Part II — Representation Theory
Theorems with proof

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Linear Algebra and Groups, Rings and Modules are essential

Representations of finite groups
Representations of groups on vector spaces, matrix representations. Equivalence of representations. Invariant subspaces and submodules. Irreducibility and Schur's Lemma. Complete reducibility for finite groups. Irreducible representations of Abelian groups.

Character theory
Determination of a representation by its character. The group algebra, conjugacy classes, and orthogonality relations. Regular representation. Permutation representations and their characters. Induced representations and the Frobenius reciprocity theorem. Mackey's theorem. Frobenius's Theorem. [12]

Arithmetic properties of characters
Divisibility of the order of the group by the degrees of its irreducible characters. Burnside's $p^a q^b$ theorem. [2]

Tensor products
Tensor products of representations and products of characters. The character ring. Tensor, symmetric and exterior algebras. [3]

Representations of S^1 and SU_2
The groups S^1, SU_2 and $\mathrm{SO}(3)$, their irreducible representations, complete reducibility. The Clebsch-Gordan formula. *Compact groups.* [4]

Further worked examples
The characters of one of $\mathrm{GL}_2(F_q), S_n$ or the Heisenberg group. [3]

Contents

0 Introduction

1 Group actions

Proposition. As groups, $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$, with the isomorphism given by $\theta \mapsto A_\theta$.

Proposition. Matrices A_1, A_2 represent the same element of $\mathrm{GL}(V)$ with respect to different bases if and only if they are *conjugate*, namely there is some $X \in \mathrm{GL}_n(\mathbb{F})$ such that

$$A_2 = XA_1X^{-1}.$$

Proposition.

$$\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A).$$

Proposition. Let $\alpha \in \mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$ and $\alpha^m = \mathrm{id}$ for some positive integer m. Then α is diagonalizable.

Proposition. Let V be a finite-dimensional vector space over $\mathbb{C}$, and $\alpha \in \mathrm{End}(V)$, not necessarily invertible. Then α is diagonalizable if and only if there is a polynomial f with distinct linear factors such that $f(\alpha) = 0$.

Proposition. A finite family of individually diagonalizable endomorphisms of a vector space over $\mathbb{C}$ can be simultaneously diagonalized if and only if they commute.

Lemma. Given an action of G on X, we obtain a homomorphism $\theta : G \to \mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the set of all permutations of X.

Proof. For $g \in G$, define $\theta(g) = \theta_g \in \mathrm{Sym}(X)$ as the function $X \to X$ by $x \mapsto gx$. This is indeed a permutation of X because $\theta_{g^{-1}}$ is an inverse.

Moreover, for any $g_1, g_2 \in G$, we get $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$, since $(g_1 g_2)x = g_1(g_2 x)$. $\square$

2 Basic definitions

Lemma. The relation of "being isomorphic" is an equivalence relation on the set of all linear representations of G over $\mathbb{F}$.

Lemma. If ρ, ρ' are isomorphic representations, then they have the same dimension.

Proof. Trivial since isomorphisms between vector spaces preserve dimension. $\square$

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and W be a G-subspace of V. If $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis containing a basis $\mathcal{B}_1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ of W (with $0 < m < n$), then the matrix of $\rho(g)$ with respect to $\mathcal{B}$ has the block upper triangular form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

for each $g \in G$.

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a decomposable representation with G-invariant decomposition $V = U \oplus W$. Let $\mathcal{B}_1 = \{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \cdots, \mathbf{w}_\ell\}$ be bases for U and W, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ be the corresponding basis for V. Then with respect to $\mathcal{B}$, we have

$$[\rho(g)]_{\mathcal{B}} = \begin{pmatrix} [\rho_u(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_u(g)]_{\mathcal{B}_2} \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Theorem. Every finite-dimensional representation V of a finite group over a field of characteristic 0 is completely reducible, namely, $V \cong V_1 \oplus \cdots \oplus V_r$ is a direct sum of irreducible representations.

Theorem (Maschke's theorem)**.** Let G be a finite group, and $\rho : G \to \mathrm{GL}(V)$ a representation over a finite-dimensional vector space V over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} = 0$. If W is a G-subspace of V, then there exists a G-subspace U of V such that $V = W \oplus U$.

Proof. From linear algebra, we know W has a complementary sub*space*. Let W' be any vector subspace complement of W in V, i.e. $V = W \oplus W'$ *as vector spaces.*

Let $q : V \to W$ be the projection of V onto W along W', i.e. if $\mathbf{v} = \mathbf{w} + \mathbf{w}'$ with $\mathbf{w} \in W, \mathbf{w}' \in W'$, then $q(\mathbf{v}) = \mathbf{w}$.

The clever bit is to take this q and tweak it a little bit. Define

$$\bar{q} : \mathbf{v} \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g) q(\rho(g^{-1})\mathbf{v}).$$

This is in some sense an averaging operator, averaging over what $\rho(g)$ does. Here we need the field to have characteristic zero such that $\frac{1}{|G|}$ is well-defined. In fact, this theorem holds as long as $\operatorname{char} F \nmid |G|$.

For simplicity of expression, we drop the ρ's, and simply write

$$\bar{q} : \mathbf{v} \mapsto \frac{1}{|G|} \sum_{g \in G} g q(g^{-1}\mathbf{v}).$$

We first claim that $\bar{q}$ has image in W. This is true since for $\mathbf{v} \in V$, $q(g^{-1}\mathbf{v}) \in W$, and $gW \leq W$. So this is a little bit like a projection.

Next, we claim that for $\mathbf{w} \in W$, we have $\bar{q}(\mathbf{w}) = \mathbf{w}$. This follows from the fact that q itself fixes W. Since W is G-invariant, we have $g^{-1}\mathbf{w} \in W$ for all $\mathbf{w} \in W$. So we get

$$\bar{q}(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g q(g^{-1}\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g g^{-1}\mathbf{w} = \frac{1}{|G|} \sum_{g \in G} \mathbf{w} = \mathbf{w}.$$

Putting these together, this tells us $\bar{q}$ is a projection onto W.

Finally, we claim that for $h \in G$, we have $h\bar{q}(\mathbf{v}) = \bar{q}(h\mathbf{v})$, i.e. it is invariant under the G-action. This follows easily from definition:

$$\begin{aligned} h\bar{q}(\mathbf{v}) &= h \frac{1}{|G|} \sum_{g \in G} g q(g^{-1}\mathbf{v}) \\ &= \frac{1}{|G|} \sum_{g \in G} h g q(g^{-1}\mathbf{v}) \\ &= \frac{1}{|G|} \sum_{g \in G} (hg) q((hg)^{-1} h\mathbf{v}) \end{aligned}$$

We now put $g' = hg$. Since h is invertible, summing over all g is the same as summing over all g'. So we get

$$\begin{aligned} &= \frac{1}{|G|}\sum_{g' \in G} g' q(g'^{-1}(h\mathbf{v}))\\ &= \bar{q}(h\mathbf{v}). \end{aligned}$$

We are pretty much done. We finally show that $\ker \bar{q}$ is G-invariant. If $\mathbf{v} \in \ker \bar{q}$ and $h \in G$, then $\bar{q}(h\mathbf{v}) = h\bar{q}(\mathbf{v}) = 0$. So $h\mathbf{v} \in \ker \bar{q}$.

Thus

$$V = \operatorname{im} \bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$$

is a G-subspace decomposition. □

Proposition. Let W be G-invariant subspace of V, and V have a G-invariant inner product. Then $W^\perp$ is also G-invariant.

Proof. To prove this, we have to show that for all $\mathbf{v} \in W^\perp$, $g \in G$, we have $g\mathbf{v} \in W^\perp$.

This is not hard. We know $\mathbf{v} \in W^\perp$ if and only if $\langle \mathbf{v}, \mathbf{w}\rangle = 0$ for all $\mathbf{w} \in W$. Thus, using the definition of G-invariance, for $\mathbf{v} \in W^\perp$, we know

$$\langle g\mathbf{v}, g\mathbf{w}\rangle = 0$$

for all $g \in G, \mathbf{w} \in W$.

Thus for all $\mathbf{w}' \in W$, pick $\mathbf{w} = g^{-1}\mathbf{w}' \in W$, and this shows $\langle g\mathbf{v}, \mathbf{w}'\rangle = 0$. Hence $g\mathbf{v} \in W^\perp$. □

Theorem (Weyl's unitary trick)**.** Let ρ be a complex representation of a finite group G on the complex vector space V. Then there is a G-invariant Hermitian inner product on V.

Corollary. Every finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{U}(n)$.

Proof. We start by defining an arbitrary inner product on V: take a basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$. Define $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$, and extend it sesquilinearly. Define a new inner product

$$\langle \mathbf{v}, \mathbf{w}\rangle = \frac{1}{|G|}\sum_{g \in G}(g\mathbf{v}, g\mathbf{w}).$$

We now check this is sesquilinear, positive-definite and G-invariant. Sesquilinearity and positive-definiteness are easy. So we just check G-invariance: we have

$$\begin{aligned} \langle h\mathbf{v}, h\mathbf{w}\rangle &= \frac{1}{|G|}\sum_{g \in G}((gh)\mathbf{v}, (gh)\mathbf{w})\\ &= \frac{1}{|G|}\sum_{g' \in G}(g'\mathbf{v}, g'\mathbf{w})\\ &= \langle \mathbf{v}, \mathbf{w}\rangle. \end{aligned}$$ □

Proposition. Let ρ be an irreducible representation of the finite group G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.

Proof. Take $\rho : G \to \mathrm{GL}(V)$ be irreducible, and pick our favorite $0 \neq \mathbf{v} \in V$. Now define $\theta : \mathbb{F}G \to V$ by

$$\sum_g a_g \mathbf{e}_g \mapsto \sum a_g (g\mathbf{v}).$$

It is not hard to see this is a G-homomorphism. We are now going to exploit the fact that V is irreducible. Thus, since $\operatorname{im}\theta$ is a G-subspace of V and non-zero, we must have $\operatorname{im}\theta = V$. Also, $\ker\theta$ is a G-subspace of $\mathbb{F}G$. Now let W be the G-complement of $\ker\theta$ in $\mathbb{F}G$, which exists by Maschke's theorem. Then $W \leq \mathbb{F}G$ is a G-subspace and

$$\mathbb{F}G = \ker\theta \oplus W.$$

Then the isomorphism theorem gives

$$W \cong \mathbb{F}G/\ker\theta \cong \operatorname{im}\theta = V. \qquad \square$$

4 Schur's lemma

Theorem (Schur's lemma)**.**

(i) Assume V and W are irreducible G-spaces over a field $\mathbb{F}$. Then any G-homomorphism $\theta : V \to W$ is either zero or an isomorphism.

(ii) If $\mathbb{F}$ is algebraically closed, and V is an irreducible G-space, then any G-endomorphism $V \to V$ is a scalar multiple of the identity map ι_V.

Proof.

(i) Let $\theta : V \to W$ be a G-homomorphism between irreducibles. Then $\ker \theta$ is a G-subspace of V, and since V is irreducible, either $\ker \theta = 0$ or $\ker \theta = V$. Similarly, $\operatorname{im} \theta$ is a G-subspace of W, and as W is irreducible, we must have $\operatorname{im} \theta = 0$ or $\operatorname{im} \theta = W$. Hence either $\ker \theta = V$, in which case $\theta = 0$, or $\ker \theta = 0$ and $\operatorname{im} \theta = W$, i.e. θ is a bijection.

(ii) Since $\mathbb{F}$ is algebraically closed, θ has an eigenvalue λ. Then $\theta - \lambda \iota_V$ is a singular G-endomorphism of V. So by (i), it must be the zero map. So $\theta = \lambda \iota_V$. □

Corollary. If V, W are irreducible complex G-spaces, then

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \begin{cases} 1 & V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V and W are not isomorphic, then the only possible map between them is the zero map by Schur's lemma.

Otherwise, suppose $V \cong W$ and let $\theta_1, \theta_2 \in \operatorname{Hom}_G(V, W)$ be both non-zero. By Schur's lemma, they are isomorphisms, and hence invertible. So $\theta_2^{-1}\theta_1 \in \operatorname{End}_G(V)$. Thus $\theta_2^{-1}\theta_1 = \lambda \iota_V$ for some $\lambda \in \mathbb{C}$. Thus $\theta_1 = \lambda \theta_2$. □

Corollary. If G is a finite group and has a faithful complex irreducible representation, then its center $Z(G)$ is cyclic.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful irreducible complex representation. Let $z \in Z(G)$. So $zg = gz$ for all $g \in G$. Hence $\phi_z : \mathbf{v} \mapsto z\mathbf{v}$ is a G-endomorphism on V. Hence by Schur's lemma, it is multiplication by a scalar μ_z, say. Thus $z\mathbf{v} = \mu_z \mathbf{v}$ for all $\mathbf{v} \in V$.

Then the map

$$\begin{aligned} \sigma : Z(G) &\to \mathbb{C}^\times \\ z &\mapsto \mu_g \end{aligned}$$

is a representation of $Z(G)$. Since ρ is faithful, so is σ. So $Z(G) = \{\mu_z : z \in Z(G)\}$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$, hence cyclic. □

Corollary. The irreducible complex representations of a finite abelian group G are all 1-dimensional.

Proof. We can use the fact that commuting diagonalizable matrices are simultaneously diagonalizable. Thus for every irreducible V, we can pick some $\mathbf{v} \in V$ that is an eigenvector for each $g \in G$. Thus $\langle \mathbf{v} \rangle$ is a G-subspace. As V is irreducible, we must have $V = \langle \mathbf{v} \rangle$.

Alternatively, we can prove this in a representation-theoretic way. Let V be an irreducible complex representation. For each $g \in G$, the map

$$\begin{aligned} \theta_g : V &\to V \\ \mathbf{v} &\mapsto g\mathbf{v} \end{aligned}$$

is a G-endomorphism of V, since it commutes with the other group elements. Since V is irreducible, $\theta_g = \lambda_g \iota_V$ for some $\lambda_g \in \mathbb{C}$. Thus

$$g\mathbf{v} = \lambda_g \mathbf{v}$$

for any g. As V is irreducible, we must have $V = \langle \mathbf{v} \rangle$. □

Proposition. The finite abelian group $G = C_{n_1} \times \cdots \times C_{n_r}$ has precisely $|G|$ irreducible representations over $\mathbb{C}$.

Proof. Write

$$G = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle,$$

where $|x_j| = n_j$. Any irreducible representation ρ must be one-dimensional. So we have

$$\rho : G \to \mathbb{C}^\times.$$

Let $\rho(1, \cdots, x_j, \cdots, 1) = \lambda_j$. Then since ρ is a homomorphism, we must have $\lambda_j^{n_j} = 1$. Therefore λ_j is an n_jth root of unity.

Now the values $(\lambda_1, \cdots, \lambda_r)$ determine ρ completely, namely

$$\rho(x_1^{j_1}, \cdots, x_r^{j_r}) = \lambda_1^{j_1} \cdots \lambda_r^{j_r}.$$

Also, whenever λ_i is an n_ith root of unity for each i, then the above formula gives a well-defined representation. So there is a one-to-one correspondence $\rho \leftrightarrow (\lambda_1, \cdots, \lambda_r)$, with $\lambda_j^{n_j} = 1$.

Since for each j, there are n_j many n_jth roots of unity, it follows that there are $|G| = n_1 \cdots n_r$ many choices of the λ_i. Thus the proposition. □

Lemma. Let V, V_1, V_2 be G-vector spaces over $\mathbb{F}$. Then

(i) $\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$

(ii) $\mathrm{Hom}_G(V_1 \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$.

Proof. The proof is to write down the obvious homomorphisms and inverses.

Define the projection map

$$\pi_i : V_1 \oplus V_2 \to V_i,$$

which is the G-linear projection onto V_i.

Then we can define the G-homomorphism

$$\begin{aligned} \mathrm{Hom}_G(V, V_1 \oplus V_2) &\mapsto \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2) \\ \varphi &\mapsto (\pi_1 \varphi, \pi_2 \varphi). \end{aligned}$$

Then the map $(\psi_1, \psi_2) \mapsto \psi_1 + \psi_2$ is an inverse.

For the second part, we have the homomorphism $\varphi \mapsto (\varphi|_{V_1}, \varphi|_{V_2})$ with inverse $(\psi_1, \psi_2) \mapsto \psi_1 \pi_1 + \psi_2 \pi_2$. □

Lemma. Let $\mathbb{F}$ be an algebraically closed field, and V be a representation of G. Suppose $V = \bigoplus_{i=1}^n V_i$ is its decomposition into irreducible components. Then for each irreducible representation S of G,

$$|\{j : V_j \cong S\}| = \dim \operatorname{Hom}_G(S, V).$$

Proof. We induct on n. If $n = 0$, then this is a trivial space. If $n = 1$, then V itself is irreducible, and by Schur's lemma, $\dim \operatorname{Hom}_G(S, V) = 1$ if $V = S$, 0 otherwise. Otherwise, for $n > 1$, we have

$$V = \left(\bigoplus_{i=1}^{n-1} V_i\right) \oplus V_n.$$

By the previous lemma, we know

$$\dim \hom_G\left(S, \left(\bigoplus_{i=1}^{n-1} V_i\right) \oplus V_n\right) = \dim \operatorname{Hom}_G\left(S, \bigoplus_{i=1}^{n-1} V_i\right) + \dim \hom_G(S, V_n).$$

The result then follows by induction. $\square$

5 Character theory

Theorem.

(i) $\chi_V(1) = \dim V$.

(ii) χ_V is a *class function*, namely it is conjugation invariant, i.e.

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

for all $g, h \in G$. Thus χ_V is constant on conjugacy classes.

(iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(iv) For two representations V, W, we have

$$\chi_{V \oplus W} = \chi_V + \chi_W.$$

Proof.

(i) Obvious since $\rho_V(1) = \mathrm{id}_V$.

(ii) Let R_g be the matrix representing g. Then

$$\chi(hgh^{-1}) = \mathrm{tr}(R_h R_g R_h^{-1}) = \mathrm{tr}(R_g) = \chi(g),$$

as we know from linear algebra.

(iii) Since $g \in G$ has finite order, we know $\rho(g)$ is represented by a diagonal matrix

$$R_g = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

and $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by

$$R_{g^{-1}} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix},$$

Noting that each λ_i is an nth root of unity, hence $|\lambda_i| = 1$, we know

$$\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi(g)}.$$

(iv) Suppose $V = V_1 \oplus V_2$, with $\rho : G \to \mathrm{GL}(V)$ splitting into $\rho_i : G \to \mathrm{GL}(V_i)$. Pick a basis $\mathcal{B}_i$ for V_i, and let $\mathcal{B} = B_1 \cup \mathcal{B}_2$. Then with respect to $\mathcal{B}$, we have

$$[\rho(g)]_{\mathcal{B}} = \begin{pmatrix} [\rho_1(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_2(g)]_{\mathcal{B}_2} \end{pmatrix}.$$

So $\chi(g) = \mathrm{tr}(\rho(g)) = \mathrm{tr}(\rho_1(g)) + \mathrm{tr}(\rho_2(g)) = \chi_1(g) + \chi_2(g)$. $\square$

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation affording the character χ. Then

$$|\chi(g)| \leq \chi(1),$$

with equality if and only if $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ if and only if $g \in \ker \rho$.

Proof. Fix g, and pick a basis of eigenvectors of $\rho(g)$. Then the matrix of $\rho(g)$ is diagonal, say

$$\rho(g) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

Hence

$$|\chi(g)| = \left|\sum \lambda_i\right| \leq \sum |\lambda_i| = \sum 1 = \dim V = \chi(1).$$

In the triangle inequality, we have equality if and only if all the λ_i's are equal, to λ, say. So $\rho(g) = \lambda I$. Since all the λ_i's are roots of unity, so is λ.

And, if $\chi(g) = \chi(1)$, then since $\rho(g) = \lambda I$, taking the trace gives $\chi(g) = \lambda\chi(1)$. So $\lambda = 1$, i.e. $\rho(g) = I$. So $g \in \ker \rho$. □

Lemma.

(i) If χ is a complex (irreducible) character of G, then so is $\bar{\chi}$.

(ii) If χ is a complex (irreducible) character of G, then so is $\varepsilon\chi$ for any linear (1-dimensional) character ε.

Proof.

(i) If $R : G \to \mathrm{GL}_n(\mathbb{C})$ is a complex matrix representation, then so is $\bar{R} : G \to \mathrm{GL}_n(\mathbb{C})$, where $g \mapsto \overline{R(g)}$. Then the character of $\bar{R}$ is $\bar{\chi}$

(ii) Similarly, $R' : g \mapsto \varepsilon(g)R(g)$ for $g \in G$ is a representation with character $\varepsilon\chi$.

It is left as an exercise for the reader to check the details. □

Theorem (Completeness of characters)**.** The complex irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$, namely

(i) If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are two complex irreducible representations affording characters χ, χ' respectively, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho \text{ and } \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

This is the (row) orthogonality of characters.

(ii) Each class function of G can be expressed as a linear combination of irreducible characters of G.

Corollary. Complex representations of finite groups are characterised by their characters.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ afford the character χ. We know we can write $\rho = m_1\rho_1 \oplus \cdots \oplus m_k\rho_k$, where $\rho_1, \cdots, \rho_k$ are (distinct) irreducible and $m_j \geq 0$ are the multiplicities. Then we have

$$\chi = m_1\chi_1 + \cdots + m_k\chi_k,$$

where χ_j is afforded by ρ_j. Then by orthogonality, we know

$$m_j = \langle \chi, \chi_j \rangle.$$

So we can obtain the multiplicity of each ρ_j in ρ just by looking at the inner products of the characters. $\square$

Corollary (Irreducibility criterion)**.** If $\rho : G \to \mathrm{GL}(V)$ is a complex representation of G affording the character χ, then ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof. If ρ is irreducible, then orthogonality says $\langle \chi, \chi \rangle = 1$. For the other direction, suppose $\langle \chi, \chi \rangle = 1$. We use complete reducibility to get

$$\chi = \sum m_j \chi_j,$$

with χ_j irreducible, and $m_j \geq 0$ the multiplicities. Then by orthogonality, we get

$$\langle \chi, \chi \rangle = \sum m_j^2.$$

But $\langle \chi, \chi \rangle = 1$. So exactly one of m_j is 1, while the others are all zero, and $\chi = \chi_j$. So χ is irreducible. $\square$

Theorem. Let $\rho_1, \cdots, \rho_k$ be the irreducible complex representations of G, and let their dimensions be $n_1, \cdots, n_k$. Then

$$|G| = \sum n_i^2.$$

Proof. Recall that $\rho_{\mathrm{reg}} : G \to \mathrm{GL}(\mathbb{C}G)$, given by G acting on itself by multiplication, is the regular representation of G of dimension $|G|$. Let its character be π_{reg}, the *regular character* of G.

First note that we have $\pi_{\mathrm{reg}}(1) = |G|$, and $\pi_{\mathrm{reg}}(h) = 0$ if $h \neq 1$. The first part is obvious, and the second is easy to show, since we have only 0s along the diagonal.

Next, we decompose π_{reg} as

$$\pi_{\mathrm{reg}} = \sum a_j \chi_j,$$

We now want to find a_j. We have

$$a_j = \langle \pi_{\mathrm{reg}}, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\pi_{\mathrm{reg}}(g)} \chi_j(g) = \frac{1}{|G|} \cdot |G| \chi_j(1) = \chi_j(1).$$

Then we get

$$|G| = \pi_{\mathrm{reg}}(1) = \sum a_j \chi_j(1) = \sum \chi_j(1)^2 = \sum n_j^2.$$ $\square$

Corollary. The number of irreducible characters of G (up to equivalence) is k, the number of conjugacy classes.

Proof. The irreducible characters and the characteristic functions of the conjugacy classes are both bases of $\mathcal{C}(G)$. □

Corollary. Two elements g_1, g_2 are conjugate if and only if $\chi(g_1) = \chi(g_2)$ for all irreducible characters χ of G.

Proof. If g_1, g_2 are conjugate, since characters are class functions, we must have $\chi(g_1) = \chi(g_2)$.

For the other direction, let δ be the characteristic function of the class of g_1. Then since δ is a class function, we can write

$$\delta = \sum m_j \chi_j,$$

where χ_j are the irreducible characters of G. Then

$$\delta(g_2) = \sum m_j \chi_j(g_2) = \sum m_j \chi_j(g_1) = \delta(g_1) = 1.$$

So g_2 is in the same conjugacy class as g_1. □

6 Proof of orthogonality

Theorem (Row orthogonality relations)**.** If $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are two complex irreducible representations affording characters χ, χ' respectively, then

$$\langle \chi, \chi'\rangle = \begin{cases} 1 & \text{if } \rho \text{ and } \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise}\end{cases}.$$

Proof. We fix a basis of V and of V'. Write $R(g), R'(g)$ for the matrices of $\rho(g)$ and $\rho'(g)$ with respect to these bases respectively. Then by definition, we have

$$\begin{aligned}\langle \chi', \chi\rangle &= \frac{1}{|G|}\sum_{g \in G} \chi'(g^{-1})\chi(g)\\ &= \frac{1}{|G|}\sum_{g \in G}\sum_{\substack{1 \leq i \leq n'\\ 1 \leq j \leq n}} R'(g^{-1})_{ii} R(g)_{jj}.\end{aligned}$$

For any linear map $\varphi : V \to V'$, we define a new map by averaging by ρ' and ρ.

$$\begin{aligned}\tilde{\varphi} : V &\to V'\\ \mathbf{v} &\mapsto \frac{1}{|G|}\sum \rho'(g^{-1})\varphi\rho(g)\mathbf{v}\end{aligned}$$

We first check $\tilde{\varphi}$ is a G-homomorphism — if $h \in G$, we need to show

$$\rho'(h^{-1})\tilde{\varphi}\rho(h)(\mathbf{v}) = \tilde{\varphi}(\mathbf{v}).$$

We have

$$\begin{aligned}\rho'(h^{-1})\tilde{\varphi}\rho(h)(\mathbf{v}) &= \frac{1}{|G|}\sum_{g \in G}\rho'((gh)^{-1})\varphi\rho(gh)\mathbf{v}\\ &= \frac{1}{|G|}\sum_{g' \in G}\rho'(g'^{-1})\varphi\rho(g')\mathbf{v}\\ &= \tilde{\varphi}(\mathbf{v}).\end{aligned}$$

(i) Now we first consider the case where ρ, ρ' is *not* isomorphic. Then by Schur's lemma, we must have $\tilde{\varphi} = 0$ for any linear $\varphi : V \to V'$.

We now pick a very nice φ, where everything disappears. We let $\varphi = \varepsilon_{\alpha\beta}$, the operator having matrix $E_{\alpha\beta}$ with entries 0 everywhere except 1 in the (α, β) position.

Then $\tilde{\varepsilon}_{\alpha\beta} = 0$. So for each i, j, we have

$$\frac{1}{|G|}\sum_{g \in G}(R'(g^{-1})E_{\alpha\beta}R(g))_{ij} = 0.$$

Using our choice of $\varepsilon_{\alpha\beta}$, we get

$$\frac{1}{|G|}\sum_{g \in G} R'(g^{-1})_{i\alpha}R(g)_{\beta j} = 0$$

for all i, j. We now pick $\alpha = i$ and $\beta = j$. Then

$$\frac{1}{|G|}\sum_{g \in G} R'(g^{-1})_{ii} R(g)_{jj} = 0.$$

We can sum this thing over all i and j to get that $\langle \chi', \chi \rangle = 0$.

(ii) Now suppose ρ, ρ' are isomorphic. So we might as well take $\chi = \chi'$, $V = V'$ and $\rho = \rho'$. If $\varphi : V \to V$ is linear, then $\tilde{\varphi} \in \mathrm{End}_G(V)$.

We first claim that $\operatorname{tr} \tilde{\varphi} = \operatorname{tr} \varphi$. To see this, we have

$$\operatorname{tr} \tilde{\varphi} = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr}(\rho(g^{-1})\varphi\rho(g)) = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr} \varphi = \operatorname{tr} \varphi,$$

using the fact that traces don't see conjugacy (and $\rho(g^{-1}) = \rho(g)^{-1}$ since ρ is a group homomorphism).

By Schur's lemma, we know $\tilde{\varphi} = \lambda \iota_v$ for some $\lambda \in \mathbb{C}$ (which depends on φ). Then if $n = \dim V$, then

$$\lambda = \frac{1}{n} \operatorname{tr} \varphi.$$

Let $\varphi = \varepsilon_{\alpha\beta}$. Then $\operatorname{tr} \varphi = \delta_{\alpha\beta}$. Hence

$$\tilde{\varepsilon}_{\alpha\beta} = \frac{1}{|G|}\sum_{g} \rho(g^{-1})\varepsilon_{\alpha\beta}\rho(g) = \frac{1}{n}\delta_{\alpha\beta}\iota.$$

In terms of matrices, we take the (i, j)th entry to get

$$\frac{1}{|G|}\sum R(g^{-1})_{i\alpha} R(g)_{\beta j} = \frac{1}{n}\delta_{\alpha\beta}\delta_{ij}.$$

We now put $\alpha = i$ and $\beta = j$. Then we are left with

$$\frac{1}{|G|}\sum_{g} R(g^{-1})_{ii} R(g)_{jj} = \frac{1}{n}\delta_{ij}.$$

Summing over all i and j, we get $\langle \chi, \chi \rangle = 1$. □

Alternative proof. Consider two representation spaces V and W. Then

$$\langle \chi_W, \chi_V \rangle = \frac{1}{|G|}\sum \overline{\chi_W(g)}\chi_V(g) = \frac{1}{|G|}\sum \chi_{V \otimes W^*}(g).$$

We notice that there is a natural isomorphism $V \otimes W^* \cong \mathrm{Hom}(W, V)$, and the action of g on this space is by conjugation. Thus, a G-invariant element is just a G-homomorphism $W \to V$. Thus, we can decompose $\mathrm{Hom}(V, W) = \mathrm{Hom}_G(V, W) \oplus U$ for some G-space U, and U has no G-invariant element. Hence in the decomposition of $\mathrm{Hom}(V, W)$ into irreducibles, we know there are exactly $\dim \mathrm{Hom}_G(V, W)$ copies of the trivial representation. By Schur's lemma, this number is 1 if $V \cong W$, and 0 if $V \not\cong W$.

So it suffices to show that if χ is a non-trivial irreducible character, then

$$\sum_{g \in G} \chi(g) = 0.$$

But if ρ affords χ, then any element in the image of $\sum_{g \in G} \rho(g)$ is fixed by G. By irreducibility, the image must be trivial. So $\sum_{g \in G} \rho(g) = 0$. □

Theorem (Column orthogonality relations)**.** We have

$$\sum_{i=1}^{k} \overline{\chi_i(g_j)}\chi_i(g_\ell) = \delta_{j\ell}|C_G(g_\ell)|.$$

Corollary.

$$|G| = \sum_{i=1}^{k} \chi_i^2(1).$$

Proof of column orthogonality. Consider the character table $X = (\chi_i(g_j))$. We know

$$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum_\ell \frac{1}{|C_G(g_\ell)|} \overline{\chi_i(g_\ell)}\chi_k(g_\ell).$$

Then

$$\bar{X} D^{-1} X^T = I_{k\times k},$$

where

$$D = \begin{pmatrix} |C_G(g_1)| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |C_G(g_k)| \end{pmatrix}.$$

Since X is square, it follows that $D^{-1}\bar{X}^T$ is the inverse of X. So $\bar{X}^T X = D$, which is exactly the theorem. $\square$

Theorem. Each class function of G can be expressed as a linear combination of irreducible characters of G.

Proof. We list all the irreducible characters $\chi_1, \cdots, \chi_\ell$ of G. Note that we don't know the number of irreducibles is k. This is essentially what we have to prove here. We now claim these generate $\mathcal{C}(G)$, the ring of class functions.

Now recall that $\mathcal{C}(G)$ has an inner product. So it suffices to show that the orthogonal complement to the span $\langle \chi_1, \cdots, \chi_\ell \rangle$ in $\mathcal{C}(G)$ is trivial. To see this, assume $f \in \mathcal{C}(G)$ satisfies

$$\langle f, \chi_j \rangle = 0$$

for all χ_j irreducible. We let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation affording $\chi \in \{\chi_1, \cdots, \chi_\ell\}$. Then $\langle f, \chi \rangle = 0$.

Consider the function

$$\varphi = \frac{1}{|G|}\sum_g \bar{f}(g)\rho(g) : V \to V.$$

For any $h \in G$, we can compute

$$\rho(h)^{-1}\varphi\rho(h) = \frac{1}{|G|}\sum_g \bar{f}(g)\rho(h^{-1}gh) = \frac{1}{|G|}\sum_g \bar{f}(h^{-1}gh)\rho(h^{-1}gh) = \varphi,$$

using the fact that $\bar{f}$ is a class function. So this is a G-homomorphism. So as ρ is irreducible, Schur's lemma says it must be of the form $\lambda\iota_V$ for some $\lambda \in \mathbb{C}$.

Now we take the trace of this thing. So we have

$$n\lambda = \mathrm{tr}\left(\frac{1}{|G|}\sum_g \overline{f(g)}\rho(g)\right) = \frac{1}{|G|}\sum_g \overline{f(g)}\chi(g) = \langle f, \chi \rangle = 0.$$

So $\lambda = 0$, i.e. $\sum_g \overline{f(g)}\rho(g) = 0$, the zero endomorphism on V. This is valid for any irreducible representation, and hence for every representation, by complete reducibility.

In particular, take $\rho = \rho_{\mathrm{reg}}$, where $\rho_{\mathrm{reg}}(g) : \mathbf{e}_1 \mapsto \mathbf{e}_g$ for each $g \in G$. Hence

$$\sum \overline{f(g)}\rho_{\mathrm{reg}}(g) : \mathbf{e}_1 \mapsto \sum_g \overline{f(g)}\mathbf{e}_g.$$

Since this is zero, it follows that we must have $\sum \overline{f(g)}\mathbf{e}_g = 0$. Since the $\mathbf{e}_g$'s are linearly independent, we must have $\overline{f(g)} = 0$ for all $g \in G$, i.e. $f = 0$. $\square$

7 Permutation representations

Lemma. π_X always contains the trivial character 1_G (when decomposed in the basis of irreducible characters). In particular, $\operatorname{span}\{\mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_n}\}$ is a trivial G-subspace of $\mathbb{C}X$, with G-invariant complement $\{\sum_x a_x \mathbf{e}_x : \sum a_x = 0\}$.

Lemma. $\langle \pi_X, 1\rangle$, which is the multiplicity of 1 in π_X, is the number of orbits of G on X.

Proof. We write X as the disjoint union of orbits, $X = X_1 \cup \cdots \cup X_\ell$. Then it is clear that the permutation representation on X is just the sum of the permutation representations on the X_i, i.e.

$$\pi_X = \pi_{X_1} + \cdots + \pi_{x_\ell},$$

where π_{X_j} is the permutation character of G on X_j. So to prove the lemma, it is enough to consider the case where the action is transitive, i.e. there is just one orbit.

So suppose G acts transitively on X. We want to show $\langle \pi_X, 1\rangle = 1$. By definition, we have

$$\begin{aligned}\langle \pi_X, 1\rangle &= \frac{1}{|G|}\sum_g \pi_X(g)\\ &= \frac{1}{|G|}\left|\{(g, x) \in G \times X : gx = x\}\right|\\ &= \frac{1}{|G|}\sum_{x \in X} |G_x|,\end{aligned}$$

where G_x is the stabilizer of x. By the orbit-stabilizer theorem, we have $|G_x||X| = |G|$. So we can write this as

$$\begin{aligned}&= \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|X|}\\ &= \frac{1}{|G|}\cdot |X| \cdot \frac{|G|}{|X|}\\ &= 1.\end{aligned}$$

So done. ∎

Lemma. Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ by

$$g(x_1, x_2) = (gx_1, gx_2).$$

Then the character

$$\pi_{X_1 \times X_2} = \pi_{X_1}\pi_{X_2},$$

and so

$$\langle \pi_{X_1}, \pi_{X_2}\rangle = \text{number of orbits of } G \text{ on } X_1 \times X_2.$$

Proof. We know $\pi_{X_1\times X_2}(g)$ is the number of pairs $(x_1, x_2) \in X_1 \times X_2$ fixed by g. This is exactly the number of things in X_1 fixed by g times the number of things in X_2 fixed by g. So we have

$$\pi_{X_1\times X_2}(g) = \pi_{X_1}(g)\pi_{X_2}(g).$$

Then using the fact that π_1, π_2 are real, we get

$$\begin{aligned}
\langle \pi_{X_1}, \pi_{X_2} \rangle &= \frac{1}{|G|} \sum_g \overline{\pi_{X_1}(g)} \pi_{X_2}(g)\\
&= \frac{1}{|G|} \sum_g \overline{\pi_{X_1}(g) \pi_{X_2}(g)} 1_G(g)\\
&= \langle \pi_{X_1}\pi_{X_2}, 1\rangle\\
&= \langle \pi_{X_1 \times X_2}, 1\rangle.
\end{aligned}$$

So the previous lemma gives the desired result. $\square$

Lemma. Let G act on X, with $|X| > 2$. Then

$$\pi_X = 1_G + \chi,$$

with χ irreducible if and only if G is 2-transitive on X.

Proof. We know

$$\pi_X = m_1 1_G + m_2 \chi_2 + \cdots + m_\ell \chi_\ell,$$

with $1_G, \chi_2, \cdots, \chi_\ell$ distinct irreducible characters and $m_i \in \mathbb{N}$ are non-zero. Then by orthogonality,

$$\langle \pi_X, \pi_X\rangle = \sum_{i=1}^j m_i^2.$$

Since $\langle \pi_X, \pi_X\rangle$ is the number of orbits of $X \times X$, we know G is 2-transitive on X if and only if $\ell = 2$ and $m_1 = m_2 = 1$. $\square$

Lemma. Let $g \in A_n$, $n > 1$. If g commutes with some odd permutation in S_n, then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$. Otherwise, $\mathcal{C}_{S_n}$ splits into two conjugacy classes in A_n of equal size.

8 Normal subgroups and lifting

Lemma. Let $N \lhd G$. Let $\tilde{\rho} : G/N \to \mathrm{GL}(V)$ be a representation of G/N. Then the composition

$$\rho : G \xrightarrow{\text{natural}} G/N \xrightarrow{\tilde{\rho}} \mathrm{GL}(V)$$

is a representation of G, where $\rho(g) = \tilde{\rho}(gN)$. Moreover,

(i) ρ is irreducible if and only if $\tilde{\rho}$ is irreducible.

(ii) The corresponding characters satisfy $\chi(g) = \tilde{\chi}(gN)$.

(iii) $\deg \chi = \deg \tilde{\chi}$.

(iv) The lifting operation $\tilde{\chi} \mapsto \chi$ is a bijection

$$\{\text{irreducibles of } G/N\} \longleftrightarrow \{\text{irreducibles of } G \text{ with } N \text{ in their kernel}\}.$$

We say $\tilde{\chi}$ *lifts to* χ.

Proof. Since a representation of G is just a homomorphism $G \to \mathrm{GL}(V)$, and the composition of homomorphisms is a homomorphisms, it follows immediately that ρ as defined in the lemma is a representation.

(i) We can compute

$$\begin{aligned}
\langle \chi, \chi \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} \sum_{k \in N} \overline{\chi(gk)} \chi(gk) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} \sum_{k \in N} \overline{\tilde{\chi}(gN)} \tilde{\chi}(gN) \\
&= \frac{1}{|G|} \sum_{gN \in G/N} |N| \overline{\tilde{\chi}(gN)} \tilde{\chi}(gN) \\
&= \frac{1}{|G/N|} \sum_{gN \in G/N} \overline{\tilde{\chi}(gN)} \tilde{\chi}(gN) \\
&= \langle \tilde{\chi}, \tilde{\chi} \rangle.
\end{aligned}$$

So $\langle \chi, \chi \rangle = 1$ if and only if $\langle \tilde{\chi}, \tilde{\chi} \rangle = 1$. So ρ is irreducible if and only if $\tilde{\rho}$ is irreducible.

(ii) We can directly compute

$$\chi(g) = \operatorname{tr} \rho(g) = \operatorname{tr}(\tilde{\rho}(gN)) = \tilde{\chi}(gN)$$

for all $g \in G$.

(iii) To see that χ and $\tilde{\chi}$ have the same degree, we just notice that

$$\deg \chi = \chi(1) = \tilde{\chi}(N) = \deg \tilde{\chi}.$$

Alternatively, to show they have the same dimension, just note that ρ and $\tilde{\rho}$ map to the general linear group of the same vector space.

(iv) To show this is a bijection, suppose $\tilde{\chi}$ is a character of G/N and χ is its lift to G. We need to show the kernel contains N. By definition, we know $\tilde{\chi}(N) = \chi(1)$. Also, if $k \in N$, then $\chi(k) = \tilde{\chi}(kN) = \tilde{\chi}(N) = \chi(1)$. So $N \leq \ker \chi$.

Now let χ be a character of G with $N \leq \ker \chi$. Suppose $\rho : G \to \mathrm{GL}(V)$ affords χ. Define

$$\begin{aligned} \tilde{\rho} : G/N &\to \mathrm{GL}(V) \\ gN &\mapsto \rho(g) \end{aligned}$$

Of course, we need to check this is well-defined. If $gN = g'N$, then $g^{-1}g' \in N$. So $\rho(g) = \rho(g')$ since $N \leq \ker \rho$. So this is indeed well-defined. It is also easy to see that $\tilde{\rho}$ is a homomorphism, hence a representation of G/N.

Finally, if $\tilde{\chi}$ is a character of $\tilde{\rho}$, then $\tilde{\chi}(gN) = \chi(g)$ for all $g \in G$ by definition. So $\tilde{\chi}$ lifts to χ. It is clear that these two operations are inverses to each other. □

Lemma. Given a group G, the *derived subgroup* or *commutator subgroup*

$$G' = \langle [a, b] : a, b \in G \rangle,$$

where $[a, b] = aba^{-1}b^{-1}$, is the unique minimal normal subgroup of G such that G/G' is abelian. So if G/N is abelian, then $G' \leq N$.

Moreover, G has precisely $\ell = |G : G'|$ representations of dimension 1, all with kernel containing G', and are obtained by lifting from G/G'.

In particular, by Lagrange's theorem, $\ell \mid G$.

Proof. Consider $[a, b] = aba^{-1}b^{-1} \in G'$. Then for any $h \in G$, we have

$$h(aba^{-1}b^{-1})h^{-1} = \Big((ha)b(ha)^{-1}b^{-1}\Big)\Big(bhb^{-1}h^{-1}\Big) = [ha, b][b, h] \in G'$$

So in general, let $[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] \in G'$. Then

$$h[a_1, b_1][a_2, b_2] \cdots [a_n, b_n]h^{-1} = (h[a_1, b_1]h^{-1})(h[a_2, b_2]h^{-1}) \cdots (h[a_n, b_n]h^{-1}),$$

which is in G'. So G' is a normal subgroup.

Let $N \lhd G$. Let $g, h \in G$. Then $[g, h] \in N$ if and only if $ghg^{-1}h^{-1} \in N$ if and only if $ghN = hgN$, if and only if $(gN)(hN) = (hN)(gN)$ by normality.

Since G' is generated by all $[g, h]$, we know $G' \leq N$ if and only if G/N is abelian.

Since G/G', is abelian, we know it has exactly ℓ irreducible characters, $\tilde{\chi}_1, \cdots, \tilde{\chi}_\ell$, all of degree 1. The lifts of these to G also have degree 1, and by the previous lemma, these are precisely the irreducible characters χ_i of G such that $G' \leq \ker \chi_i$.

But any degree 1 character of G is a homomorphism $\chi : G \to \mathbb{C}^\times$, hence $\chi(ghg^{-1}h^{-1}) = 1$. So for any 1-dimensional character, χ, we must have $G' \leq \ker \chi$. So the lifts $\chi_1, \cdots, \chi_\ell$ are all 1-dimensional characters of G. □

Lemma. G is not simple if and only if $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and some $1 \neq g \in G$. Any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of G, i.e. $N = \bigcap \ker \chi_i$.

Proof. Suppose $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ, and χ is afforded by ρ. Then $g \in \ker \rho$. So if $g \neq 1$, then $1 \neq \ker \rho \lhd G$, and $\ker \rho \neq G$. So G cannot be simple.

If $1 \neq N \lhd G$ is a non-trivial proper subgroup, take an irreducible character $\tilde{\chi}$ of G/N, and suppose $\tilde{\chi} \neq 1_{G/N}$. Lift this to get an irreducible character χ, afforded by the representation ρ of G. Then $N \leq \ker \rho \lhd G$. So $\chi(g) = \chi(1)$ for $g \in N$.

Finally, let $1 \neq N \lhd G$. We claim that N is the intersection of the kernels of the lifts $\chi_1, \cdots, \chi_\ell$ of all the irreducibles of G/N. Clearly, we have $N \leq \bigcap_i \ker \chi_i$. If $g \in G \setminus N$, then $gN \neq N$. So $\tilde{\chi}(gN) \neq \tilde{\chi}(N)$ for some irreducible $\tilde{\chi}$ of G/N. Lifting $\tilde{\chi}$ to χ, we have $\chi(g) \neq \chi(1)$. So g is not in the intersection of the kernels. □

9 Dual spaces and tensor products of representations

9.1 Dual spaces

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation over $\mathbb{F}$, and let $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ be the dual space of V. Then V^* is a G-space under

$$(\rho^*(g)\varphi)(\mathbf{v}) = \varphi(\rho(g^{-1})\mathbf{v}).$$

This is the *dual representation* to ρ. Its character is $\chi(\rho^*)(g) = \chi_\rho(g^{-1})$.

Proof. We have to check ρ^* is a homomorphism. We check

$$\begin{aligned}\rho^*(g_1)(\rho^*(g_2)\varphi)(\mathbf{v}) &= (\rho^*(g_2)\varphi)(\rho(g_1^{-1})(\mathbf{v}))\\ &= \varphi(\rho(g_2^{-1})\rho(g_1^{-2})\mathbf{v})\\ &= \varphi(\rho((g_1g_2)^{-1})(\mathbf{v}))\\ &= (\rho^*(g_1g_2)\varphi)(\mathbf{v}).\end{aligned}$$

To compute the character, fix a $g \in G$, and let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be a basis of eigenvectors of V of $\rho(g)$, say

$$\rho(g)\mathbf{e}_j = \lambda_j \mathbf{e}_j.$$

If we have a dual space of V, then we have a dual basis. We let $\varepsilon_1, \cdots, \varepsilon_n$ be the dual basis. Then

$$(\rho^*(g)\varepsilon_j)(\mathbf{e}_i) = \varepsilon_j(\rho(g^{-1})\mathbf{e}_i) = \varepsilon_j(\lambda_i^{-1}\mathbf{e}_i) = \lambda_i^{-1}\delta_{ij} = \lambda_j^{-1}\delta_{ij} = \lambda_j^{-1}\varepsilon_j(\mathbf{e}_i).$$

Thus we get

$$\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j.$$

So

$$\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_\rho(g^{-1}). \qquad \square$$

9.2 Tensor products

Lemma.

(i) For $\mathbf{v} \in V$, $\mathbf{w} \in W$ and $\lambda \in \mathbb{F}$, we have

$$(\lambda\mathbf{v}) \otimes \mathbf{w} = \lambda(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \otimes (\lambda\mathbf{w}).$$

(ii) If $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in V$ and $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in W$, then

$$\begin{aligned}(\mathbf{x}_1 + \mathbf{x}_2) \otimes \mathbf{y} &= (\mathbf{x}_1 \otimes \mathbf{y}) + (\mathbf{x}_2 \otimes \mathbf{y})\\ \mathbf{x} \otimes (\mathbf{y}_1 + \mathbf{y}_2) &= (\mathbf{x} \otimes \mathbf{y}_1) + (\mathbf{x} \otimes \mathbf{y}_2).\end{aligned}$$

Proof.

(i) Let $\mathbf{v} = \sum \alpha_i \mathbf{v}_i$ and $\mathbf{w} = \sum \beta_j \mathbf{w}_j$. Then

$$\begin{aligned}(\lambda \mathbf{v}) \otimes \mathbf{w} &= \sum_{ij} (\lambda \alpha_i) \beta_j \mathbf{v}_i \otimes \mathbf{w}_j \\ \lambda(\mathbf{v} \otimes \mathbf{w}) &= \lambda \sum_{ij} \alpha_i \beta_j \mathbf{v}_i \otimes \mathbf{w}_j \\ \mathbf{v} \otimes (\lambda \mathbf{w}) &= \sum \alpha_i (\lambda \beta_j) \mathbf{v}_i \otimes \mathbf{w}_j,\end{aligned}$$

and these three things are obviously all equal.

(ii) Similar nonsense. □

Lemma. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be any other basis of V, and $\{\mathbf{f}_1, \cdots, \mathbf{f}_m\}$ be another basis of W. Then

$$\{\mathbf{e}_i \otimes \mathbf{f}_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

is a basis of $V \otimes W$.

Proof. Writing

$$\mathbf{v}_k = \sum \alpha_{ik} \mathbf{e}_i, \quad \mathbf{w}_\ell = \sum \beta_{j\ell} \mathbf{f}_\ell,$$

we have

$$\mathbf{v}_k \otimes \mathbf{w}_\ell = \sum \alpha_{ik} \beta_{jl} \mathbf{e}_i \otimes \mathbf{f}_j.$$

Therefore $\{\mathbf{e}_i \otimes \mathbf{f}_j\}$ spans $V \otimes W$. Moreover, there are nm of these. Therefore they form a basis of $V \otimes W$. □

Proposition. Let $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$. We define

$$\rho \otimes \rho' : G \to \mathrm{GL}(V \otimes V')$$

by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} \mathbf{v}_i \otimes \mathbf{w}_j \mapsto \sum \lambda_{ij} (\rho(g) \mathbf{v}_i) \otimes (\rho'(g) \mathbf{w}_j).$$

Then $\rho \otimes \rho'$ is a representation of g, with character

$$\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g) \chi_{\rho'}(g)$$

for all $g \in G$.

Proof. It is clear that $(\rho \otimes \rho')(g) \in \mathrm{GL}(V \otimes V')$ for all $g \in G$. So $\rho \otimes \rho'$ is a homomorphism $G \to \mathrm{GL}(V \otimes V')$.

To check the character is indeed as stated, let $g \in G$. Let $\mathbf{v}_1, \cdots, \mathbf{v}_m$ be a basis of V of eigenvectors of $\rho(g)$, and let $\mathbf{w}_1, \cdots, \mathbf{w}_n$ be a basis of V' of eigenvectors of $\rho'(g)$, say

$$\rho(g) \mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad \rho'(g) \mathbf{w}_j = \mu_j \mathbf{w}_j.$$

Then

$$\begin{aligned}(\rho \otimes \rho')(g)(\mathbf{v}_i \otimes \mathbf{w}_j) &= \rho(g) \mathbf{v}_i \otimes \rho'(g) \mathbf{w}_j \\ &= \lambda_i \mathbf{v}_i \otimes \mu_j \mathbf{w}_j \\ &= (\lambda_i \mu_j)(\mathbf{v}_i \otimes \mathbf{w}_j).\end{aligned}$$

So

$$\chi_{\rho \otimes \rho'}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum \lambda_i\right) \left(\sum \mu_j\right) = \chi_\rho(g) \chi_{\rho'}(g).$$ □

9.3 Powers of characters

Lemma. For any G-space V, S^2V and Λ^2V are G-subspaces of $V^{\otimes 2}$, and

$$V^{\otimes 2} = S^2V \oplus \Lambda^2 V.$$

The space S^2V has basis

$$\{\mathbf{v}_i\mathbf{v}_j = \mathbf{v}_i \otimes \mathbf{v}_j + \mathbf{v}_j \otimes \mathbf{v}_i : 1 \leq i \leq j \leq n\},$$

while Λ^2V has basis

$$\{\mathbf{v}_i \wedge \mathbf{v}_j = \mathbf{v}_i \otimes \mathbf{v}_j - \mathbf{v}_j \otimes \mathbf{v}_i : 1 \leq i < j \leq n\}.$$

Note that we have a strict inequality for $i < j$, since $\mathbf{v}_i \otimes \mathbf{v}_j - \mathbf{v}_j \otimes \mathbf{v}_i = 0$ if $i = j$. Hence

$$\dim S^2V = \frac{1}{2}n(n+1), \quad \dim \Lambda^2 V = \frac{1}{2}n(n-1).$$

Proof. This is elementary linear algebra. For the decomposition $V^{\otimes 2}$, given $\mathbf{x} \in V^{\otimes 2}$, we can write it as

$$\mathbf{x} = \underbrace{\frac{1}{2}(\mathbf{x} + \tau(\mathbf{x}))}_{\in S^2V} + \underbrace{\frac{1}{2}(\mathbf{x} - \tau(\mathbf{x}))}_{\in \Lambda^2 V}. \qquad \square$$

Lemma. Let $\rho : G \to \mathrm{GL}(V)$ be a representation affording the character χ. Then $\chi^2 = \chi_S + \chi_\Lambda$ where $\chi_S = S^2\chi$ is the character of G in the subrepresentation on S^2V, and $\chi_\Lambda = \Lambda^2\chi$ the character of G in the subrepresentation on Λ^2V. Moreover, for $g \in G$,

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)), \quad \chi_\Lambda(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

Proof. The fact that $\chi^2 = \chi_S + \chi_\Lambda$ is immediate from the decomposition of G-spaces.

We now compute the characters χ_S and χ_Λ. For $g \in G$, we let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis of V of eigenvectors of $\rho(g)$, say

$$\rho(g)\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

We'll be lazy and just write $g\mathbf{v}_i$ instead of $\rho(g)\mathbf{v}_i$. Then, acting on Λ^2V, we get

$$g(\mathbf{v}_i \wedge \mathbf{v}_j) = \lambda_i\lambda_j \mathbf{v}_i \wedge \mathbf{v}_j.$$

Thus

$$\chi_\Lambda(g) = \sum_{1 \leq i < j \leq n} \lambda_i\lambda_j.$$

Since the answer involves the square of the character, let's write that down:

$$\begin{aligned}(\chi(g))^2 &= \left(\sum \lambda_i\right)^2\\ &= \sum \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j\\ &= \chi(g^2) + 2\sum_{i<j}\lambda_i\lambda_j\\ &= \chi(g^2) + 2\chi_\Lambda(g).\end{aligned}$$

Then we can solve to obtain

$$\chi_\Lambda(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

Then we can get

$$\chi_S = \chi^2 - \chi_\Lambda = \frac{1}{2}(\chi^2(g) + \chi(g^2)).$$

□

9.4 Characters of $G \times H$

Proposition. Let G and H be two finite groups with irreducible characters $\chi_1, \cdots, \chi_k$ and $\psi_1, \cdots, \psi_r$ respectively. Then the irreducible characters of the direct product $G \times H$ are precisely

$$\{\chi_i \psi_j : 1 \leq i \leq k, 1 \leq j \leq r\},$$

where

$$(\chi_i \psi_j)(g, h) = \chi_i(g)\psi_j(h).$$

Proof. Take $\rho : G \to \mathrm{GL}(V)$ affording χ, and $\rho' : H \to \mathrm{GL}(W)$ affording ψ. Then define

$$\begin{aligned} \rho \otimes \rho' : G \times H &\to \mathrm{GL}(V \otimes W) \\ (g, h) &\mapsto \rho(g) \otimes \rho'(h), \end{aligned}$$

where

$$(\rho(g) \otimes \rho'(h))(\mathbf{v}_i \otimes \mathbf{w}_j) \mapsto \rho(g)\mathbf{v}_i \otimes \rho'(h)\mathbf{w}_j.$$

This is a representation of $G \times H$ on $V \otimes W$, and $\chi_{\rho \otimes \rho'} = \chi\psi$. The proof is similar to the case where ρ, ρ' are both representations of G, and we will not repeat it here.

Now we need to show $\chi_i \psi_j$ are distinct and irreducible. It suffices to show they are orthonormal. We have

$$\begin{aligned} \langle \chi_i \psi_j, \chi_r \psi_s \rangle_{G \times H} &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \overline{\chi_i \psi_j(g, h)} \chi_r \psi_s(g, h) \\ &= \left(\frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_r(g) \right) \left(\frac{1}{|H|} \sum_{h \in H} \overline{\psi_j(h)} \psi_s(h) \right) \\ &= \delta_{ir} \delta_{js}. \end{aligned}$$

So it follows that $\{\chi_i \psi_j\}$ are distinct and irreducible. We need to show this is complete. We can consider

$$\sum_{i,j} \chi_i \psi_j(1)^2 = \sum \chi_i^2(1) \psi_j^2(1) = \left(\sum \chi_i^2(1) \right) \left(\sum \psi_j^2(1) \right) = |G||H| = |G \times H|.$$

So done. □

9.5 Symmetric and exterior powers

9.6 Tensor algebra

9.7 Character ring

Lemma. Suppose α is a generalized character and $\langle \alpha, \alpha \rangle = 1$ and $\alpha(1) > 0$. Then α is actually a character of an irreducible representation of G.

Proof. We list the irreducible characters as $\chi_1, \cdots, \chi_k$. We then write

$$\alpha = \sum n_i \chi_i.$$

Since the χ_i's are orthonormal, we get

$$\langle \alpha, \alpha \rangle = \sum n_i^2 = 1.$$

So exactly one of n_i is ± 1, while the others are all zero. So $\alpha = \pm \chi_i$ for some i. Finally, since $\alpha(1) > 0$ and also $\chi(1) > 0$, we must have $n_i = +1$. So $\alpha = \chi_i$. $\square$

10 Induction and restriction

Lemma. Let $H \leq G$. If ψ is any non-zero irreducible character of H, then there exists an irreducible character χ of G such that ψ is a constituent of $\operatorname{Res}^G_H \chi$, i.e.

$$\langle \operatorname{Res}^G_H \chi, \psi \rangle \neq 0.$$

Proof. We list the irreducible characters of G as $\chi_1, \cdots, \chi_k$. Recall the regular character π_{reg}. Consider

$$\langle \operatorname{Res}^G_H \pi_{\mathrm{reg}}, \psi \rangle = \frac{|G|}{|H|} \psi(1) \neq 0.$$

On the other hand, we also have

$$\langle \operatorname{Res}^G_H \pi_{\mathrm{reg}}, \psi \rangle_H = \sum_1^k \deg \chi_i \langle \operatorname{Res}^G_H \chi_i, \psi \rangle.$$

If this sum has to be non-zero, then there must be some i such that $\langle \operatorname{Res}^G_H \chi_i, \psi \rangle \neq 0$. □

Lemma. Let χ be an irreducible character of G, and let

$$\operatorname{Res}^G_H \chi = \sum_i c_i \chi_i,$$

with χ_i irreducible characters of H, and c_i non-negative integers. Then

$$\sum c_i^2 \leq |G : H|,$$

with equality iff $\chi(g) = 0$ for all $g \in G \setminus H$.

Proof. We have

$$\langle \operatorname{Res}^G_H \chi, \operatorname{Res}^G_H \chi \rangle_H = \sum c_i^2.$$

However, by definition, we also have

$$\langle \operatorname{Res}^G_H \chi, \operatorname{Res}^G_H \chi \rangle_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2.$$

On the other hand, since χ is irreducible, we have

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle_G \\
&= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\
&= \frac{1}{|G|} \left(\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right) \\
&= \frac{|H|}{|G|} \sum c_i^2 + \frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2 \\
&\geq \frac{|H|}{|G|} \sum c_i^2.
\end{aligned}$$

So the result follows. □

Lemma. Let $\psi \in \mathcal{C}_H$. Then $\operatorname{Ind}_H^G \psi \in \mathcal{C}(G)$, and $\operatorname{Ind}_H^G \psi(1) = |G : H|\psi(1)$.

Proof. The fact that $\operatorname{Ind}_H^G \psi$ is a class function follows from direct inspection of the formula. Then we have

$$\operatorname{Ind}_H^G \psi(1) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(1) = \frac{|G|}{|H|}\psi(1) = |G : H|\psi(1). \qquad \square$$

Lemma. Given a (left) transversal $t_1, \cdots, t_n$ of H, we have

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i).$$

Proof. We can express every $x \in G$ as $x = t_i h$ for some $h \in H$ and i. We then have

$$\mathring{\psi}((t_i h)^{-1} g (t_i h)) = \mathring{\psi}(h^{-1}(t_i^{-1} g t_i) h) = \mathring{\psi}(t_i^{-1} g t_i),$$

since ψ is a class function of H, and $h^{-1}(t_i^{-1} g t_i) h \in H$ if and only if $t_i^{-1} g t_i \in H$, as $h \in H$. So the result follows. $\square$

Theorem (Frobenius reciprocity)**.** Let $\psi \in \mathcal{C}(H)$ and $\varphi \in \mathcal{C}(G)$. Then

$$\langle \operatorname{Res}_H^G \varphi, \psi \rangle_H = \langle \varphi, \operatorname{Ind}_H^G \psi \rangle_G.$$

Proof. We have

$$\begin{aligned}
\langle \varphi, \psi^G \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi^G(g) \\
&= \frac{1}{|G||H|} \sum_{x, g \in G} \overline{\varphi(g)} \mathring{\psi}(x^{-1} g x)
\end{aligned}$$

We now write $y = x^{-1} g x$. Then summing over g is the same as summing over y. Since φ is a G-class function, this becomes

$$= \frac{1}{|G||H|} \sum_{x, y \in G} \overline{\varphi(y)} \mathring{\psi}(y)$$

Now note that the sum is independent of x. So this becomes

$$= \frac{1}{|H|} \sum_{y \in G} \overline{\varphi}(y) \mathring{\psi}(y)$$

Now this only has contributions when $y \in H$, by definition of $\mathring{\psi}$. So

$$\begin{aligned}
&= \frac{1}{|H|} \sum_{y \in H} \overline{\varphi(y)} \psi(y) \\
&= \langle \varphi_H, \psi \rangle_H. \qquad \square
\end{aligned}$$

Corollary. Let ψ be a character of H. Then $\operatorname{Ind}_H^G \psi$ is a character of G.

Proof. Let χ be an irreducible character of G. Then

$$\langle \operatorname{Ind}_H^G \psi, \chi\rangle = \langle \psi, \operatorname{Res}_H^G \chi\rangle.$$

Since ψ and $\operatorname{Res}_H^G \chi$ are characters, the thing on the right is in $\mathbb{Z}_{\geq 0}$. Hence Ind_H^G is a linear combination of irreducible characters with non-negative coefficients, and is hence a character. $\square$

Proposition. Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$\mathcal{C}_G(g) \cap H = \bigcup_{i=1}^m \mathcal{C}_H(x_i),$$

where the x_i are the representatives of the H conjugacy classes of elements of H conjugate to g. If $m = 0$, then $\operatorname{Ind}_H^G \psi(g) = 0$. Otherwise,

$$\operatorname{Ind}_H^G \psi(g) = |C_G(g)| \sum_{i=1}^m \frac{\psi(x_i)}{|C_H(x_i)|}.$$

Proof. If $m = 0$, then $\{x \in G : x^{-1}gx \in H\} = \emptyset$. So $\mathring{\psi}(x^{-1}gx) = 0$ for all x. So $\operatorname{Ind}_H^G \psi(g) = 0$ by definition.

Now assume $m > 0$. We let

$$X_i = \{x \in G : x^{-1}gx \in H \text{ and is conjugate in } H \text{ to } x_i\}.$$

By definition of x_i, we know the X_i's are pairwise disjoint, and their union is $\{x \in G : x^{-1}gx \in H\}$. Hence by definition,

$$\begin{aligned}
\operatorname{Ind}_H^G \psi(g) &= \frac{1}{|H|}\sum_{x \in G} \mathring{\psi}(x^{-1}gx)\\
&= \frac{1}{|H|}\sum_{i=1}^m \sum_{x \in X_i} \psi(x^{-1}gx)\\
&= \frac{1}{|H|}\sum_{i=1}^m \sum_{x \in X_i} \psi(x_i)\\
&= \sum_{i=1}^m \frac{|X_i|}{|H|}\psi(x_i).
\end{aligned}$$

So we now have to show that in fact

$$\frac{|X_i|}{|H|} = \frac{|C_G(g)|}{|C_H(x_i)|}.$$

We fix some $1 \leq i \leq m$. Choose some $g_i \in G$ such that $g_i^{-1}gg_i = x_i$. This exists by definition of x_i. So for every $c \in C_G(g)$ and $h \in H$, we have

$$(cg_ih)^{-1}g(cg_ih) = h^{-1}g_i^{-1}c^{-1}gcg_ih$$

We now use the fact that c commutes with g, since $c \in C_G(g)$, to get

$$\begin{aligned}
&= h^{-1}g_i^{-1}c^{-1}cgg_ih\\
&= h^{-1}g_i^{-1}gg_ih\\
&= h^{-1}x_ih.
\end{aligned}$$

Hence by definition of X_i, we know $cg_ih \in X_i$. Hence

$$C_G(g)g_iH \subseteq X_i.$$

Conversely, if $x \in X_i$, then $x^{-1}gx = h^{-1}x_ih = h^{-1}(g_i^{-1}gg_i)h$ for some h. Thus $xh^{-1}g_i^{-1} \in C_G(g)$, and so $x \in C_G(g)g_ih$. So

$$x \in C_G(g)g_iH.$$

So we conclude

$$X_i = C_G(g)g_iH.$$

Thus, using some group theory magic, which we shall not prove, we get

$$|X_i| = |C_G(g)g_iH| = \frac{|C_G(g)||H|}{|H \cap g_i^{-1}C_G(g)g_i|}$$

Finally, we note

$$g_i^{-1}C_G(g)g_i = C_G(g_i^{-1}gg_i) = C_G(x_i).$$

Thus

$$|X_i| = \frac{|H||C_G(g)|}{|H \cap C_G(x_i)|} = \frac{|H||C_G(g)|}{|C_H(x_i)|}.$$

Dividing, we get

$$\frac{|X_i|}{|H|} = \frac{|C_G(g)|}{|C_H(x_i)|}.$$

So done. □

Lemma. Let $\psi = 1_H$, the trivial character of H. Then $\operatorname{Ind}_H^G 1_H = \pi_X$, the permutation character of G on the set X, where $X = G/H$ is the set of left cosets of H.

Proof. We let $n = |G : H|$, and $t_1, \cdots, t_n$ be representatives of the cosets. By definition, we know

$$\begin{aligned}\operatorname{Ind}_H^G 1_H(g) &= \sum_{i=1}^n \mathring{1}_H(t_i^{-1}gt_i)\\ &= |\{i : t_i^{-1}gt_i \in H\}|\\ &= |\{i : g \in t_iHt_i^{-1}\}|\end{aligned}$$

But $t_iHt_i^{-1}$ is the stabilizer in G of the coset $t_iH \in X$. So this is equal to

$$\begin{aligned}&= |\operatorname{fix}_X(g)|\\ &= \pi_X(g).\end{aligned}$$

□

11 Frobenius groups

Theorem (Frobenius' theorem (1891))**.** Let G be a transitive permutation group on a finite set X, with $|X| = n$. Assume that each non-identity element of G fixes *at most one* element of X. Then the set of fixed point-free elements ("derangements")

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \text{ for all } \alpha \in X\}$$

is a normal subgroup of G with order n.

Proof. The idea of the proof is to construct a character Θ whose kernel is K. First note that by definition of K, we have

$$G = K \cup \bigcup_{\alpha \in X} G_\alpha,$$

where G_α is, as usual, the stabilizer of α. Also, we know that $G_\alpha \cap G_\beta = \{1\}$ if $\alpha \neq \beta$ by assumption, and by definition of K, we have $K \cap G_\alpha = \{1\}$ as well.

Next note that all the G_α are conjugate. Indeed, we know G is transitive, and $gG_\alpha g^{-1} = G_{g\alpha}$. We set $H = G_\alpha$ for some arbitrary choice of α. Then the above tells us that

$$|G| = |K| - |X|(|H| - 1).$$

On the other hand, by the orbit-stabilizer theorem, we know $|G| = |X||H|$. So it follows that we have

$$|K| = |X| = n.$$

We first compute what induced characters look like.

Claim. Let ψ be a character of H. Then

$$\operatorname{Ind}_H^G \psi(g) = \begin{cases} n\psi(1) & g = 1 \\ \psi(g) & g \in H \setminus \{1\} \\ 0 & g \in K \setminus \{1\} \end{cases}.$$

Since every element in G is either in K or conjugate to an element in H, this uniquely specifies what the induced character is.

This is a matter of computation. Since $[G : H] = n$, the case $g = 1$ immediately follows. Using the definition of the induced character, since any non-identity in K is not conjugate to any element in H, we know the induced character vanishes on $K \setminus \{1\}$.

Finally, suppose $g \in H \setminus \{1\}$. Note that if $x \in G$, then $xgx^{-1} \in G_{x\alpha}$. So this lies in H if and only if $x \in H$. So we can write the induced character as

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{g \in G} \mathring{\psi}(xgx^{-1}) = \frac{1}{|H|} \sum_{h \in H} \psi(hgh^{-1}) = \psi(g).$$

Claim. Let ψ be an irreducible character of H, and define

$$\theta = \psi^G - \psi(1)(1_H)^G + \psi(1)1_G.$$

Then θ is a character, and

$$\theta(g) = \begin{cases} \psi(h) & h \in H \\ \psi(1) & k \in K \end{cases}.$$

Note that we chose the coefficients exactly so that the final property of θ holds. This is a matter of computation:

	1	$h \in H \setminus \{1\}$	$K \setminus \{1\}$
ψ^G	$n\psi(1)$	$\psi(h)$	0
$\psi(1)(1_H)^G$	$n\psi(1)$	$\psi(1)$	0
$\psi(1)1_G$	$\psi(1)$	$\psi(1)$	$\psi(1)$
θ_i	$\psi(1)$	$\psi(h)$	$\psi(1)$

The less obvious part is that θ is a character. From the way we wrote it, we already know it is a virtual character. We then compute the inner product

$$\begin{aligned}
\langle \theta, \theta \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\theta(g)|^2 \\
&= \frac{1}{|G|} \left(\sum_{g \in K} |\theta(g)|^2 + \sum_{g \in G \setminus K} |\theta(g)|^2 \right) \\
&= \frac{1}{|G|} \left(n|\psi(1)|^2 + n \sum_{h \neq 1 \in H} |\psi(h)|^2 \right) \\
&= \frac{1}{|G|} \left(n \sum_{h \in H} |\psi(h)|^2 \right) \\
&= \frac{1}{|G|} (n|H| \langle \psi, \psi \rangle_H) \\
&= 1.
\end{aligned}$$

So either θ or $-\theta$ is a character. But $\theta(1) = \psi(1) > 0$. So θ is a character.

Finally, we have

Claim. Let $\psi_1, \cdots, \psi_t$ be the irreducible representations of H, and θ_i be the corresponding representations of G constructed above. Set

$$\Theta = \sum_{i=1}^{t} \theta_i(1) \theta_i.$$

Then we have

$$\theta(g) = \begin{cases} |H| & g \in K \\ 0 & g \notin K \end{cases}.$$

From this, it follows that the kernel of the representation affording θ is K, and in particular K is a normal subgroup of G.

This is again a computation using column orthogonality. For $1 \neq h \in H$, we have

$$\Theta(h) = \sum_{i=1}^{t} \psi_i(1) \psi_i(h) = 0,$$

and for any $y \in K$, we have

$$\Theta(y) = \sum_{i=1}^{t} \psi_i(1)^2 = |H|.$$

□

Proposition. The left action of any finite Frobenius group on the cosets of the Frobenius complement satisfies the hypothesis of Frobenius' theorem.

Proof. Let G be a Frobenius group, having a complement H. Then the action of G on the cosets G/H is transitive. Furthermore, if $1 \neq g \in G$ fixes xH and yH, then we have $g \in xHx^{-1} \cap yHy^{-1}$. This implies $H \cap (y^{-1}x)H(y^{-1}x)^{-1} \neq 1$. Hence $xH = yH$. □

12 Mackey theory

Proposition. Let G be a finite group and $H, K \leq G$. Let $g_1, \cdots, g_k$ be the representatives of the double cosets $K \backslash G / H$. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G 1_H \cong \bigoplus_{i=1}^k \operatorname{Ind}_{g_i H g_i^{-1} \cap K}^K 1.$$

Theorem (Mackey's restriction formula)**.** In general, for $K, H \leq G$, we let $\mathcal{S} = \{1, g_1, \cdots, g_r\}$ be a set of double coset representatives, so that

$$G = \bigcup K g_i H.$$

We write $H_g = gHg^{-1} \cap K \leq G$. We let (ρ, W) be a representation of H. For each $g \in G$, we define (ρ_g, W_g) to be a representation of H_g, with the same underlying vector space W, but now the action of H_g is

$$\rho_g(x) = \rho(g^{-1}xg),$$

where $h = g^{-1}xg \in H$ by construction.

This is clearly well-defined. Since $H_g \leq K$, we obtain an induced representation $\operatorname{Ind}_{H_g}^K W_g$.

Let G be finite, $H, K \leq G$, and W be a H-space. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K W_g.$$

Corollary. Let ψ be a character of a representation of H. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \psi = \sum_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K \psi_g,$$

where ψ_g is the class function (and a character) on H_g given by

$$\psi_g(x) = \psi(g^{-1}xg).$$

Corollary (Mackey's irreducibility criterion)**.** Let $H \leq G$ and W be a H-space. Then $V = \operatorname{Ind}_H^G W$ is irreducible if and only if

(i) W is irreducible; and

(ii) For each $g \in \mathcal{S} \backslash H$, the two H_g spaces W_g and $\operatorname{Res}_{H_g}^H W$ have no irreducible constituents in common, where $H_g = gHg^{-1} \cap H$.

Proof. We use characters, and let W afford the character ψ. We take $K = H$ in Mackey's restriction formula. Then we have $H_g = gHg^{-1} \cap H$.

Using Frobenius reciprocity, we can compute the inner product as

$$\begin{aligned}
\langle \operatorname{Ind}_H^G \psi, \operatorname{Ind}_H^G \psi \rangle_G &= \langle \psi, \operatorname{Res}_H^G \operatorname{Ind}_H^G \psi \rangle_H \\
&= \sum_{g \in \mathcal{S}} \langle \psi, \operatorname{Ind}_{H_g}^H \psi_g \rangle_H \\
&= \sum_{g \in \mathcal{S}} \langle \operatorname{Res}_{H_g}^H \psi, \psi_g \rangle_{H_g} \\
&= \langle \psi, \psi \rangle + \sum_{g \in \mathcal{S} \backslash H} \langle \operatorname{Res}_{H_g}^H \psi, \psi_g \rangle_{H_g}
\end{aligned}$$

We can write this because if $g = 1$, then $H_g = H$, and $\psi_g = \psi$.

This is a sum of non-negative integers, since the inner products of characters always are. So $\operatorname{Ind}_H^G \psi$ is irreducible if and only if $\langle \psi, \psi \rangle = 1$, and all the other terms in the sum are 0. In other words, W is an irreducible representation of H, and for all $g \notin H$, W and W_g are disjoint representations of H_g. $\square$

Corollary. Let $H \lhd G$, and suppose ψ is an irreducible character of H. Then $\operatorname{Ind}_H^G \psi$ is irreducible if and only if ψ is distinct from all its conjugates ψ_g for $g \in G \setminus H$ (where $\psi_g(h) = \psi(g^{-1}hg)$ as before).

Proof. We take $K = H \lhd G$. So the double cosets are just left cosets. Also, $H_g = H$ for all g. Moreover, W_g is irreducible since W is irreducible.

So, by Mackey's irreducible criterion, $\operatorname{Ind}_H^G W$ is irreducible precisely if $W \not\cong W_g$ for all $g \in G \setminus H$. This is equivalent to $\psi \neq \psi_g$. $\square$

Theorem (Mackey's restriction formula)**.** In general, for $K, H \leq G$, we let $\mathcal{S} = \{1, g_1, \cdots, g_r\}$ be a set of double coset representatives, so that

$$G = \bigcup K g_i H.$$

We write $H_g = gHg^{-1} \cap K \leq G$. We let (ρ, W) be a representation of H. For each $g \in G$, we define (ρ_g, W_g) to be a representation of H_g, with the same underlying vector space W, but now the action of H_g is

$$\rho_g(x) = \rho(g^{-1}xg),$$

where $h = g^{-1}xg \in H$ by construction.

This is clearly well-defined. Since $H_g \leq K$, we obtain an induced representation $\operatorname{Ind}_{H_g}^K W_g$.

Let G be finite, $H, K \leq G$, and W be a H-space. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{g \in \mathcal{S}} \operatorname{Ind}_{H_g}^K W_g.$$

Proof. Write $V = \operatorname{Ind}_H^G W$. Pick $g \in G$, so that $KgH \in K \backslash G / H$. Given a left transversal $\mathcal{T}$ of H in G, we can obtain V explicitly as a direct sum

$$V = \bigoplus_{t \in \mathcal{T}} t \otimes W.$$

The idea is to "coarsen" this direct sum decomposition using double coset representatives, by collecting together the $t \otimes W$'s with $t \in KgH$. We define

$$V(g) = \bigoplus_{t \in KgH \cap \mathcal{T}} t \otimes W.$$

Now each $V(g)$ is a K-space — given $k \in K$ and $t \otimes \mathbf{w} \in t \otimes W$, since $t \in KgH$, we have $kt \in KgH$. So there is some $t' \in \mathcal{T}$ such that $ktH = t'H$. Then $t' \in ktH \subseteq KgH$. So we can define

$$k \cdot (t \otimes \mathbf{w}) = t' \otimes (\rho(t'^{-1}kt)\mathbf{w}),$$

where $t'kt \in H$.

Viewing V as a K-space (forgetting its whole G-structure), we have

$$\operatorname{Res}^G_K V = \bigoplus_{g\in\mathcal{S}} V(g).$$

The left hand side is what we want, but the right hand side looks absolutely nothing like $\operatorname{Ind}^K_{H_g} W_g$. So we need to show

$$V(g) = \bigoplus_{t\in KgH\cap\mathcal{T}} t\otimes W \cong \operatorname{Ind}^K_{H_g} W_g,$$

as K representations, for each $g\in\mathcal{S}$.

Now for g fixed, each $t\in KgH$ can be represented by some kgh, and by restricting to elements in the traversal $\mathcal{T}$ of H, we are really summing over cosets kgH. Now cosets kgH are in bijection with cosets $k(gHg^{-1})$ in the obvious way. So we are actually summing over elements in $K/(gHg^{-1}\cap K) = K/H_g$. So we write

$$V(g) = \bigoplus_{k\in K/H_g} (kg)\otimes W.$$

We claim that there is a isomorphism that sends $k\otimes W_g \cong (kg)\otimes W$. We define $k\otimes W_g \to (kg)\otimes W$ by $k\otimes \mathbf{w}\mapsto kg\otimes\mathbf{w}$. This is an isomorphism of vector spaces almost by definition, so we only check that it is compatible with the action. The action of $x\in K$ on the left is given by

$$\rho_g(x)(k\otimes\mathbf{w}) = k'\otimes(\rho_g(k'^{-1}xk)\mathbf{w}) = k'\otimes(\rho(g^{-1}k'^{-1}xkg)\mathbf{w}),$$

where $k'\in K$ is such that $k'^{-1}xk\in H_g$, i.e. $g^{-1}k'^{-1}xkg\in H$. On the other hand,

$$\rho(x)(kg\otimes\mathbf{w}) = k''\otimes(\rho(k''x^{-1}(kg))\mathbf{w}),$$

where $k''\in K$ is such that $k''^{-1}xkg\in H$. Since there is a unique choice of k'' (after picking a particular transversal), and $k'g$ works, we know this is equal to

$$k'g\otimes(\rho(g^{-1}k'^{-1}xkg)\mathbf{w}).$$

So the actions are the same. So we have an isomorphism.

Then

$$V(g) = \bigoplus_{k\in K/H_g} k\otimes W_g = \operatorname{Ind}^K_{H_g} W_g,$$

as required. □

13 Integrality in the group algebra

Proposition.

(i) The algebraic integers form a subring of $\mathbb{C}$.

(ii) If $a \in \mathbb{C}$ is both an algebraic integer and rational, then a is in fact an integer.

(iii) Any subring of $\mathbb{C}$ which is a finitely generated $\mathbb{Z}$-module consists of algebraic integers.

Proposition. If χ is a character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.

Proof. We know $\chi(g)$ is the sum of roots nth roots of unity (where n is the order of g). Each root of unity is an algebraic integer, since it is by definition a root of $x^n - 1$. Since algebraic integers are closed under addition, the result follows. □

Proposition. The class sums $C_1, \cdots, C_k$ form a basis of $Z(\mathbb{C}G)$. There exists non-negative *integers* $a_{ij\ell}$ (with $1 \leq i, j, \ell \leq k$) with

$$C_i C_j = \sum_{\ell=1}^{k} a_{ij\ell} C_\ell.$$

Proof. It is clear from definition that $gC_jg^{-1} = C_j$. So we have $C_j \in Z(\mathbb{C}G)$. Also, since the C_j's are produced from disjoint conjugacy classes, they are linearly independent.

Now suppose $z \in Z(\mathbb{C}G)$. So we can write

$$z = \sum_{g \in G} \alpha_g g.$$

By definition, this commutes with all elements of $\mathbb{C}G$. So for all $h \in G$, we must have

$$\alpha_{h^{-1}gh} = \alpha_g.$$

So the function $g \mapsto \alpha_g$ is constant on conjugacy classes of G. So we can write $\alpha_j = \alpha_g$ for $g \in \mathcal{C}_j$. Then

$$g = \sum_{j=1}^{k} \alpha_j C_j.$$

Finally, the center $Z(\mathbb{C}G)$ is an algebra. So

$$C_i C_j = \sum_{\ell=1}^{k} a_{ij\ell} C_\ell$$

for some complex numbers $a_{ij\ell}$, since the C_j span. The claim is that $a_{ij\ell} \in \mathbb{Z}_{\geq 0}$ for all $i, j\ell$. To see this, we fix $g_\ell \in \mathcal{C}_\ell$. Then by definition of multiplication, we know

$$a_{ij\ell} = |\{(x, y) \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_\ell\}|,$$

which is clearly a non-negative integer. □

Lemma. The values of
$$\omega_\chi(C_i) = \frac{\chi(g)}{\chi(1)}|\mathcal{C}_i|$$
are algebraic integers.

Proof. Using the definition of $a_{ij\ell} \in \mathbb{Z}_{\geq 0}$, and the fact that ω_χ is an algebra homomorphism, we get
$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{\ell=1}^k a_{ij\ell}\omega_\chi(C_\ell).$$
Thus the span of $\{\omega(C_j) : 1 \leq j \leq k\}$ is a subring of $\mathbb{C}$ and is finitely generated as a $\mathbb{Z}$-module (by definition). So we know this consists of algebraic integers. $\square$

Theorem. The degree of any irreducible character of G divides $|G|$, i.e.
$$\chi_j(1) \mid |G|$$
for each irreducible χ_j.

Proof. Let χ be an irreducible character. By orthogonality, we have
$$\begin{aligned}\frac{|G|}{\chi(1)} &= \frac{1}{\chi(1)}\sum_{g\in G}\chi(g)\chi(g^{-1})\\ &= \frac{1}{\chi(1)}\sum_{i=1}^k|\mathcal{C}_i|\chi(g_i)\chi(g_i^{-1})\\ &= \sum_{i=1}^k\frac{|\mathcal{C}_i|\chi(g_i)}{\chi(1)}\chi(g_i)^{-1}.\end{aligned}$$
Now we notice
$$\frac{|\mathcal{C}_i|\chi(g_i)}{\chi(1)}$$
is an algebraic integer, by the previous lemma. Also, $\chi(g_i^{-1})$ is an algebraic integer. So the whole mess is an algebraic integer since algebraic integers are closed under addition and multiplication.

But we also know $\frac{|G|}{\chi(1)}$ is rational. So it must be an integer! $\square$

14 Burnside's theorem

Theorem (Burside's $p^a q^b$ theorem)**.** Let p, q be primes, and let $|G| = p^a q^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then G is not simple.

Lemma. Suppose
$$\alpha = \frac{1}{m} \sum_{j=1}^{m} \lambda_j,$$
is an algebraic integer, where $\lambda_j^n = 1$ for all j and some n. Then either $\alpha = 0$ or $|\alpha| = 1$.

Proof (non-examinable). Observe $\alpha \in \mathbb{F} = \mathbb{Q}(\varepsilon)$, where $\varepsilon = e^{2\pi i/n}$ (since $\lambda_j \in \mathbb{F}$ for all j). We let $\mathcal{G} = \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$. Then
$$\{\beta \in \mathbb{F} : \sigma(\beta) = \beta \text{ for all } \sigma \in \mathcal{G}\} = \mathbb{Q}.$$
We define the "norm"
$$N(\alpha) = \prod_{\sigma \in \mathcal{G}} \sigma(\alpha).$$
Then $N(\alpha)$ is fixed by every element $\sigma \in \mathcal{G}$. So $N(\alpha)$ is rational.

Now $N(\alpha)$ is an algebraic integer, since Galois conjugates $\sigma(\alpha)$ of algebraic integers are algebraic integers. So in fact $N(\alpha)$ is an integer. But for $\alpha \in \mathcal{G}$, we know
$$|\sigma(\alpha)| = \left|\frac{1}{m} \sum \sigma(\lambda_j)\right| \leq 1.$$
So if $\alpha \neq 0$, then $N(\alpha) = \pm 1$. So $|\alpha| = 1$. □

Lemma. Suppose χ is an irreducible character of G, and $\mathcal{C}$ is a conjugacy class in G such that $\chi(1)$ and $|\mathcal{C}|$ are coprime. Then for $g \in \mathcal{C}$, we have
$$|\chi(g)| = \chi(1) \text{ or } 0.$$

Proof. Of course, we want to consider the quantity
$$\alpha = \frac{\chi(g)}{\chi(1)}.$$
Since $\chi(g)$ is the sum of $\deg \chi = \chi(1)$ many roots of unity, it suffices to show that α is an algebraic integer.

By Bézout's theorem, there exists $a, b \in \mathbb{Z}$ such that
$$a\chi(1) + b|\mathcal{C}| = 1.$$
So we can write
$$\alpha = \frac{\chi(g)}{\chi(1)} = a\chi(g) + b\frac{\chi(g)}{\chi(1)}|\mathcal{C}|.$$
Since $\chi(g)$ and $\frac{\chi(g)}{\chi(1)}|\mathcal{C}|$ are both algebraic integers, we know α is. □

Proposition. If in a finite group, the number of elements in a conjugacy class $\mathcal{C}$ is of (non-trivial) prime power order, then G is not non-abelian simple.

Proof. Suppose G is a non-abelian simple group, and let $1 \neq g \in G$ be living in the conjugacy class $\mathcal{C}$ of order p^r. If $\chi \neq 1_G$ is a non-trivial irreducible character of G, then either $\chi(1)$ and $|\mathcal{C}| = p^r$ are not coprime, in which case $p \mid \chi(1)$, or they are coprime, in which case $|\chi(g)| = \chi(1)$ or $\chi(g) = 0$.

However, it cannot be that $|\chi(g)| = \chi(1)$. If so, then we must have $\rho(g) = \lambda I$ for some λ. So it commutes with everything, i.e. for all h, we have

$$\rho(gh) = \rho(g)\rho(h) = \rho(h)\rho(g) = \rho(hg).$$

Moreover, since G is simple, ρ must be faithful. So we must have $gh = hg$ for all h. So $Z(G)$ is non-trivial. This is a contradiction. So either $p \mid \chi(1)$ or $\chi(g) = 0$.

By column orthogonality applied to $\mathcal{C}$ and 1, we get

$$0 = 1 + \sum_{1 \neq \chi \text{ irreducible, } p \mid \chi(1)} \chi(1)\chi(g),$$

where we have deleted the 0 terms. So we get

$$-\frac{1}{p} = \sum_{\chi \neq 1} \frac{\chi(1)}{p}\chi(g).$$

But this is both an algebraic integer and a rational number, but not integer. This is a contradiction. □

Theorem (Burside's $p^a q^b$ theorem)**.** Let p, q be primes, and let $|G| = p^a q^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then G is not simple.

Proof. Let $|G| = p^a q^b$. If $a = 0$ or $b = 0$, then the result is trivial. Suppose $a, b > 0$. We let $Q \in \mathrm{Syl}_q(G)$. Since Q is a p-group, we know $Z(Q)$ is non-trivial. Hence there is some $1 \neq g \in Z(Q)$. By definition of center, we know $Q \leq C_G(g)$. Also, $C_G(g)$ is not the whole of G, since the center of G is trivial. So

$$|\mathcal{C}_G(g)| = |G : C_G(g)| = p^r$$

for some $0 < r \leq a$. So done. □

15 Representations of compact groups

Theorem. Every one-dimensional (continuous) representation S^1 is of the form

$$\rho : z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

Lemma. If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continuous group homomorphism, then there exists a $c \in \mathbb{R}$ such that

$$\psi(x) = cx$$

for all $x \in \mathbb{R}$.

Proof. Given $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ continuous, we let $c = \psi(1)$. We now claim that $\psi(x) = cx$.

Since ψ is a homomorphism, for every $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, we know

$$\psi(nx) = \psi(x + \cdots + x) = \psi(x) + \cdots + \psi(x) = n\psi(x).$$

In particular, when $x = 1$, we know $\psi(n) = cn$. Also, we have

$$\psi(-n) = -\psi(n) = -cn.$$

Thus $\psi(n) = cn$ for all $n \in \mathbb{Z}$.

We now put $x = \frac{m}{n} \in \mathbb{Q}$. Then we have

$$m\psi(x) = \psi(nx) = \psi(m) = cm.$$

So we must have

$$\psi\left(\frac{m}{n}\right) = c\frac{m}{n}.$$

So we get $\psi(q) = cq$ for all $q \in \mathbb{Q}$. But $\mathbb{Q}$ is dense in $\mathbb{R}$, and ψ is continuous. So we must have $\psi(x) = cx$ for all $x \in \mathbb{R}$. $\square$

Lemma. Continuous homomorphisms $\varphi : (\mathbb{R}, +) \to S^1$ are of the form

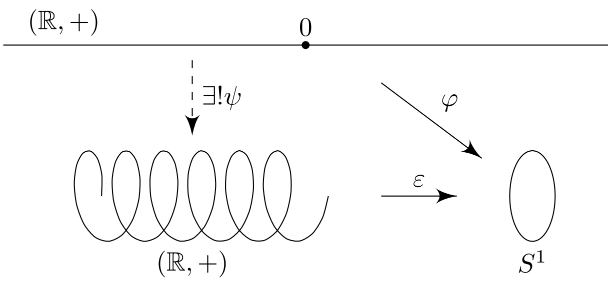
$$\varphi(x) = e^{icx}$$

for some $c \in \mathbb{R}$.

Proof. Let $\varepsilon : (\mathbb{R}, +) \to S^1$ be defined by $x \mapsto e^{ix}$. This homomorphism wraps the real line around S^1 with period 2π.

We now claim that given any continuous *function* $\varphi : \mathbb{R} \to S^1$ such that $\varphi(0) = 1$, there exists a unique continuous *lifting* homomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\varepsilon \circ \psi = \varphi, \quad \psi(0) = 0.$$



The lifting is constructed by starting with $\psi(0) = 0$, and then extending a small interval at a time to get a continuous map $\mathbb{R} \to \mathbb{R}$. We will not go into the details. Alternatively, this follows from the lifting criterion from IID Algebraic Topology.

We now claim that if in addition φ is a homomorphism, then so is its continuous lifting ψ. If this is true, then we can conclude that $\psi(x) = cx$ for some $c \in \mathbb{R}$. Hence

$$\varphi(x) = e^{icx}.$$

To show that ψ is indeed a homomorphism, we have to show that $\psi(x+y) = \psi(x) + \psi(y)$.

By definition, we know

$$\varphi(x+y) = \varphi(x)\varphi(y).$$

By definition of ψ, this means

$$\varepsilon(\psi(x+y) - \psi(x) - \psi(y)) = 1.$$

We now look at our definition of ε to get

$$\psi(x+y) - \psi(x) - \psi(y) = 2k\pi$$

for some integer $k \in \mathbb{Z}$, depending *continuously* on x and y. But k can only be an integer. So it must be constant. Now we pick our favorite x and y, namely $x = y = 0$. Then we find $k = 0$. So we get

$$\psi(x+y) = \psi(x) + \psi(y).$$

So ψ is a group homomorphism. □

Theorem. Every one-dimensional (continuous) representation S^1 is of the form

$$\rho : z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

Proof. Let $\rho : S^1 \to \mathbb{C}^\times$ be a continuous representation. We now claim that ρ actually maps S^1 to S^1. Since S^1 is compact, we know $\rho(S^1)$ has closed and bounded image. Also,

$$\rho(z^n) = (\rho(z))^n$$

for all $n \in \mathbb{Z}$. Thus for each $z \in S^1$, if $|\rho(z)| > 1$, then the image of $\rho(z^n)$ is unbounded. Similarly, if it is less than 1, then $\rho(z^{-n})$ is unbounded. So we must have $\rho(S^1) \subseteq S^1$. So we get a continuous homomorphism

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto \rho(e^{ix}). \end{aligned}$$

So we know there is some $c \in \mathbb{R}$ such that

$$\rho(e^{ix}) = e^{icx},$$

Now in particular,

$$1 = \rho(e^{2\pi i}) = e^{2\pi i c}.$$

This forces $c \in \mathbb{Z}$. Putting $n = c$, we get

$$\rho(z) = z^n.$$ □

Theorem. Let G be a compact Hausdorff topological group. Then there exists a unique Haar measure on G.

Corollary (Weyl's unitary trick)**.** Let G be a compact group. Then every representation (ρ, V) has a G-invariant Hermitian inner product.

Proof. As for the finite case, take any inner product $(\,\cdot\,,\,\cdot\,)$ on V, then define a new inner product by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \int_G (\rho(g)\mathbf{v}, \rho(g)\mathbf{w})\;\mathrm{d}g.$$

Then this is a G-invariant inner product. □

Theorem (Maschke's theoerm)**.** Let G be compact group. Then every representation of G is completely reducible.

Proof. Given a representation (ρ, V). Choose a G-invariant inner product. If V is not irreducible, let $W \leq V$ be a subrepresentation. Then $W^\perp$ is also G-invariant, and

$$V = W \oplus W^\perp.$$

Then the result follows by induction. □

Theorem (Orthogonality)**.** Let G be a compact group, and V and W be irreducible representations of G. Then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 1 & V \cong W\\ 0 & V \not\cong W\end{cases}.$$

15.1 Representations of SU(2)

Lemma (SU(2)-conjugacy classes)**.**

(i) Let $t \in T$. Then $sts^{-1} = t^{-1}$.

(ii) $s^2 = -I \in Z(\mathrm{SU}(2))$.

(iii) The normalizer

$$N_G(T) = T \cup sT = \left\{\begin{pmatrix} a & 0\\ 0 & \bar{a}\end{pmatrix}, \begin{pmatrix} 0 & a\\ -\bar{a} & 0\end{pmatrix} : a \in \mathbb{C}, |a| = 1\right\}.$$

(iv) Every conjugacy class $\mathcal{C}$ of SU(2) contains an element of T, i.e. $\mathcal{C} \cap T \neq \emptyset$.

(v) In fact,

$$\mathcal{C} \cap T = \{t, t^{-1}\}$$

for some $t \in T$, and $t = t^{-1}$ if and only if $t = \pm I$, in which case $\mathcal{C} = \{t\}$.

(vi) There is a bijection

$$\{\text{conjugacy classes in SU(2)}\} \leftrightarrow [-1, 1],$$

given by

$$A \mapsto \frac{1}{2}\operatorname{tr} A.$$

We can see that if
$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix},$$
then
$$\frac{1}{2}\operatorname{tr} A = \frac{1}{2}(\lambda + \bar{\lambda}) = \operatorname{Re}(\lambda).$$

Proof.

(i) Write it out.

(ii) Write it out.

(iii) Direct verification.

(iv) It is well-known from linear algebra that every unitary matrix X has an orthonormal basis of eigenvectors, and hence is conjugate in $U(2)$ to one in T, say
$$QXQ^\dagger \in T.$$
We now want to force Q into $\mathrm{SU}(2)$, i.e. make Q have determinant 1.

We put $\delta = \det Q$. Since Q is unitary, i.e. $QQ^\dagger = I$, we know $|\delta| = 1$. So we let ε be a square root of δ, and define
$$Q_1 = \varepsilon^{-1} Q.$$
Then we have
$$Q_1 X Q_1^\dagger \in T.$$

(v) We let $g \in G$, and suppose $g \in \mathcal{C}$. If $g = \pm I$, then $\mathcal{C} \cap T = \{g\}$. Otherwise, g has two distinct eigenvalues λ, λ^{-1}. Note that the two eigenvlaues must be inverses of each other, since it is in $\mathrm{SU}(2)$. Then we know
$$\mathcal{C} = \left\{ h \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} h^{-1} : h \in G \right\}.$$
Thus we find
$$\mathcal{C} \cap T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \right\}.$$
This is true since eigenvalues are preserved by conjugation, so if any
$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$
then $\{\mu, \mu^{-1}\} = \{\lambda, \lambda^{-1}\}$. Also, we can get the second matrix from the first by conjugating with s.

(vi) Consider the map
$$\frac{1}{2}\operatorname{tr} : \{\text{conjugacy classes}\} \to [-1, 1].$$
By (v), matrices are conjugate in G iff they have the same set of eigenvalues. Now
$$\frac{1}{2}\operatorname{tr}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \frac{1}{2}(\lambda + \bar{\lambda}) = \operatorname{Re}(\lambda) = \cos\theta,$$

where $\lambda = e^{i\theta}$. Hence the map is a surjection onto $[-1, 1]$.

Now we have to show it is injective. This is also easy. If g and g' have the same image, i.e.

$$\frac{1}{2}\operatorname{tr} g = \frac{1}{2}\operatorname{tr} g',$$

then g and g' have the same characteristic polynomial, namely

$$x^2 - (\operatorname{tr} g)x + 1.$$

Hence they have the same eigenvalues, and hence they are similar. □

Proposition. For $t \in (-1, 1)$, the class $\mathcal{C}_t \cong S^2$ as topological spaces.

Proof. Exercise! □

Lemma. A continuous class function $f : G \to \mathbb{C}$ is determined by its restriction to T, and $F|_T$ is even, i.e.

$$f\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right) = f\left(\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}\right).$$

Proof. Each conjugacy class in $\mathrm{SU}(2)$ meets T. So a class function is determined by its restriction to T. Evenness follows from the fact that the two elements are conjugate. □

Lemma. If χ is a character of a representation of $\mathrm{SU}(2)$, then its restriction $\chi|_T$ is a Laurent polynomial, i.e. a finite $\mathbb{N}$-linear combination of functions

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \mapsto \lambda^n$$

for $n \in \mathbb{Z}$.

Proof. If V is a representation of $\mathrm{SU}(2)$, then $\operatorname{Res}_T^{\mathrm{SU}(2)} V$ is a representation of T, and its character $\operatorname{Res}_T^{\mathrm{SU}(2)} \chi$ is the restriction of χ_V to T. But every representation of T has its character of the given form. So done. □

Theorem. The representations $\rho_n : \mathrm{SU}(2) \to \mathrm{GL}(V_n)$ of dimension $n + 1$ are irreducible for $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $0 \neq W \leq V_n$ be a G-invariant subspace, i.e. a subrepresentation of V_n. We will show that $W = V_n$.

All we know about W is that it is non-zero. So we take some non-zero vector of W.

Claim. Let

$$0 \neq w = \sum_{j=0}^{n} r_j x^{n-j} y^j \in W.$$

Since this is non-zero, there is some i such that $r_i \neq 0$. The claim is that $x^{n-i} y^i \in W$.

We argue by induction on the number of non-zero coefficients r_j. If there is only one non-zero coefficient, then we are already done, as w is a non-zero scalar multiple of $x^{n-i}y^i$.

So assume there is more than one, and choose one i such that $r_i \neq 0$. We pick $z \in S^1$ with $z^n, z^{n-2}, \cdots, z^{2-n}, z^{-n}$ all distinct in $\mathbb{C}$. Now

$$\rho_n\left(\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix}\right) w = \sum r_j z^{n-2j} x^{n-j}y^j \in W.$$

Subtracting a copy of w, we find

$$\rho_n\left(\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix}\right) w - z^{n-2i}w = \sum r_j (z^{n-2j} - z^{n-2i}) x^{n-j}y^j \in W.$$

We now look at the coefficient

$$r_j(z^{n-2j} - z^{n-2i}).$$

This is non-zero if and only if r_j is non-zero and $j \neq i$. So we can use this to remove any non-zero coefficient. Thus by induction, we get

$$x^{n-j}y^j \in W$$

for all j such that $r_j \neq 0$.

This gives us one basis vector inside W, and we need to get the rest.

Claim. $W = V_n$.

We now know that $x^{n-i}y^i \in W$ for some i. We consider

$$\rho_n\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1\\ 1 & 1\end{pmatrix}\right) x^{n-i}y^i = \frac{1}{\sqrt{2}}(x+y)^{n-i}(-x+y)^i \in W.$$

It is clear that the coefficient of x^n is non-zero. So we can use the claim to deduce $x^n \in W$.

Finally, for general $a, b \neq 0$, we apply

$$\rho_n\left(\begin{pmatrix} a & -\bar{b}\\ b & \bar{a}\end{pmatrix}\right) x^n = (ax+by)^n \in W,$$

and the coefficient of everything is non-zero. So basis vectors are in W. So $W = V_n$. $\square$

Theorem. Every finite-dimensional continuous irreducible representation of G is one of the $\rho_n : G \to \mathrm{GL}(V_n)$ as defined above.

Proof. Assume $\rho_V : G \to \mathrm{GL}(V)$ is an irreducible representation affording a character $\chi_V \in \mathbb{N}[z, z^{-1}]_{\mathrm{ev}}$. We will show that $\chi_V = \chi_n$ for some n. Now we see

$$\begin{aligned}\chi_0 &= 1\\ \chi_1 &= z + z^{-1}\\ \chi_2 &= z^2 + 1 + z^{-2}\\ &\vdots\end{aligned}$$

form a basis of $\mathbb{Q}[z, z^{-1}]_{\text{ev}}$, which is a non-finite dimensional vector space over $\mathbb{Q}$. Hence we can write

$$\chi_V = \sum_n a_n \chi_n,$$

a finite sum with finitely many $a_n \neq 0$. Note that it is possible that $a_n \in \mathbb{Q}$. So we clear denominators, and move the summands with negative coefficients to the left hand side. So we get

$$m\chi_V + \sum_{i \in I} m_i \chi_i = \sum_{j \in J} n_j \chi_j,$$

with I, J disjoint finite subsets of $\mathbb{N}$, and $m, m_i, n_j \in \mathbb{N}$.

We know the left and right-hand side are characters of representations of G. So we get

$$mV \oplus \bigoplus_I m_i V_i = \bigoplus_J n_j V_j.$$

Since V is irreducible and factorization is unique, we must have $V \cong V_n$ for some $n \in J$. $\square$

Proposition. Let $G = \mathrm{SU}(2)$ or $G = S^1$, and V, W are representations of G. Then

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Proof. By the previous remark, it is enough to consider the case $G = S^1$. Suppose V and W have eigenbases $\mathbf{e}_1, \cdots, \mathbf{e}_n$ and $\mathbf{f}_1, \cdots, \mathbf{f}_m$ respectively such that

$$\rho(z)\mathbf{e}_i = z^{n_i}\mathbf{e}_i, \quad \rho(z)\mathbf{f}_j = z^{m_j}\mathbf{f}_j$$

for each i, j. Then

$$\rho(z)(\mathbf{e}_i \otimes \mathbf{f}_j) = z^{n_i + m_j}\mathbf{e}_i \otimes \mathbf{f}_j.$$

Thus the character is

$$\chi_{V \otimes W}(z) = \sum_{i,j} z^{n_i + m_j} = \left(\sum_i z^{n_i}\right)\left(\sum_j z^{m_j}\right) = \chi_V(z)\chi_W(z). \qquad \square$$

Proposition (Clebsch-Gordon rule)**.** For $n, m \in \mathbb{N}$, we have

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

Proof. We just check this works for characters. Without loss of generality, we assume $n \geq m$. We can compute

$$\begin{aligned}
(\chi_n \chi_m)(z) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}(z^m + z^{m-2} + \cdots + z^{-m}) \\
&= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{2j-n-m-1}}{z - z^{-1}} \\
&= \sum_{j=0}^{m} \chi_{n+m-2j}(z).
\end{aligned}$$

Note that the condition $n \geq m$ ensures there are no cancellations in the sum. $\square$

15.2 Representations of SO(3), SU(2) and $U(2)$

Proposition. There are isomorphisms of topological groups:

(i) $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\} = \mathrm{PSU}(2)$

(ii) $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm(I, I)\}$

(iii) $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SU}(2)/\{\pm(I, I)\}$

All maps are group isomorphisms, but in fact also homeomorphisms. To show this, we can use the fact that a continuous bijection from a Hausdorff space to a compact space is automatically a homeomorphism.

Corollary. Every irreducible representation of SO(3) has the following form:

$$\rho_{2m} : \mathrm{SO}(3) \to \mathrm{GL}(V_{2m}),$$

for some $m \geq 0$, where V_n are the irreducible representations of SU(2).

Proof. Irreducible representations of SO(3) correspond to irreducible representations of SU(2) such that $-I$ acts trivially by lifting. But $-I$ acts on V_n as -1 when n is odd, and as 1 when n is even, since

$$\rho(-I) = \begin{pmatrix} (-1)^n & & & \\ & (-1)^{n-2} & & \\ & & \ddots & \\ & & & (-1)^{-n} \end{pmatrix} = (-1)^n I. \qquad \square$$

Proposition. $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\}$.

Proof sketch. Recall that SU(2) can be viewed as the sphere of unit norm quaternions $\mathbb{H} \cong \mathbb{R}^4$.

Let

$$\mathbb{H}^0 = \{A \in \mathbb{H} : \operatorname{tr} A = 0\}.$$

These are the "pure" quaternions. This is a three-dimensional subspace of $\mathbb{H}$. It is not hard to see this is

$$\mathbb{H}^0 = \mathbb{R}\left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle = \mathbb{R}\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle,$$

where $\mathbb{R}\langle\cdots\rangle$ is the $\mathbb{R}$-span of the things.

This is equipped with the norm

$$\|A\|^2 = \det A.$$

This gives a nice 3-dimensional Euclidean space, and SU(2) acts as isometries on $\mathbb{H}_0$ by conjugation, i.e.

$$X \cdot A = XAX^{-1},$$

giving a group homomorphism

$$\varphi : \mathrm{SU}(2) \to \mathrm{O}(3),$$

and the kernel of this map is $Z(\mathrm{SU}(2)) = \{\pm I\}$. We also know that SU(2) is compact, and O(3) is Hausdorff. Hence the continuous group isomorphism

$$\bar{\varphi} : \mathrm{SU}(2)/\{\pm I\} \to \operatorname{im}\varphi$$

is a homeomorphism. It remains to show that $\operatorname{im}\varphi = \mathrm{SO}(3)$.

But we know SU(2) is connected, and $\det(\varphi(X))$ is a continuous function that can only take values 1 or -1. So $\det(\varphi(X))$ is either always 1 or always -1. But $\det(\varphi(I)) = 1$. So we know $\det(\varphi(X)) = 1$ for all X. Hence $\operatorname{im}\varphi \leq \mathrm{SO}(3)$.

To show that equality indeed holds, we have to show that all possible rotations in $\mathbb{H}^0$ are possible. We first show all rotations in the $\mathbf{i}, \mathbf{j}$-plane are implemented by elements of the form $a + b\mathbf{k}$, and similarly for any permutation of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Since all such rotations generate SO(3), we are then done. Now consider

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} ai & b \\ -\bar{b} & -ai \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ai & e^{2i\theta} b \\ -\bar{b}e^{-2i\theta} & -ai \end{pmatrix}.$$

So

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

acts on $\mathbb{R}\langle \mathbf{i}, \mathbf{j}, \mathbf{k}\rangle$ by a rotation in the $(\mathbf{j}, \mathbf{k})$-plane through an angle 2θ. We can check that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

act by rotation of 2θ in the $(\mathbf{i}, \mathbf{k})$-plane and $(\mathbf{i}, \mathbf{j})$-plane respectively. So done. □

Proposition. The complete list of irreducible representations of SO(4) is $\rho_m \times \rho_n$, where $m, n > 0$ and $m \equiv n \pmod 2$.

Proposition. The complete list of irreducible representations of U(2) is

$$\det{}^{\otimes m} \otimes \rho_n,$$

where $m, n \in \mathbb{Z}$ and $n \geq 0$, and det is the obvious one-dimensional representation.