.** Let $f : Y \to X$ be a map, and $p : \tilde{X} \to X$ a covering space. A *lift* of f is a map $\tilde{f} : Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$, i.e. the following diagram commutes:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Definition (n-sheeted)**.** A covering space $p : \tilde{X} \to X$ of a path-connected space X is *n-sheeted* if $|p^{-1}(x)| = n$ for any (and hence all) $x \in X$.

Definition (Universal cover)**.** A covering map $p : \tilde{X} \to X$ is a *universal cover* if $\tilde{X}$ is simply connected.

3.2 The fundamental group of the circle and its applications

3.3 Universal covers

Definition (Locally simply connected)**.** X is *locally simply connected* if for all $x_0 \in X$, there is some neighbourhood U of x_0 such that U is simply connected.

Definition (Semi-locally simply connected)**.** X is *semi-locally simply connected* if for all $x_0 \in X$, there is some neighbourhood U of x_0 such that any loop γ based at x_0 is homotopic to c_{x_0} as paths *in* X.

Definition (Locally path connected)**.** A space X is *locally path connected* if for any point x and any neighbourhood V of x, there is some open path connected $U \subseteq V$ such that $x \in U$.

3.4 The Galois correspondence

4 Some group theory

4.1 Free groups and presentations

Definition (Alphabet and words)**.** We let $S = \{s_\alpha : \alpha \in \Lambda\}$ be our *alphabet*, and we have an extra set of symbols $S^{-1} = \{s_\alpha^{-1} : \alpha \in \Lambda\}$. We assume that $S \cap S^{-1} = \emptyset$. What do we do with alphabets? We write words with them!

We define S^* to be the set of *words* over $S \cup S^{-1}$, i.e. it contains n-tuples $x_1 \cdots x_n$ for any $0 \leq n < \infty$, where each $x_i \in S \cup S^{-1}$.

Definition (Elementary reduction)**.** An *elementary reduction* takes a word $us_\alpha s_\alpha^{-1}v$ and gives uv, or turns $us_\alpha^{-1}s_\alpha v$ into uv.

Definition (Reduced word)**.** A word is *reduced* if it does not admit an elementary reduction.

Definition (Free group)**.** The *free group* on the set S, written $F(S)$, is the set of reduced words on S^* together with some operations:

(i) Multiplication is given by concatenation followed by elementary reduction to get a reduced word. For example, $(aba^{-1}b^{-1}) \cdot (bab) = aba^{-1}b^{-1}bab = ab^2$

(ii) The identity is the empty word $\emptyset$.

(iii) The inverse of $x_1 \cdots x_n$ is $x_n^{-1} \cdots x_1^{-1}$, where, of course, $(s_\alpha^{-1})^{-1} = s_\alpha$.

The elements of S are called the *generators* of $F(S)$.

Definition (Presentation of a group)**.** Let S be a set, and let $R \subseteq F(S)$ be any sub*set*. We denote by $\langle\langle R\rangle\rangle$ the *normal closure* of R, i.e. the smallest normal subgroup of $F(S)$ containing R. This can be given explicitly by

$$\langle\langle R\rangle\rangle = \left\{\prod_{i=1}^{n} g_i r_i g_i^{-1} : n \in \mathbb{N}, r_i \in R, g_i \in F(S)\right\}.$$

Then we write

$$\langle S \mid R\rangle = F(S)/\langle\langle R\rangle\rangle.$$

4.2 Another view of free groups

4.3 Free products with amalgamation

Definition (Free product)**.** Suppose we have groups $G_1 = \langle S_1 \mid R_1\rangle, G_2 = \langle S_2 \mid R_2\rangle$, where we assume $S_1 \cap S_2 = \emptyset$. The *free product* $G_1 * G_2$ is defined to be

$$G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2\rangle.$$

Definition (Free product with amalgamation)**.** Suppose we have groups G_1, G_2 and H, with the following homomorphisms:

$$\begin{tikzcd} H \arrow[r, "i_2"] \arrow[d, "i_1"] & G_2 \\ G_1 \end{tikzcd}$$

The *free product with amalgamation* is defined to be

$$G_1 \underset{H}{*} G_2 = G_1 * G_2 / \langle\langle \{(j_2 \circ i_2(h))^{-1}(j_1 \circ i_1)(h) : h \in H\} \rangle\rangle.$$

5 Seifert-van Kampen theorem

5.1 Seifert-van Kampen theorem

5.2 The effect on π_1 of attaching cells

5.3 A refinement of the Seifert-van Kampen theorem

Definition (Neighbourhood deformation retract)**.** A subset $A \subseteq X$ is a *neighbourhood deformation retract* if there is an open set $A \subseteq U \subseteq X$ such that A is a strong deformation retract of U, i.e. there exists a retraction $r : U \to A$ and $r \simeq \mathrm{id}_U \operatorname{rel} A$.

5.4 The fundamental group of all surfaces

Definition (Surface)**.** A *surface* is a Hausdorff topological space such that every point has a neighbourhood U that is homeomorphic to $\mathbb{R}^2$.

6 Simplicial complexes

6.1 Simplicial complexes

Definition (Affine independence)**.** A finite set of points $\{a_1, \cdots, a_n\} \subseteq \mathbb{R}^m$ is *affinely independent* iff

$$\sum_{i=1}^n t_i a_i = 0 \text{ with } \sum_{i=1}^n t_i = 0 \Leftrightarrow t_i = 0 \text{ for all } i.$$

Definition (n-simplex)**.** An *n-simplex* is the convex hull of $(n+1)$ affinely independent points $a_0, \cdots, a_n \in \mathbb{R}^m$, i.e. the set

$$\sigma = \langle a_0, \cdots, a_n \rangle = \left\{ \sum_{i=0}^n t_i a_i : \sum_{i=0}^n t_i = 1, t_i \geq 0 \right\}.$$

The points $a_0, \cdots, a_n$ are the *vertices*, and are said to *span* σ. The $(n+1)$-tuple $(t_0, \cdots, t_n)$ is called the *barycentric coordinates* for the point $\sum t_i a_i$.

Definition (Face, boundary and interior)**.** A *face* of a simplex is a subset (or subsimplex) spanned by a subset of the vertices. The *boundary* is the union of the proper faces, and the *interior* is the complement of the boundary.

The boundary of σ is usually denoted by $\partial\sigma$, while the interior is denoted by $\mathring{\sigma}$, and we write $\tau \leq \sigma$ when τ is a face of σ.

Definition. A *(geometric) simplicial complex* is a finite set K of simplices in $\mathbb{R}^n$ such that

(i) If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.

(ii) If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a face of both σ and τ.

Definition (Vertices)**.** The *vertices* of K are the zero simplices of K, denoted V_K.

Definition (Polyhedron)**.** The *polyhedron* defined by K is the union of the simplices in K, and denoted by $|K|$.

Definition (Dimension and skeleton)**.** The *dimension* of K is the highest dimension of a simplex of K. The *d-skeleton* $K^{(d)}$ of K is the union of the n-simplices in K for $n \leq d$.

Definition (Triangulation)**.** A *triangulation* of a space X is a homeomorphism $h : |K| \to X$, where K is some simplicial complex.

Definition (Simplicial map)**.** A *simplicial map* $f : K \to L$ is a function $f : V_K \to V_L$ such that if $\langle a_0, \cdots, a_n \rangle$ is a simplex in K, then $\{f(a_0), \cdots, f(a_n)\}$ spans a simplex of L.

6.2 Simplicial approximation

Definition (Open star and link)**.** Let $x \in |K|$. The *open star* of x is the union of all the interiors of the simplices that contain x, i.e.

$$\mathrm{St}_K(x) = \bigcup_{x \in \sigma \in K} \mathring{\sigma}.$$

The *link* of x, written $\mathrm{Lk}_K(x)$, is the union of all those simplices that do not contain x, but are faces of a simplex that does contain x.

Definition (Simplicial approximation)**.** Let $f : |K| \to |L|$ be a continuous map between the polyhedra. A function $g : V_K \to V_L$ is a *simplicial approximation* to f if for all $v \in V_K$,

$$f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(g(v)). \tag{$*$}$$

Definition (Barycenter)**.** The *barycenter* of $\sigma = \langle a_0, \cdots, a_n\rangle$ is

$$\hat{\sigma} = \sum_{i=0}^n \frac{1}{n+1} a_i.$$

Definition (Barycentric subdivision)**.** The *(first) barycentric subdivision* K' of K is the simplicial complex:

$$K' = \{\langle \hat{\sigma}_0, \cdots, \hat{\sigma}_n\rangle : \sigma_i \in K \text{ and } \sigma_0 < \sigma_1 < \cdots < \sigma_n\}.$$

If you stare at this long enough, you will realize this is exactly what we have drawn above.

The rth barycentric subdivision $K^{(r)}$ is defined inductively as the barycentric subdivision of the $r - 1$th barycentric subdivision, i.e.

$$K^{(r)} = (K^{(r-1)})'.$$

Definition (Mesh)**.** Let K be a simplicial complex. The *mesh* of K is

$$\mu(K) = \max\{\|v_0 - v_1\| : \langle v_0, v_1\rangle \in K\}.$$

7 Simplicial homology

7.1 Simplicial homology

Definition (Oriented n-simplex)**.** An *oriented n-simplex* in a simplicial complex K is an $(n+1)$-tuple $(a_0, \cdots, a_n)$ of vertices $a_i \in V_k$ such that $\langle a_0, \cdots, a_n\rangle \in K$, where we think of two $(n+1)$-tuples $(a_0, \cdots, a_n)$ and $(a_{\pi(0)}, \cdots, a_{\pi(n)})$ as the same *oriented* simplex if $\pi \in S_n$ is an *even* permutation.

We often denote an oriented simplex as σ, and then $\bar{\sigma}$ denotes the same simplex with the opposite orientation.

Definition (Chain group $C_n(K)$)**.** Let K be a simplicial complex. For each $n \geq 0$, we define $C_n(K)$ as follows:

Let $\{\sigma_1, \cdots, \sigma_\ell\}$ be the set of n-simplices of K. For each i, choose an orientation on σ_i. That is, choose an order for the vertices (up to an even permutation). This choice is not important, but we need to make it. Now when we say σ_i, we mean the oriented simplex with this particular orientation.

Now let $C_n(K)$ be the free abelian group with basis $\{\sigma_1, \cdots, \sigma_\ell\}$, i.e. $C_n(K) \cong \mathbb{Z}^\ell$. So an element in $C_n(K)$ might look like

$$\sigma_3 - 7\sigma_1 + 52\sigma_{64} - 28\sigma_{1000000}.$$

In other words, an element of $C_n(K)$ is just a formal sum of n-simplices.

For convenience, we define $C_\ell(K) = 0$ for $\ell < 0$. This will save us from making exceptions for $n = 0$ cases later.

Definition (Boundary homomorphisms)**.** We define *boundary homomorphisms*

$$d_n : C_n(K) \to C_{n-1}(K)$$

by

$$(a_0, \cdots, a_n) \mapsto \sum_{i=0}^{n} (-1)^i (a_0, \cdots, \hat{a}_i, \cdots, a_n),$$

where $(a_0, \cdots, \hat{a}_i, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ is the simplex with a_i removed.

Definition (Simplicial homology group $H_n(K)$)**.** The *nth simplicial homology group* $H_n(K)$ is defined as

$$H_n(K) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

Definition (Chains, cycles and boundaries)**.** The elements of $C_k(K)$ are called *k-chains* of K, those of $\ker d_k$ are called *k-cycles* of K, and those of $\operatorname{im} d_{k+1}$ are called *k-boundaries* of K.

7.2 Some homological algebra

Definition (Chain complex and differentials)**.** A *chain complex* $C_\bullet$ is a sequence of abelian groups $C_0, C_1, C_2, \cdots$ equipped with maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$ for all n. We call these maps the *differentials* of $C_\bullet$.

Definition (Cycles and boundaries)**.** The space of n*-cycles* is

$$Z_n(C) = \ker d_n.$$

The space of n*-boundaries* is

$$B_n(C) = \operatorname{im} d_{n+1}.$$

Definition (Homology group)**.** The n*-th homology group* of $C_\bullet$ is defined to be

$$H_n(C) = \frac{\ker d_n}{\operatorname{im} d_{n+1}} = \frac{Z_n(C)}{B_n(C)}.$$

Definition (Chain map)**.** A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that

$$f_{n-1} \circ d_n = d_n \circ f_n$$

for all n. In other words, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[d, equal] & C_0 \arrow[l, "d_0"'] \arrow[d, "f_0"] & C_1 \arrow[l, "d_1"'] \arrow[d, "f_1"] & C_2 \arrow[l, "d_2"'] \arrow[d, "f_2"] & \cdots \arrow[l, "d_3"'] \\
0 & D_0 \arrow[l, "d_0"] & D_1 \arrow[l, "d_1"] & D_2 \arrow[l, "d_2"] & \cdots \arrow[l, "d_3"]
\end{tikzcd}$$

Definition (Chain homotopy)**.** A *chain homotopy* between chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n.$$

We write $f_\bullet \simeq g_\bullet$ if there is a chain homotopy between $f_\bullet$ and $g_\bullet$.

Definition (Chain homotopy equivalence)**.** Chain complexes $C_\bullet$ and $D_\bullet$ are *chain homotopy equivalent* if there exist $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ such that

$$f_\bullet \circ g_\bullet \simeq \mathrm{id}_{D_\bullet}, \quad g_\bullet \circ f_\bullet \simeq \mathrm{id}_{C_\bullet}.$$

7.3 Homology calculations

Definition (Cone)**.** A simplicial complex is a *cone* if, for some $v_0 \in V_k$,

$$|K| = \mathrm{St}_K(v_0) \cup \mathrm{Lk}_K(v_0).$$

7.4 Mayer-Vietoris sequence

Definition (Exact sequence)**.** A pair of homomorphisms of abelian groups

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact (at B)* if

$$\operatorname{im} f = \ker g.$$

A collection of homomorphisms

$$\cdots \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \xrightarrow{f_{i+2}} \cdots$$

is *exact at* A_i if

$$\ker f_i = \operatorname{im} f_{i-1}.$$

We say it is *exact* if it is exact at every A_i.

Definition (Short exact sequence)**.** A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Definition (Short exact sequence of chain complexes)**.** A *short exact sequence of chain complexes* is a pair of *chain* maps $i_\bullet$ and $j_\bullet$

$$0 \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0$$

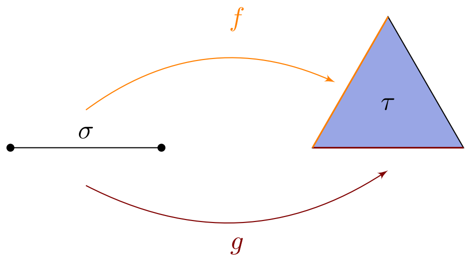
such that for each k,

$$0 \longrightarrow A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \longrightarrow 0$$

is exact.

7.5 Continuous maps and homotopy invariance

Definition (Contiguous maps)**.** Simplicial maps $f, g : K \to L$ are *contiguous* if for each $\sigma \in K$, the simplices $f(\sigma)$ and $g(\sigma)$ (i.e. the simplices spanned by the image of the vertices of σ) are faces of some some simplex $\tau \in L$.



Definition (h-triangulation and homology groups)**.** An *h-triangulation* of a space X is a simplicial complex K and a homotopy equivalence $h : |K| \to X$. We define $H_n(X) = H_n(K)$ for all n.

7.6 Homology of spheres and applications

7.7 Homology of surfaces

7.8 Rational homology, Euler and Lefschetz numbers

Definition (Rational homology group)**.** For a simplicial complex K, we can define the *rational n-chain group* $C_n(K, \mathbb{Q})$ in the same way as $C_n(K) = C_n(K, \mathbb{Z})$.

That is, $C_n(K, \mathbb{Q})$ is the vector space over $\mathbb{Q}$ with basis the n-simplices of K (with a choice of orientation).

We can define d_n, Z_n, B_n as before, and the *rational nth homology group* is

$$H_n(K; \mathbb{Q}) \cong \frac{Z_n(K; \mathbb{Q})}{B_n(K; \mathbb{Q})}.$$

Definition (Euler characteristic)**.** The *Euler characteristic* of a triangulable space X is

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}).$$

Definition (Lefschetz number)**.** Given any map $f : X \to X$, we define the *Lefschetz number* of f as

$$L(f) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(f_* : H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})).$$