

Statistical Physics: Example Sheet 1

H. Reall, Lent 2017

1. Establish Stirling's formula. Start with

$$N! = \int_0^\infty e^{-x} x^N dx \equiv \int_0^\infty e^{-F(x)} dx.$$

Let the minimum of F be at x_0 . Approximate $F(x)$ by $F(x_0) + F''(x_0)(x - x_0)^2/2$ and, using one further approximation, show that

$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$

We will mostly be interested in $N \sim 10^{23}$. But what is the accuracy of Stirling's formula for the paltry value of $N = 5$?

2i. Show that two coupled systems in the microcanonical ensemble maximize the total entropy at equal temperature if the heat capacity of both systems is positive.

ii. In the canonical ensemble, show that the fluctuations in energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$ are proportional to the heat capacity.

iii. Show that in the canonical ensemble the Gibbs entropy can be written as

$$S = k \frac{\partial}{\partial T} (T \log Z)$$

3. Consider a system consisting of N spin- $\frac{1}{2}$ particles, each of which can be in one of two quantum states, 'up' and 'down'. In a magnetic field B , the energy of a spin in the up/down state is $\pm\mu B/2$ where μ is the magnetic moment. Show that the partition function is

$$Z = 2^N \cosh^N \left(\frac{\beta\mu B}{2} \right)$$

Find the average energy E and entropy S . Check that your results for both quantities make sense in at $T = 0$ and $T \rightarrow \infty$.

Compute the *magnetisation* of the system, defined by $M = N_\uparrow - N_\downarrow$ where $N_{\uparrow/\downarrow}$ are the number of up/down spins. The magnetic susceptibility is defined as $\chi \equiv \partial M / \partial B$. Derive *Curie's Law* which states that at high temperatures $\chi \sim 1/T$.

4. Consider a system of N interacting spins. At low temperatures, the interactions ensure that all spins are either aligned or anti-aligned with the z axis, even in the absence of an external field. At high temperatures, the interactions become less important and spins can point in either $\pm\hat{z}$ direction. If the heat capacity takes the form,

$$C_V = C_{\max} \left(\frac{2T}{T_0} - 1 \right) \text{ for } \frac{T_0}{2} < T < T_0 \text{ and } C_V = 0 \text{ otherwise.}$$

determine C_{\max} .

5. Compute the partition function of a quantum harmonic oscillator with frequency ω and energy levels

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad n \in \mathbf{Z}$$

Find the average energy E and entropy S as a function of temperature T .

Einstein constructed a simple model of a solid as N atoms, each of which vibrates with the same frequency ω . Treating these vibrations as a harmonic oscillator, show that at high temperatures, $kT \gg \hbar\omega$, the Einstein model correctly predicts the *Dulong-Petit* law for the heat capacity of a solid,

$$C_V = 3Nk$$

At low temperatures, the heat capacity of many solids is experimentally observed to tend to zero as $C_V \sim T^3$. Was Einstein right about this?

6i. A quantum violin string can vibrate at frequencies ω , 2ω , 3ω and so on. Each vibration mode can be treated as an independent harmonic oscillator. Ignore the zero point energy, so that the mode with frequency $p\omega$ has energy $E = n\hbar p\omega$, $n \in \mathbf{Z}$. Write an expression for the average energy of the string at temperature T . Show that at large temperatures the free energy is given by,

$$F = -\frac{\pi^2 k^2 T^2}{6 \hbar\omega}$$

(Hint: You may need the value $\zeta(2) = \pi^2/6$)

ii*. Show that the partition function of the quantum violin string can be written as

$$Z = \sum_N p(N) e^{-\beta E_N}$$

where $E_N = N\hbar\omega$ and $p(N)$ counts the number of partitions of N . It can be shown that this formula also applies to a relativistic string if we use $E_N = \sqrt{N}\hbar\omega$. Show that the

relativistic string has a maximum temperature, known as the Hagedorn temperature, $kT_{\max} = \sqrt{6}\hbar\omega/2\pi$.

(Hint: Google the Hardy-Ramanujan formula).

7. The purpose of this question is to provide a universal way of looking at all the ensembles, starting from the Gibbs entropy for a probability distribution $p(n)$,

$$S = -k \sum_n p(n) \log p(n)$$

i. By implementing the constraint $\sum_n p(n) = 1$ through the use of a Lagrange multiplier show that, when restricted to states of fixed energy E , the entropy is maximised by the microcanonical ensemble in which all such states are equally likely. Further show that in this case the Gibbs entropy coincides with the Boltzmann entropy.

ii. Show that at fixed *average* energy $\langle E \rangle = \sum_n p(n)E_n$, the entropy is maximised by the canonical ensemble. Moreover, show that the Lagrange multiplier imposing the constraint is proportional to β , the inverse temperature. Confirm that maximizing the entropy is equivalent to minimizing the free energy.

iii. Show that at fixed average energy $\langle E \rangle$ and average particle number $\langle N \rangle$, the entropy is maximised by the grand canonical ensemble. What is the interpretation of the Lagrange multiplier in this case?

8. Let Z_N be the canonical partition function for N particles. Show that the grand partition function \mathcal{Z} can be written as

$$\mathcal{Z}(\mu, V, T) = \sum_{N=0}^{\infty} \xi^N Z_N(V, T)$$

where $\xi = e^{\mu\beta}$ is called the *fugacity*. (It will be denoted z in the lecture notes but I wanted to save you from having to write three different types of z). Show that

$$\langle N \rangle = \xi \frac{\partial}{\partial \xi} \log \mathcal{Z} \quad , \quad (\Delta N)^2 = \left(\xi \frac{\partial}{\partial \xi} \right)^2 \log \mathcal{Z} \quad .$$

If $Z_N = Z_1^N/N!$ show that $\mathcal{Z}(\xi, V, T) = e^{\xi Z_1(V, T)}$. For this case, show also that

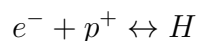
$$\frac{\Delta N}{\langle N \rangle} = \frac{1}{\langle N \rangle^{1/2}} \quad .$$

9. Make use of the fact that the free energy $F(T, V, N)$ of a thermodynamic system must be extensive, to explain why

$$F = V \left. \frac{\partial F}{\partial V} \right|_{T, N} + N \left. \frac{\partial F}{\partial N} \right|_{T, V} \quad .$$

The *Gibbs free energy* is defined as $G = F + pV$. Use the result above for F to show that the Gibbs free energy can be expressed as $G = \mu N$. Explain why this result was to be expected from the scaling behaviour of G .

10. A neutral gas consists of N_e electrons e^- , N_p protons p^+ and N_H Hydrogen atoms H . An electron and proton can combine to form Hydrogen,



At fixed temperature and volume, the free energy of the system is $F(T, V; N_e, N_p, N_H)$. We can define a chemical potential for each of the three species as

$$\mu_i = \frac{\partial F}{\partial N_i}$$

By minimizing the free energy, together with suitable constraints on the particle numbers, show that the condition for equilibrium is

$$\mu_e + \mu_p = \mu_H$$

Such reactions usually take place at constant pressure, rather than constant volume. What quantity should you consider instead of F in this case?

Statistical Physics: Example Sheet 2

H. Reall, Lent 2017

1. A particle moving in one dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^4$$

Show that the heat capacity for a gas of N such particles is $C_V = 3Nk/4$. Explain why the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable.

2. Derive the Sackur-Tetrode formula for the entropy of an ideal monatomic gas with $Z = Z_1^N/N!$. Show that the entropy is not extensive if we fail to include the $N!$ factor.

3. Consider an ultra-relativistic gas of N spinless particles obeying the energy-momentum relation $E = pc$, where c is the speed of light. (Here ultra-relativistic means that $pc \gg mc^2$ where m is the mass of the particle). Show that the canonical partition function is given by

$$Z(V, T) = \frac{1}{N!} \left[\frac{V}{\pi^2} \left(\frac{kT}{\hbar c} \right)^3 \right]^N$$

Hence show that an ultra-relativistic gas also obeys the familiar ideal gas law $pV = NkT$.

4*. Consider a perfect classical gas of diatomic molecules for which each molecule has a magnetic moment m aligned along its axis. Let there be a magnetic field B , so that each molecule has a potential energy $-mB \cos \theta$ (θ being the angle between the axis of the molecule and the magnetic field). Show that the rotational part of the partition function is $Z_{\text{rot}} = (z_{\text{rot}})^N$ where

$$z_{\text{rot}} = \left[\frac{2I}{\hbar^2 m B \beta^2} \right] \sinh(m\beta B) \quad (1)$$

Evaluate the total magnetisation, $M = -\partial F/\partial B$ and sketch its dependence upon $m\beta B$. Show that, for large $m\beta B$, the average value of the potential energy is $NkT - NmB(1 + 2e^{-2m\beta B} + \dots)$.

5. A classical gas in three dimensions is constrained by a wall to move in the $x \geq 0$ region of space. A potential

$$V(x) = \frac{1}{2}\alpha x^2 \quad x \geq 0$$

attracts the atoms to the wall. The atoms are free to move in an area A in the y and z directions. If the gas is at uniform temperature T , show that the number of particles varies as

$$N(x) = 2N \sqrt{\frac{\alpha\beta}{2\pi}} e^{-\alpha\beta x^2/2}$$

By considering a slab of gas between x and $x + \Delta x$, show that locally the gas continues to obey the ideal gas law. Hence determine the pressure that the gas exerts on the wall.

6. Consider the neutral gas of electrons, protons and Hydrogen discussed in Question 10 of Examples Sheet 1. You know from Quantum Mechanics that the Hydrogen atom has binding energy $E = -\Delta$ (where $\Delta = 13.6 \text{ eV}$). Let the number of Hydrogen atoms be $N_H = (1 - x)N$ and the number of electrons and protons be $N_e = N_p = xN$ with $x \in [0, 1]$. By treating the system as three ideal gases in the grand canonical ensemble, use the equilibrium condition $\mu_H = \mu_e + \mu_p$ to show that

$$\frac{x^2}{1 - x} = \frac{V}{N} \left(\frac{m_e m_p}{2\pi\hbar^2 m_H} \right)^{3/2} (kT)^{3/2} e^{-\Delta/kT}$$

7. Compute the equation of state, including the second virial coefficient, for a gas of non-interacting hard discs of radius $r_0/2$ in two dimensions.

8. Determine the density of states for non-relativistic particles in $d = 2$ and $d = 1$ dimensions. (You should find that the density is constant for particles on a plane and decreases with energy for particles on a line)

9. In many experiments, particles are not trapped in a box, but instead in a quadratic potential. In d -dimensions, the potential energy felt by a single particle is

$$V(\vec{x}) = \frac{1}{2} \sum_{i=1}^d \omega_i^2 x_i^2$$

Compute the density of states $g(E)$ in $d = 3$ and $d = 2$ dimensions assuming that E is large enough that the spectrum may be treated as a continuum (i.e. $E \gg \hbar\omega_i$).

(Hint: First determine $G(E)$, the number of states with energy less than E)

10. Consider blackbody radiation at temperature T . Show that the average number of photons grows as T^3 . What is the mean photon energy? What is the most likely energy of a photon?

11. Recall from the lectures that a black body at temperature T absorbs all the radiation that falls on it and emits radiation at the rate $\mathcal{E} = \sigma T^4$ per unit area, where σ is Stefan's constant. A black, perfectly conducting sphere orbits a star of radius $7 \times 10^5 \text{ km}$ at a distance of $1.5 \times 10^8 \text{ km}$. The star radiates like a black body at temperature 6000 K . Can you make a gin and tonic on this sphere?

12. The purpose of this question is to explain why the microwave background radiation still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history.

Consider a region of volume V in the cosmos containing black body radiation of temperature T . Suppose the cosmos expands (slowly) by a scale factor α , so that the wavevector \vec{k} and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/\alpha$. Explain why you should expect the mean number of photons in each mode not to change. Show that the Planck distribution is valid after the expansion provided the temperature is also rescaled by $1/\alpha$.

Verify, from the formula for the entropy of black body radiation, that the entropy in the expanded volume is the same as the original entropy, thus confirming the adiabatic character of the expansion.

13. Suppose that you don't know the value of Boltzmann's constant. What experiments could you do on a box of gas to determine how many atoms it contained?

Statistical Physics: Example Sheet 3

H. Reall, Lent 2017

1. A *Wigner crystal* is a triangular lattice of electrons in a two dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha\sqrt{k}$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

2. Use the fact that the density of states is constant in $d = 2$ dimensions to show that Bose-Einstein condensation does not occur no matter how low the temperature.

3. Consider N non-interacting, non-relativistic bosons, each of mass m , in a cubic box of side L . Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $E_n \propto 1/mL^2$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $\mathcal{O}(N)$.

4. Consider an ideal gas of bosons whose density of states is given by $g(E) = CE^{\alpha-1}$ for some constants C and $\alpha > 1$. Derive an expression for the critical temperature T_c , below which the gas experiences Bose-Einstein condensation.

In BEC experiments, atoms are confined in magnetic traps which can be modelled by a quadratic potential of the type discussed in Question 9 of Example Sheet 2. Determine T_c for bosons in a three dimensional trap. Show that bosons in a two dimensional trap will condense at suitably low temperatures. In each case, calculate the number of particles in the condensate as a function of $T < T_c$,

5. A system has two energy levels with energies 0 and ϵ . These can be occupied by (spinless) fermions from a particle and heat bath with temperature T and chemical potential μ . The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$\mathcal{Z}(\mu, V, T) = 1 + z + ze^{-\beta\epsilon} + z^2e^{-\beta\epsilon}$$

where $z = e^{\beta\mu}$. Evaluate the average occupation number of the state of energy ϵ , and show that this is compatible with the result of the calculation of the average energy of the system using the Fermi-Dirac distribution. How could you take account of fermion interactions?

6. In an ideal Fermi gas the average occupation numbers of the single particle state $|r\rangle$ is n_r . Show that the entropy

$$S = \frac{\partial}{\partial T} (kT \log \mathcal{Z})_{\mu, V}.$$

can be written as

$$S = -k \sum_r [(1 - n_r) \log(1 - n_r) + n_r \log n_r]$$

Find the corresponding expression for an ideal Bose gas.

Show that $(\Delta n_r)^2 = n_r(1 - n_r)$ for the ideal Fermi gas. Comment on this result, especially for very low T . What is the corresponding result for an ideal Bose gas? How does $\Delta n_0/n_0$ behave at low T for the Bose gas?

7. As a simple model of a semiconductor, suppose that there are N bound electron states, each having energy $-\Delta < 0$, which are filled at zero temperature. At non-zero temperature some electrons are excited into the conduction band, which is a continuum of positive energy states. The density of these states is given by $g(E)dE = A\sqrt{E}dE$ where A is a constant. Show that at temperature T the mean number \bar{n}_c of excited electrons is determined by the pair of equations

$$n_c = \frac{N}{e^{(\mu+\Delta)/kT} + 1} = \int_0^\infty \frac{g(E) dE}{e^{(E-\mu)/kT} + 1}.$$

Show also that, if $n_c \ll N$ and $kT \ll \Delta$ and $e^{\mu/kT} \ll 1$, then

$$2\mu \approx -\Delta + kT \log \left[\frac{2N}{A\sqrt{\pi}(kT)^3} \right].$$

8. Consider an almost degenerate Fermi gas of electrons with spin degeneracy $g_s = 2$. At high temperatures, show that the equation of state is given by

$$pV = NkT \left(1 + \frac{\lambda^3 N}{4\sqrt{2}g_s V} + \dots \right)$$

At low temperatures, show that the chemical potential is

$$\mu = E_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{E_F} \right)^2 + \dots \right)$$

and the average energy is

$$E = \frac{3NE_F}{5} \left(1 + \frac{5\pi^2}{12} \left(\frac{kT}{E_F} \right)^2 + \dots \right)$$

9. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential Φ . Show that $3pV = E$. Show that at zero temperature $pV^{4/3} = \text{const}$. Show that at high temperatures $E = 3NkT$ and the equation of state coincides with that of a classical ultra-relativistic gas.

10. A crude non-relativistic model of a white dwarf star consists of a sphere of radius R of free electrons at zero temperature together with a sufficient number of protons to make the star electrically neutral. Determine the energy E_{el} of all the electrons. Assuming the gravitational energy of the star is given by $E_{\text{grav}} = -\gamma M^2/R$, where M is the total mass of the star, show that if the state of equilibrium of the star is given by minimising the total energy ($E_{\text{grav}} + E_{\text{el}}$) then R is proportional to $M^{-1/3}$. What justification can be given for neglecting the proton zero-point energy?

Statistical Physics: Example Sheet 4

H. Reall, Lent 2017

1i. By examining variations in E , F , H and G , derive the four different Maxwell relations for the partial derivatives of S, p, T and V .

ii. Obtain the partial derivative identity

$$\left. \frac{\partial S}{\partial T} \right|_p = \left. \frac{\partial S}{\partial T} \right|_V + \left. \frac{\partial S}{\partial V} \right|_T \left. \frac{\partial V}{\partial T} \right|_p$$

iii. Obtain the partial derivative identity

$$\left. \frac{\partial p}{\partial T} \right|_V \left. \frac{\partial T}{\partial V} \right|_p \left. \frac{\partial V}{\partial p} \right|_T = -1$$

2. Consider a gas with a fixed number of molecules. Two experimentally accessible quantities are C_V , the heat capacity at fixed volume and C_p , the heat capacity at fixed pressure, defined as

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_V, \quad C_p = T \left. \frac{\partial S}{\partial T} \right|_p$$

Using the results of the previous question, show that:

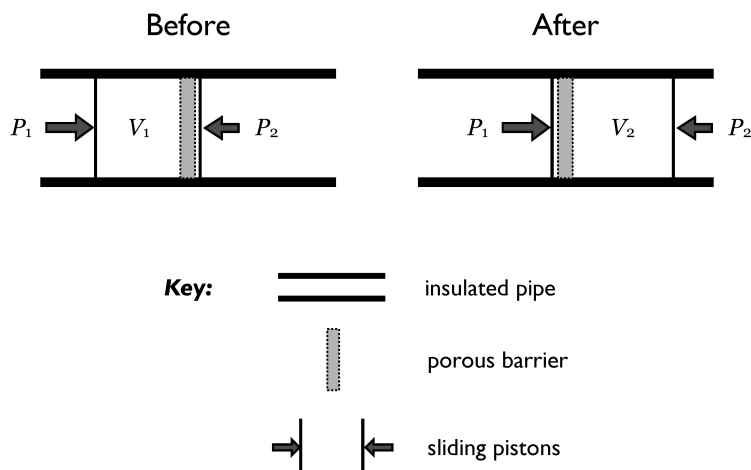
- i.** $C_p - C_V = T \left. \frac{\partial V}{\partial T} \right|_p \left. \frac{\partial p}{\partial T} \right|_V = -T \left. \frac{\partial V}{\partial T} \right|_p^2 \left. \frac{\partial p}{\partial V} \right|_T$
- ii.** $\left. \frac{\partial E}{\partial V} \right|_T = T \left. \frac{\partial p}{\partial T} \right|_V - p$
- iii.** $\left. \frac{\partial E}{\partial p} \right|_T = -T \left. \frac{\partial V}{\partial T} \right|_p - p \left. \frac{\partial V}{\partial p} \right|_T$
- iv.** $\left. \frac{\partial C_V}{\partial V} \right|_T = T \left. \frac{\partial^2 p}{\partial T^2} \right|_V$
- v.** $\left. \frac{\partial C_p}{\partial p} \right|_T = -T \left. \frac{\partial^2 V}{\partial T^2} \right|_p$

3. Consider a classical ideal gas with equation of state $pV = NkT$ and constant heat capacity $C_V = Nk\alpha$ for some α . Use the results above to show that $C_p = Nk(\alpha + 1)$, and that the entropy is

$$S = Nk \log \left(\frac{V}{N} \right) + Nk\alpha \log T + \text{const.}$$

Deduce that, for a reversible adiabatic process (with $dS = 0$), VT^α is constant and, equivalently, pV^γ is constant, where $\gamma = C_p/C_V$.

4. This question describes the Joule-Thomson process (also known as the Joule-Kelvin process). The figure shows a thermally insulated pipe which has a porous barrier separating two halves of the pipe. A gas of volume V_1 , initially on the left-hand side of the pipe, is forced by a piston to go through the porous barrier using a constant pressure p_1 . Assume the process can be treated quasistatically. As a result the gas flows to the right-hand side, resisted by another piston which applies a constant pressure p_2 ($p_2 < p_1$). Eventually all of the gas occupies a volume V_2 on the right-hand side.



- i. Show that enthalpy, $H = E + pV$, is conserved.
- ii. Find the Joule-Thomson coefficient $\mu_{JT} \equiv \left(\frac{\partial T}{\partial p} \right)_H$ in terms of T , V , the heat capacity at constant pressure C_p , and the volume coefficient of expansion $\alpha \equiv \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p$. (Hint: You will need to use a Maxwell relation).
- iii. What is μ_{JT} for an ideal gas?
- iv. If we wish to use the Joule-Thomson process to cool a real (non-ideal) gas, what must the sign of μ_{JT} be?
- v. Derive μ_{JT} for a gas obeying the van der Waals equation of state to leading order in the density N/V . For what values of temperature T can the gas be cooled?

5. A (non-ideal) gas has constant heat capacities C_V and C_p . Using the results of Question 2, show that its equation of state can be written as

$$(C_p - C_V)T = (p + a)(V + b)$$

where a and b are constants. Show also that E is of the form $E = C_V T + f(V)$, find $f(V)$ and calculate the entropy as a function of V and T .

6. In this question we will determine the critical exponent β for the van der Waals equation of state. Starting from the law of corresponding states

$$\bar{p} = \frac{8\bar{T}}{3\bar{v} - 1} - \frac{3}{\bar{v}^2}$$

let $\bar{T} = 1 + t$ and $\bar{v} = 1 + \phi$ where t and ϕ are small. Expand in ϕ to show that

$$\bar{p} = 1 - \frac{3}{2}\phi^3 + \mathcal{O}(\phi^4) + t [4 - 6\phi + \mathcal{O}(\phi^2)]$$

Now, at fixed small t , let $\phi_l(t)$ and $\phi_g(t)$ be the values of ϕ corresponding to the liquid and gas phases respectively. The Maxwell construction gives

$$0 = \int_l^g \bar{v} d\bar{p} = \int_l^g (1 + \phi) d\bar{p} = \int_l^g \phi d\bar{p}$$

where the integral is evaluated along an isotherm and we used $\bar{p}_l(t) = \bar{p}_g(t)$ in the final step. Now change integration variable from \bar{p} to ϕ and evaluate the integral to leading order to show that either $\phi_l(t) = -\phi_g(t)$ or $\phi_l(t)^2 + \phi_g(t)^2 + 8t/3 = 0$. Finally use $\bar{p}_l(t) = \bar{p}_g(t)$ to exclude the latter case, and in the former case to deduce that $\phi_g(t) \propto (-t)^{1/2}$ and hence

$$\bar{v}_g - \bar{v}_l \propto (T_c - T)^{1/2}$$

7. The Dieterici equation of state for a gas is

$$p = \frac{kT}{v - b} \exp\left(-\frac{a}{kTv}\right)$$

where $v = V/N$. Find the critical point and compute the ratio $p_c v_c / kT_c$. Calculate the critical exponents β , δ and γ .

8. The q -state Potts model is a generalisation of the Ising model. At each lattice site lives a variable $\sigma_i \in \{1, 2, \dots, q\}$. The Hamiltonian is given by the sum over nearest neighbours

$$H_{\text{Potts}} = -\frac{3J}{2} \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$$

How many ground states does the system have at $T = 0$?

Show that the 3-state Potts model is equivalent to the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

where \vec{s}_i take values in the set

$$\vec{s}_i \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\}$$

By developing a mean field theory for H determine the self-consistency requirement for the magnetisation $\vec{m} = \langle \vec{s}_i \rangle$. Compute the mean field free energy and show that theory undergoes a first order phase transition even in the absence of an external field.

[Hint: This calculation will be simpler if you argue that you can focus on magnetisation vectors of the form $\vec{m} = (m, 0)$.]

9. Consider the free energy

$$F = a(T)m^2 + b(T)m^4 + c(T)m^6$$

where $b(T) < 0$ and, for stability, $c(T) > 0$ for all T . Sketch the possible behaviours of the free energy as $a(T)$ varies and, in each case, identify the ground state and metastable states. Show that the system undergoes a first order phase transition at some temperature T_c . Determine the value $a(T_c)$ and the discontinuity in m at the transition.