

Part II — Logic and Set Theory

Theorems

Based on lectures by I. B. Leader

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

No specific prerequisites.

Ordinals and cardinals

Well-orderings and order-types. Examples of countable ordinals. Uncountable ordinals and Hartogs' lemma. Induction and recursion for ordinals. Ordinal arithmetic. Cardinals; the hierarchy of alephs. Cardinal arithmetic. [5]

Posets and Zorn's lemma

Partially ordered sets; Hasse diagrams, chains, maximal elements. Lattices and Boolean algebras. Complete and chain-complete posets; fixed-point theorems. The axiom of choice and Zorn's lemma. Applications of Zorn's lemma in mathematics. The well-ordering principle. [5]

Propositional logic

The propositional calculus. Semantic and syntactic entailment. The deduction and completeness theorems. Applications: compactness and decidability. [3]

Predicate logic

The predicate calculus with equality. Examples of first-order languages and theories. Statement of the completeness theorem; *sketch of proof*. The compactness theorem and the Lowenheim-Skolem theorems. Limitations of first-order logic. Model theory. [5]

Set theory

Set theory as a first-order theory; the axioms of ZF set theory. Transitive closures, epsilon-induction and epsilon-recursion. Well-founded relations. Mostowski's collapsing theorem. The rank function and the von Neumann hierarchy. [5]

Consistency

Problems of consistency and independence [1]

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0 Introduction

1 Propositional calculus

1.1 Propositions

1.2 Semantic entailment

Proposition.

- (i) If v and v' are valuations with $v(p) = v'(p)$ for all $p \in P$, then $v = v'$.
- (ii) For any function $w : P \rightarrow \{0, 1\}$, we can extend it to a valuation v such that $v(p) = w(p)$ for all $p \in L$.

1.3 Syntactic implication

Proposition (Deduction theorem). Let $S \subset L$ and $p, q \in L$. Then we have

$$S \vdash (p \Rightarrow q) \quad \Leftrightarrow \quad S \cup \{p\} \vdash q.$$

This says that \vdash behaves like the connective \Rightarrow in the language.

Proposition (Soundness theorem). If $S \vdash t$, then $S \models t$.

Theorem (Model existence theorem). If $S \models \perp$, then $S \vdash \perp$. i.e. if S has no model, then S is inconsistent. Equivalently, if S is consistent, then S has a model.

Lemma. For consistent $S \subset L$ and $p \in L$, at least one of $S \cup \{p\}$ and $S \cup \{\neg p\}$ is consistent.

Corollary (Adequacy theorem). Let $S \subset L$, $t \in L$. Then $S \models t$ implies $S \vdash t$.

Theorem (Completeness theorem). Let $S \subset L$ and $t \in L$. Then $S \models t$ if and only if $S \vdash t$.

Corollary (Compactness theorem). Let $S \subset L$ and $t \in L$ with $S \models t$. Then there is some finite $S' \subset S$ has $S' \models t$.

Corollary (Decidability theorem). Let $S \subset L$ be a finite set and $t \in L$. Then there exists an algorithm that determines, in finite and bounded time, whether or not $S \vdash t$.

2 Well-orderings and ordinals

2.1 Well-orderings

Proposition. A total order is a well-ordering if and only if it has no infinite strictly decreasing sequence.

Proposition (Principle by induction). Let X be a well-ordered set. Suppose $S \subseteq X$ has the property:

$$(\forall x) \left(((\forall y) y < x \Rightarrow y \in S) \Rightarrow x \in S \right),$$

then $S = X$.

In particular, if a property $P(x)$ satisfies

$$(\forall x) \left(((\forall y) y < x \Rightarrow P(y)) \Rightarrow P(x) \right),$$

then $P(x)$ for all x .

Proposition. Let X and Y be isomorphic well-orderings. Then there is a unique isomorphism between X and Y .

Proposition. Every initial segment Y of a well-ordered set X is of the form $I_x = \{y \in X : y < x\}$.

Theorem (Definition by recursion). Let X be a well-ordered set and Y be any set. Then for any function $G : \mathbb{P}(X \times Y) \rightarrow Y$, there exists a function $f : X \rightarrow Y$ such that

$$f(x) = G(f|_{I_x})$$

for all x .

This is a rather weird definition. Intuitively, it means that G takes previous values of $f(x)$ and returns the desired output. This means that in defining f at x , we are allowed to make use of values of f on I_x . For example, we define $f(n) = nf(n-1)$ for the factorial function, with $f(0) = 1$.

Lemma (Subset collapse). Let X be a well-ordering and let $Y \subseteq X$. Then Y is isomorphic to an initial segment of X . Moreover, this initial segment is unique.

Theorem. Let X, Y be well-orderings. Then $X \leq Y$ or $Y \leq X$.

Theorem. Let X, Y be well-orderings with $X \leq Y$ and $Y \leq X$. Then X and Y are isomorphic.

2.2 New well-orderings from old

Proposition. Let $\{X_i : i \in I\}$ be a nested set of well-orderings. Then there exists a well-ordering X with $X_i \leq X$ for all i .

2.3 Ordinals

Proposition. Let α be an ordinal. Then the ordinals $< \alpha$ form a well-ordering of order type α .

Proposition. Let S be a non-empty set of ordinals. Then S has a least element.

Theorem (Burali-Forti paradox). The ordinals do not form a set.

Theorem. There is an uncountable ordinal.

Theorem (Hartogs' lemma). For any set X , there is an ordinal that does not inject into X .

2.4 Successors and limits

2.5 Ordinal arithmetic

Proposition. Addition is associative, i.e. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Proposition. The inductive and synthetic definition of $+$ coincide.

2.6 Normal functions*

Lemma. Let f be a normal function. Then f is strictly increasing.

Lemma. Let f be a normal function, and α an ordinal. Then $f(\alpha) \geq \alpha$.

Lemma. If f is a normal function, then for any non-empty set $\{\alpha_i\}_{i \in I}$, we have

$$f(\sup\{\alpha_i : i \in I\}) = \sup\{f(\alpha_i) : i \in I\}.$$

Lemma (Fixed-point lemma). Let f be a normal function. Then for each ordinal α , there is some $\beta \geq \alpha$ such that $f(\beta) = \beta$.

Lemma (Division algorithm for normal functions). Let f be a normal function. Then for all α , there is some maximal γ such that $\alpha \geq f(\gamma)$.

3 Posets and Zorn's lemma

3.1 Partial orders

Theorem (Knaster-Tarski fixed point theorem). Let X be a complete poset, and $f : X \rightarrow X$ be a order-preserving function. Then f has a fixed point.

Corollary (Cantor-Schröder-Bernstein theorem). Let A, B be sets. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections. Then there is a bijection $h : A \rightarrow B$.

Theorem (Zorn's lemma). Assuming Axiom of Choice, let X be a (non-empty) poset in which every chain has an upper bound. Then it has a maximal element.

Theorem. Every vector space V has a basis.

Theorem (Model existence theorem (uncountable case)). Let $S \subseteq L(P)$ for any set of primitive propositions P . Then if S is consistent, S has a model.

3.2 Zorn's lemma and axiom of choice

Axiom (Axiom of choice). Given any family $\{A_i : i \in I\}$ of non-empty sets, there is a *choice function* $f : i \rightarrow \bigcup A_i$ such that $f(i) \in A_i$.

Theorem. Zorn's Lemma \Leftrightarrow Axiom of choice.

Theorem (Well-ordering theorem). Axiom of choice \Rightarrow every set X can be well-ordered.

Theorem. Well-ordering theorem \Rightarrow Axiom of Choice.

3.3 Bourbaki-Witt theorem*

Theorem (Bourbaki-Witt theorem). If X is chain-complete and $f : X \rightarrow X$ is inflationary, then f has a fixed point.

4 Predicate logic

4.1 Language of predicate logic

4.2 Semantic entailment

4.3 Syntactic implication

Proposition (Deduction theorem). Let $S \subseteq L$, and $p, q \in L$. Then $S \cup \{p\} \vdash q$ if and only if $S \vdash p \Rightarrow q$.

Proposition (Soundness theorem). Let S be a set of sentences, p a sentence. Then $S \vdash p$ implies $S \models p$.

Theorem (Model existence lemma). Let S be a consistent set of sentences. Then S has a model.

Corollary (Adequacy theorem). Let S be a theory, and p a sentence. Then $S \models p$ implies $S \vdash p$.

Theorem (Gödel's completeness theorem (for first order logic)). Let S be a theory, p and sentence. Then $S \vdash p$ if and only if $S \models p$.

Corollary (Compactness theorem). Let S be a theory such that every finite subset of S has a model. Then so does S .

Corollary. The theory of finite groups cannot be axiomatized (in the language of groups).

Corollary. Let S be a theory with arbitrarily large models. Then S has an infinite model.

“Finiteness is not a first-order property”

Corollary (Upward Löwenheim-Skolem theorem). Let S be a theory with an infinite model. Then S has an uncountable model.

Theorem (Downward Löwenheim-Skolem theorem). Let L be a countable language (i.e. Ω and Π are countable). Then if S has a model, then it has a countable model.

4.4 Peano Arithmetic

4.5 Completeness and categoricity*

Proposition. Let T be a theory that is κ categorical for some κ , and suppose T has no finite models. Then T is complete.

Theorem (Ax-Grothendieck theorem). Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a complex polynomial. If f is injective, then it is in fact a bijection.

Lemma. Any two uncountable algebraically closed fields with the same dimension and same characteristic are isomorphic. In other words, the theory of algebraically closed fields of characteristic p (for p a prime or 0) is κ -categorical for all uncountable cardinals κ , and in particular complete.

Theorem (Morley's categoricity theorem). Let T be a theory with a countable language. If T is κ -categorical for *some* uncountable cardinal κ , then it is μ -categorical for *all* uncountable cardinals μ .

5 Set theory

5.1 Axioms of set theory

Axiom (Axiom of extension). “If two sets have the same elements, they are the same set”.

$$(\forall x)(\forall y)((\forall z)(z \in x \Leftrightarrow z \in y) \Rightarrow x = y).$$

Axiom (Axiom of separation). “Can form subsets of sets”. More precisely, for any set x and a formula p , we can form $\{z \in x : p(z)\}$.

$$(\forall t_1) \cdots (\forall t_n)(\forall x)(\exists y)(\forall z)(z \in y \Leftrightarrow (z \in x \wedge p)).$$

This is an axiom scheme, with one instance for each formula p with free variables t_1, \dots, t_n, z .

Note again that we have those funny $(\forall t_i)$. We do need them to form, e.g. $\{z \in x : t \in z\}$, where t is a parameter.

This is sometimes known as Axiom of comprehension.

Axiom (Axiom of empty set). “The empty-set exists”

$$(\exists x)(\forall y)(y \notin x).$$

We write \emptyset for the (unique, by extension) set with no members. This is an abbreviation: $p(\emptyset)$ means $(\exists x)(x \text{ has no members} \wedge p(x))$. Similarly, we tend to write $\{z \in x : p(z)\}$ for the set given by separation.

Axiom (Axiom of pair set). “Can form $\{x, y\}$ ”.

$$(\forall x)(\forall y)(\exists z)(\forall t)(t \in z \Leftrightarrow (t = x \vee t = y)).$$

We write $\{x, y\}$ for this set. We write $\{x\}$ for $\{x, x\}$.

Axiom (Axiom of union). “We can form unions” Intuitively, we have $a \cup b \cup c = \{x : x \in a \text{ or } x \in b \text{ or } x \in c\}$. but instead of $a \cup b \cup c$, we write $\bigcup\{a, b, c\}$ so that we can express infinite unions as well.

$$(\forall x)(\exists y)(\forall z)(z \in y \Leftrightarrow (\exists t)(t \in x \wedge z \in t)).$$

We tend to write $\bigcup x$ for the set given above. We also write $x \cup y$ for $\bigcup\{x, y\}$.

Axiom (Axiom of power set). “Can form power sets”.

$$(\forall x)(\exists y)(\forall z)(z \in y \Leftrightarrow z \subseteq x),$$

where $z \subseteq x$ means $(\forall t)(t \in z \Rightarrow t \in x)$.

We tend to write $\mathbb{P}(x)$ for the set generated above.

Axiom (Axiom of infinity). “There is an infinite set”.

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \Rightarrow y^+ \in x)).$$

We say any set that satisfies the above axiom is a successor set.

Axiom (Axiom of foundation). “Every (non-empty) set has an \in -minimal member”

$$(\forall x)(x \neq \emptyset \Rightarrow (\exists y)(y \in x \wedge (\forall z)(z \in x \Rightarrow z \notin y))).$$

This is sometimes known as the Axiom of regularity.

Axiom (Axiom of replacement). “The image of a set under a function-class is a set”. This is an axiom scheme, with an instance for each first-order formula p :

$$\underbrace{(\forall t_1) \cdots (\forall t_n)}_{\text{parameters}} \left(\underbrace{[(\forall x)(\forall y)(\forall z)((p \wedge p[z/y]) \Rightarrow y = z)]}_{p \text{ defines a function-class}} \right) \\ \Rightarrow \underbrace{[(\forall x)(\exists y)(z \in y \Leftrightarrow (\exists t)(t \in x \wedge p[t/x, z/y]))]}_{\text{image of } x \text{ under } F \text{ is a set}}.$$

5.2 Properties of ZF

Lemma. Every x is contained in a transitive set.

Theorem (Principle of \in -induction). For each formula p , with free variables t_1, \dots, t_n, x ,

$$(\forall t_1) \cdots (\forall t_n) \left([(\forall x)((\forall y)(y \in x \Rightarrow p(y))) \Rightarrow p(x)] \Rightarrow (\forall x)(p(x)) \right)$$

Note that officially, $p(y)$ means $p[y/x]$ and $p(x)$ is simply x .

Proposition. \in -induction \Rightarrow Foundation.

Theorem (\in -recursion theorem). Let G be a function-class, everywhere defined. Then there is a function-class F such that $F(x) = G(F|_x)$ for all x . Moreover, F is unique (cf. definition of recursion on well-orderings).

Proposition. p -induction and p -recursion are well-defined and valid for any $p(x, y)$ that is well-founded and local.

Theorem (Mostowski collapse theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive b and a bijection $f : a \rightarrow b$ such that $(\forall x, y \in a)(x r y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

5.3 Picture of the universe

Lemma. Each V_α is transitive.

Lemma. If $\alpha \leq \beta$, then $V_\alpha \subseteq V_\beta$.

Theorem. Every x belongs to some V_α . Intuitively, we want to say

$$V = \bigcup_{\alpha \in \text{On}} V_\alpha,$$

Proposition. $\text{rank}(x)$ is the first α such that $x \subseteq V_\alpha$.

6 Cardinals

6.1 Definitions

Theorem. The \aleph_α are the cardinals of all infinite sets (or, in ZF, the cardinals of all infinite well-orderable sets). For example, $\text{card}(\omega) = \aleph_0$, $\text{card}\omega_1 = \aleph_1$.

6.2 Cardinal arithmetic

Proposition.

- (i) $m + n = n + m$ since $N \sqcup M \leftrightarrow N \sqcup N$ with the obvious bijection.
- (ii) $mn = nm$ using the obvious bijection
- (iii) $(m^n)^p = m^{np}$ as $(M^N)^P \leftrightarrow M^{N \times P}$ since both objects take in a P and an N and returns an M .

Theorem. For every ordinal α ,

$$\aleph_\alpha \aleph_\alpha = \aleph_\alpha.$$

This is the best we could ever ask for. What can be simpler?

Corollary. Let $\alpha \leq \beta$. Then

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \aleph_\beta = \aleph_\beta.$$

7 Incompleteness*

Theorem (Gödel's incompleteness theorem). PA is incomplete.

Theorem. "Truth is not definable"

$T = \{p : p \text{ holds in } \mathbb{N}\}$ is not definable. This officially means

$$\{m : m \text{ codes a member of } T\}$$

is not a definable set.

Theorem. $\text{PA} \not\vdash \text{con}(\text{PA})$.