PART II REPRESENTATION THEORY
SHEET 1

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let ρ be a representation of the group G.
(a) Show that $\delta : g \mapsto \det \rho(g)$ is a 1-dimensional representation of G.
(b) Prove that $G/\ker \delta$ is abelian.
(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

2 Let $\theta : G \to F^\times$ be a 1-dimensional representation of the group G, and let $\rho : G \to \mathrm{GL}(V)$ be another representation. Show that $\theta \otimes \rho : G \to \mathrm{GL}(V)$ given by $\theta \otimes \rho : g \mapsto \theta(g) \cdot \rho(g)$ is a representation of G, and that it is irreducible if and only if ρ is irreducible.

3 Find an example of a representation of some finite group over some field of characteristic p, which is not completely reducible. Find an example of such a representation in characteristic 0 for an infinite group.

4 Let N be a normal subgroup of the group G. Given a representation of the quotient G/N, use it to obtain a representation of G. Which representations of G do you get this way?

Recall that the derived subgroup G' of G is the unique smallest normal subgroup of G such that G/G' is abelian. Show that the 1-dimensional complex representations of G are precisely those obtained from G/G'.

5 Describe Weyl's unitary trick.

Let G be a finite group acting on a complex vector space V, and let $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ be a skew-symmetric form, i.e. $\langle\ y,\ x\ \rangle = -\langle\ x,\ y\ \rangle$ for all x, y in V.

Show that the form $(x, y) = \frac{1}{|G|} \sum \langle gx, gy \rangle$, where the sum is over all elements $g \in G$, is a G-invariant skew-symmetric form.

Does this imply that every finite subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ is conjugate to a subgroup of the symplectic group[1] $\mathrm{Sp}_{2m}(\mathbb{C})$?

6 Let G be a cyclic group of order n.
(i) G acts on $\mathbb{R}^2$ as symmetries of the regular n-gon. Choose a basis of $\mathbb{R}^2$, and write the matrix $R(1)$ representing the action of $1 \in G$ in this basis. Is this an irreducible representation?
(ii) Now regard $R(1)$ above as a complex matrix, so that we get a representation of G on $\mathbb{C}^2$. Decompose $\mathbb{C}^2$ into its irreducible summands.

7 Let G be a cyclic group of order n. Explicitly decompose the complex regular representation of G as a direct sum of 1-dimensional representations, by giving the matrix of change of coordinates from the natural basis $\{e_g\}_{g \in G}$ to a basis where the group action is diagonal.

[1] the group of all linear transformations of a $2m$-dimensional vector space over $\mathbb{C}$ that preserve a non-degenerate, skew-symmetric, bilinear form.

8 Let G be the dihedral group D_{10} of order 10,

$$D_{10} = \langle x, y \,:\, x^5 = 1 = y^2,\ yxy^{-1} = x^{-1} \rangle.$$

Show that G has precisely two 1-dimensional representations. By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to one of two representations of dimension 2. Show that all these representations can be realised over $\mathbb{R}$.

9 Let G be the quaternion group Q_8 of order 8,

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2,\ yxy^{-1} = x^{-1} \rangle.$$

By considering the effect of y on an eigenvector of x show that any complex irreducible representation of G of dimension at least 2 is isomorphic to the standard representation of Q_8 of dimension 2.

Show that this 2-dimensional representation cannot be realised over $\mathbb{R}$; that is, Q_8 is not a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

10 Suppose that F is algebraically closed. Using Schur's lemma, show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H. What happens for $F = \mathbb{R}$?

11 Let G be a subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of degree 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible representations.

12 Show that if ρ is a homomorphism from the finite group G to $\mathrm{GL}_n(\mathbb{R})$, then there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$. (Recall that the real matrix A is orthogonal if $A^t A = I$.)

Determine all finite groups which have a faithful 2-dimensional representation over $\mathbb{R}$.

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PART II REPRESENTATION THEORY
SHEET 2

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Let χ be the character of a representation V of G and let g be an element of G. If g is an involution (i.e. $g^2 = 1 \neq g$), show that $\chi(g)$ is an integer and $\chi(g) \equiv \chi(1) \bmod 2$. If G is simple (but not C_2), show that in fact $\chi(g) \equiv \chi(1) \bmod 4$. (Hint: consider the determinant of g acting on V.) If g has order 3 and is conjugate to g^{-1}, show that $\chi(g) \equiv \chi(1) \bmod 3$.

2 Construct the character table of the dihedral group D_8 and of the quaternion group Q_8. You should notice something interesting.

3 Construct the character table of the dihedral group D_{10}.

Each irreducible representation of D_{10} may be regarded as a representation of the cyclic subgroup C_5. Determine how each irreducible representation of D_{10} decomposes into irreducible representations of C_5.

Repeat for $D_{12} \cong S_3 \times C_2$ and the cyclic subgroup C_6 of D_{12}.

4 Construct the character tables of A_4, S_4, S_5, and A_5.

The group S_n acts by conjugation on the set of elements of A_n. This induces an action on the set of conjugacy classes and on the set of irreducible characters of A_n. Describe the actions in the cases where $n = 4$ and $n = 5$.

5 A certain group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β. The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

6 The table below is a part of the character table of a certain finite group, with some of the rows missing. The columns are labelled by the sizes of the conjugacy classes, and $\gamma = (-1 + i\sqrt{7})/2$, $\zeta = (-1 + i\sqrt{3})/2$. Complete the character table. Describe the group in terms of generators and relations.

	1	3	3	7	7
χ_1	1	1	1	ζ	$\bar{\zeta}$
χ_2	3	γ	$\bar{\gamma}$	0	0

7 Let x be an element of order n in a finite group G. Say, without detailed proof, why
(a) if χ is a character of G, then $\chi(x)$ is a sum of nth roots of unity;
(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};
(c) x and x^{-1} have the same number of conjugates in G.

Prove that the number of irreducible characters of G which take only real values (so-called *real characters*) is equal to the number of self-inverse conjugacy classes (so-called *real classes*).

8 A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.
[You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of unity.]

The character table thus obtained is in fact the character table of the group $G = \mathrm{PSL}_2(7)$ of 2×2 matrices with determinant 1 over the field $\mathbb{F}_7$ (of seven elements) modulo the two scalar matrices. Deduce directly from the character table that G is simple[1].

9 The group M_9 is a certain subgroup of the symmetric group S_9 generated by the two elements $(1,4,9,8)(2,5,3,6)$ and $(1,6,5,2)(3,7,9,8)$. You are given the following facts about M_9:

• there are six conjugacy classes:
— C_1 contains the identity.
— For $2 \leqslant i \leqslant 4$, $|C_i| = 18$ and C_i contains g_i, where $g_2 = (2,3,8,6)(4,7,5,9)$, $g_3 = (2,4,8,5)(3,9,6,7)$ and $g_4 = (2,7,8,9)(3,4,6,5)$.
— $|C_5| = 9$, and C_5 contains $g_5 = (2,8)(3,6)(4,5)(7,9)$
— $|C_6| = 8$, and C_6 contains $g_6 = (1,2,8)(3,9,4)(5,7,6)$.
• every element of M_9 is conjugate to its inverse.

Calculate the character table of M_9. [Hint: You may find it helpful to notice that $g_2^2 = g_3^2 = g_4^2 = g_5$.]

10 Let a finite group G act on itself by conjugation. Find the character of the corresponding permutation representation.

11 Consider the character table Z of G as a matrix of complex numbers (as we did when deriving the column orthogonality relations from the row orthogonality relations).

(a) Using the fact that the complex conjugate of an irreducible character is also an irreducible character, show that the determinant $\det Z$ is $\pm \det \bar{Z}$, where $\bar{Z}$ is the complex conjugate of Z.

(b) Deduce that either $\det Z \in \mathbb{R}$ or $\det Z \in i\mathbb{R}$.

(c) Use the column orthogonality relations to calculate the product $\bar{Z}^T Z$, where $\bar{Z}^T$ is the transpose of the complex conjugate of Z.

(d) Calculate $|\det Z|$.

[1]It is known that there are precisely five non-abelian simple groups of order less than 1000. The smallest of these is $A_5 \cong \mathrm{PSL}_2(5)$, while G is the second smallest. The others are A_6, $\mathrm{PSL}_2(8)$ and $\mathrm{PSL}_2(11)$. It is also known that for $p \geqslant 5$, $\mathrm{PSL}_2(p)$ is simple.

12 The group $\mathrm{SL}_2(\mathbb{F}_q)$ acts on the projective line $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Show that $\mathrm{SL}_2(\mathbb{F}_q)$ has an irreducible representation of dimension q.

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PART II REPRESENTATION THEORY
SHEET 3

Unless otherwise stated, all groups here are finite, and all vector spaces are finite-dimensional over a field F of characteristic zero, usually $\mathbb{C}$.

1 Recall the character table of S_4 from Sheet 2. Find all the characters of S_5 induced from the irreducible characters of S_4. Hence find the complete character table of S_5.

Repeat, replacing S_4 by the subgroup $\langle (12345), (2354) \rangle$ of order 20 in S_5.

2 Recall the construction of the character table of the dihedral group D_{10} of order 10 from Sheet 2.

(a) Use induction from the subgroup D_{10} of A_5 to A_5 to obtain the character table of A_5.

(b) Let G be the subgroup of $\mathrm{SL}_2(\mathbb{F}_5)$ consisting of upper triangular matrices. Compute the character table of G.

Hint: bear in mind that there is an isomorphism $G/Z \to D_{10}$.

3 Let H be a subgroup of the group G. Show that for every irreducible representation ρ for G there is an irreducible representation ρ' for H with ρ a component of the induced representation $\mathrm{Ind}_H^G\, \rho'$.

Prove that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G : A|$.

4 Obtain the character table of the dihedral group D_{2m} of order $2m$, by using induction from the cyclic subgroup C_m. [Hint: consider the cases m odd and m even separately, as for m even there are two conjugacy classes of reflections, whereas for m odd there is only one.]

5 Prove the transitivity of induction: if $H < K < G$ then

$$\mathrm{Ind}_K^G\, \mathrm{Ind}_H^K\, \rho \cong \mathrm{Ind}_H^G \rho$$

for any representation ρ of H.

6 (a) Let $V = U \oplus W$ be a direct sum of $\mathbb{C}G$-modules. Prove that both the symmetric square and the exterior square of V have submodules isomorphic to $U \otimes W$.

(b) Calculate $\chi_{\Lambda^2\rho}$ and $\chi_{S^2\rho}$, where ρ is the irreducible representation of dimension 2 of D_8; repeat this for Q_8. Which of these characters contains the trivial character in the two cases?

7 Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G of dimension d.

(a) Compute the dimension of S^nV and Λ^nV for all n.

(b) Let $g \in G$ and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of g on V. What are the eigenvalues of g on S^nV and Λ^nV?

(c) Let $f(t) = \det(g - tI)$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of f and χ_{Λ^nV}?

(d) Find a relationship between χ_{S^nV} and f.

8 Let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$. For any integer $r \leqslant \frac{n}{2}$, write X_r for the set of all r-element subsets of X, and let π_r be the permutation character of the action of G on X_r. Observe $\pi_r(1) = |X_r| = \binom{n}{r}$. If $0 \leqslant \ell \leqslant k \leqslant n/2$, show that

$$\langle \pi_k, \pi_\ell \rangle = \ell + 1.$$

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G$, $\chi^{(n-1,1)}, \chi^{(n-2,2)}, \ldots, \chi^{(n-m,m)}$ such that for all $r \leqslant m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \cdots + \chi^{(n-r,r)}.$$

In particular the class functions $\pi_r - \pi_{r-1}$ are irreducible characters of S_n for $1 \leqslant r \leqslant n/2$ and equal to $\chi^{(n-r,r)}$.

9 Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation for G affording the character χ. Give the characters of the representations $V \otimes V$, S^2V and $\Lambda^2 V$ in terms of χ.

(i) Let W be another finite-dimensional representation with character ψ. Show that

$$\dim W^G = \frac{1}{|G|} \sum_{g \in G} \psi(g)$$

where $W^G = \{w \in W : gw = w \text{ for all } g \in G\}$.

(ii) Prove that if V is irreducible, $V \otimes V$ contains the trivial representation at most once.

(iii) Given any irreducible character χ of G, the *indicator* $\iota\chi$ of χ is defined by

$$\iota\chi = \frac{1}{|G|} \sum_{x \in G} \chi(x^2).$$

By using the decomposition $V \otimes V = S^2V \oplus \Lambda^2 V$, deduce that

$$\iota\chi = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued} \\ \pm 1, & \text{if } \chi \text{ is real-valued.} \end{cases}$$

Deduce that if $|G|$ is odd then G has only one real-valued irreducible character.
[Remark. The sign $+$, resp. $-$, indicates whether $\rho(G)$ preserves an orthogonal, respectively symplectic form on V, and whether or not the representation can be realised over the reals. You can read about it in Ch. 23 of James and Liebeck.]

10 Suppose that G is a Frobenius group with Frobenius kernel K. Show that

(i) $C_G(k) \leqslant K$ for all $1 \neq k \in K$.

(ii) if χ is a non-trivial irreducible character of K then $\mathrm{Ind}_K^G \chi$ is also irreducible with K not lying in its kernel. Hence explain how to construct the character table of G, given the character tables of K and G/K.

[Hints for (ii):

(a) First, show each element of $G \setminus K$ permutes the conjugacy classes in K, and fixes only the identity.

(b) Deduce that each element of $G \setminus K$ fixes only the trivial character of K.

(c) Use the Orbit-Stabilizer theorem to deduce that if χ is a non-trivial irreducible character of K then the number of distinct conjugates of χ is $|G : K|$.

(d) Use Frobenius reciprocity to show that if χ is as above and ϕ is an irreducible constituent of $\mathrm{Ind}_K^G \chi$, then all $|G : K|$ conjugates of χ are constituents of $\mathrm{Res}_K^G \phi$. Finally compare degrees to get the result.]

11 Construct the character table of the symmetric group S_6. Identify which of your characters are equal to the characters $\chi^{(6)}, \chi^{(5,1)}, \chi^{(4,2)}, \chi^{(3,3)}$ constructed in question 8.

12 If θ is a faithful character of the group G, which takes r distinct values on G, prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.
[Hint: assume that $\langle \chi, \theta^i \rangle = 0$ for all $i < r$; use the fact that the Vandermonde $r \times r$ matrix involving the row of the distinct values $a_1, ..., a_r$ of θ is nonsingular to obtain a contradiction.]

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PART II REPRESENTATION THEORY
SHEET 4

Unless otherwise stated, all vector spaces are finite-dimensional over $\mathbb{C}$. In the first seven questions we let $G = \mathrm{SU}(2)$. Questions 9 onwards deal with a variety of topics at Tripos standard.

1 Let V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y. Describe a representation ρ_n of G on V_n and show that it is irreducible. What is its character? Show that V_n is isomorphic to its dual V_n^*.

2 Decompose the representation $V_4 \otimes V_3$ into irreducible G-spaces (that is, find a direct sum of irreducible representations which is isomorphic to $V_4 \otimes V_3$; in this and the following questions, you are not being asked to find such an isomorphism explicitly). Decompose $V_1^{\otimes n}$ into irreducibles.

3 Determine the character of $S^n V_1$ for $n \geqslant 1$.
Decompose $S^2 V_n$ and $\Lambda^2 V_n$ into irreducibles for $n \geqslant 1$.
Decompose $S^3 V_2$ into irreducibles.

4 Let G act on the space $\mathrm{M}_3(\mathbb{C})$ of 3×3 complex matrices, by conjugation:

$$A : X \mapsto A_1 X A_1^{-1},$$

where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this gives a representation of G and decompose it into irreducibles.

5 Let χ_n be the character of the irreducible representation ρ_n of G on V_n of dimension $n + 1$.

Show that

$$\frac{1}{2\pi} \int_0^{2\pi} K(z) \chi_n \overline{\chi_m} d\theta = \delta_{nm},$$

where $z = e^{i\theta}$ and $K(z) = \frac{1}{2}(z - z^{-1})(z^{-1} - z)$.
[Note that all you need to know about integrating on the circle is orthogonality of characters: $\frac{1}{2\pi} \int_0^{2\pi} z^n d\theta = \delta_{n,0}$. This is really a question about Laurent polynomials.]

6 Check that the usual formula for integrating functions defined on $S^3 \subseteq \mathbf{R}^4$ defines a G-invariant inner product on the vector space of integrable functions on

$$G = \mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\},$$

and normalize it so that the integral over the group is one.

7 Compute the character of the representation $S^n V_2$ of G for any $n \geqslant 0$. Calculate $\dim_{\mathbb{C}}(S^n V_2)^G$ (by which we mean the subspace of $S^n V_2$ where G acts trivially).

Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of $\mathrm{SO}(3)$ is a polynomial ring. Find a generator for this polynomial ring.

8 (a) Let G be a compact group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive definite) symmetric bilinear form.
(b) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the vector space of matrices of the form

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation. Show that this subspace is isomorphic to $\mathbb{R}^3$, that $(A, B) \mapsto -\mathrm{tr}(AB)$ is an invariant positive definite symmetric bilinear form, and that ρ is surjective with kernel $\{\pm I\}$.

9 The *Heisenberg group* of order p^3 is the (non-abelian) subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, x \in \mathbb{F}_p \right\}.$$

of matrices over the finite field $\mathbb{F}_p$ (p prime). Let H be the subgroup of G comprising matrices with $a = 0$ and Z be the subgroup of G of matrices with $a = b = 0$.

(a) Show that $Z = Z(G)$, the centre of G, and that $G/Z = \mathbb{F}_p^2$. Note that this implies that the derived subgroup G' is contained in Z. [You can check by explicit computation that it equals Z, or you can deduce this from the list of irreducible representations found in (d) below.]

(b) Find all 1-dimensional representations of G.

(c) Let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial 1-dimensional representation of the cyclic group $\mathbb{F}_p = \mathbb{Z}/p$, and define a 1-dimensional representation ρ_ψ of H by

$$\rho_\psi \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $\mathrm{Ind}_H^G \rho_\psi$ is an irreducible representation of G.

(d) Prove that the collection of representations constructed in (b) and (c) gives a complete list of all irreducible representations.

(e) Determine the character of the irreducible representation $\mathrm{Ind}_H^G \rho_\psi$.

10 Recall the character table of $G = \mathrm{PSL}_2(7)$ from Sheet 2, q.8. Identify the columns corresponding to the elements x and y where x is an element of order 7 (eg the unitriangular matrix with 1 above the diagonal) and y is an element of order 3 (eg the diagonal matrix with entries 4 and 2).

The group G acts as a permutation group of degree 8 on the set of Sylow 7-subgroups (or the set of 1-dimensional subspaces of the vector space $(\mathbb{F}_7)^2$). Obtain the permutation character of this action and decompose it into irreducible characters.

*(Harder) Show that the group G is generated by an element of order 2 and an element of order 3 whose product has order 7.
[Hint: for the last part use the formula that the number of pairs of elements conjugate to x and y respectively, whose product is conjugate to t, equals $c \sum \chi(x)\chi(y)\chi(t^{-1})/\chi(1)$, where the sum runs over all the irreducible characters of G, and $c = |G|^2(|C_G(x)||C_G(y)||C_G(t)|)^{-1}$.]

11 Let $J_{\lambda,n}$ be the $n \times n$ Jordan block with eigenvalue $\lambda \in K$ (K is any field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}.$$

(a) Compute $J_{\lambda,n}^r$ for each $r \geqslant 0$.

(b) Let G be cyclic of order N, and let K be an algebraically closed field of characteristic $p > 0$. Determine *all* the representations of G on vector spaces over K, up to equivalence. Which are irreducible? Which are indecomposable?

Remark: Over $\mathbb{C}$ irreducibility and indecomposability coincide but this can fail for modular representations.

12 [For enthusiasts only. Part (a) requires knowledge of Galois Theory.]

(a) Let G be a cyclic group and let χ be a (possibly reducible) character of G. Let $S = \{g \in G : G = \langle g \rangle\}$ and assume that $\chi(s) \neq 0$ for all $s \in S$. Show that

$$\sum_{s \in S} |\chi(s)|^2 \geqslant |S|.$$

(b) Deduce a theorem of Burnside: namely, let χ be an irreducible character of G with $\chi(1) > 1$. Show that $\chi(g) = 0$ for some $g \in G$. [Hint: partition G into equivalence classes by calling two elements of G equivalent if they generate the same cyclic subgroup of G.]

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