Lent 2015 **LOGIC AND SET THEORY – EXAMPLES 1** IBL

1. Which of the following propositions are tautologies?
(i) $(p_1 \Rightarrow (p_2 \Rightarrow p_3)) \Rightarrow (p_2 \Rightarrow (p_1 \Rightarrow p_3))$
(ii) $((p_1 \vee p_2) \wedge (p_1 \vee p_3)) \Rightarrow (p_2 \vee p_3)$
(iii) $(p_1 \Rightarrow (\neg p_2)) \Rightarrow (p_2 \Rightarrow (\neg p_1))$

2. Write down a proof of $\perp \Rightarrow q$. Use this to write down a proof of $p \Rightarrow q$ from $\neg p$.

3. Use the Deduction Theorem to show that $p \vdash \neg\neg p$.

4. Show that $\{p, q\} \vdash p \wedge q$ in three different ways: by writing down a proof, by using the Deduction Theorem, and by using the Completeness Theorem.

5. Give propositions p and q for which $(p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$ is a tautology.

6. Explain carefully why the set of all propositions is countable.

7. Three people each have a set of beliefs: a consistent deductively closed set. Show that the set of propositions that they all believe is also consistent and deductively closed. Must the set of propositions that a majority believe be consistent? Must it be deductively closed?

8. Show that the third axiom cannot be deduced from the first two. In other words, show that (for some p) there is no proof of $(\neg\neg p) \Rightarrow p$ that uses only the first two axioms and modus ponens.

9. Let $t_1, t_2, \ldots$ be propositions such that, for every valuation v, there exists n with $v(t_n) = 1$. Use the Compactness Theorem to show that the values of n are bounded: there must be an N such that, for every valuation v, there exists $n \leq N$ with $v(t_n) = 1$.

10. Two sets S, T of propositions are *equivalent* if $S \vdash t$ for every $t \in T$ and $T \vdash s$ for every $s \in S$. A set S of propositions is *independent* if for every $s \in S$ we have $S - \{s\} \nvdash s$. Show that every finite set of propositions has an independent subset equivalent to it. Give an infinite set of propositions that has no independent subset equivalent to it. Show, however, that for every set of propositions there exists an independent set equivalent to it.

11. Let t be a tautology not containing the symbol $\perp$. Must there exist a proof of t that does not involve the third axiom?

12. Give an explicit function f from natural numbers to natural numbers such that every tautology of length n has a proof that is at most $f(n)$ lines long.

13. A set S of propositions is a *chain* if for any distinct $p, q \in S$ we have $p \vdash q$ or $q \vdash p$ but not both. Write down an infinite chain. If the set of primitive propositions is allowed to be uncountable, can there exist an uncountable chain?

$^+$14. Suppose that the set of primitive propositions is allowed to be uncountable. Is it true that for every set of propositions there exists an independent set equivalent to it?

Lent 2015 **LOGIC AND SET THEORY – EXAMPLES 2** IBL

1. Write down subsets of the reals that have order-types $\omega + \omega$, ω^2 and ω^3.

2. Let α and β be non-zero ordinals. Must we have $\alpha + \beta > \alpha$? Must we have $\alpha + \beta > \beta$?

3. Is there a non-zero ordinal α with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

4. Show that the inductive and the synthetic definitions of ordinal multiplication coincide.

5. Let α, β, γ be ordinals. Prove that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

6. Let α, β, γ be ordinals. Must we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$? Must we have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$?

7. Let α and β be ordinals with $\alpha \geq \beta$. Show that there is a unique ordinal γ such that $\beta + \gamma = \alpha$. Must there exist an ordinal γ with $\gamma + \beta = \alpha$?

8. An ordinal written as $\omega^{\alpha_1} n_1 + \ldots + \omega^{\alpha_k} n_k$, where $\alpha_1 > \ldots > \alpha_k$ are ordinals (and k and $n_1, \ldots, n_k$ are non-zero natural numbers), is said to be in *Cantor Normal Form*. Show that every non-zero ordinal has a unique Cantor Normal Form. What is the Cantor Normal Form for the ordinal ϵ_0?

9. Is ω_1 a successor or a limit?

10. Let α be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with supremum equal to α. Is this result true for $\alpha = \omega_1$?

11. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

12. An operation $\alpha * \beta$ is defined on ordinals as follows. We set $0 * \beta$ to be ω^β, and $\alpha^+ * \beta$ to be the β-th ordinal γ such that $\alpha * \gamma = \gamma$, and $\lambda * \beta$ to be the supremum of the set $\{\alpha * \beta : \alpha < \lambda\}$ for λ a non-zero limit. Explain why this definition makes sense. Describe the ordinals $1 * 0$, $1 * 1$, $1 * 2$ and $2 * 0$. Is there a countable ordinal α such that $\alpha * 0 = \alpha$?

13. Is it possible to select for each countable (non-zero) limit ordinal α an ordinal $x_\alpha < \alpha$ in such a way that the x_α are distinct?

$^+$14. Let X be a totally ordered set such that the only order-preserving injection from X to itself is the identity. Must X be finite?

Lent 2015 **LOGIC AND SET THEORY – EXAMPLES 3** IBL

1. How many different partial orders (up to isomorphism) are there on a set of 4 elements? How many of these are complete?

2. Which of the following posets (ordered by inclusion) are complete?
(i) The set of all subsets of $\mathbb{N}$ that are finite or have finite complement
(ii) The set of all independent subsets of a vector space V
(iii) The set of all subspaces of a vector space V

3. Let X be a complete poset, and let $f : X \to X$ be order-reversing (meaning that $x \leq y$ implies $f(x) \geq f(y)$). Give an example to show that f need not have a fixed point. Show, however, that there must exist either a fixed point of f or two distinct points x and y with $f(x) = y$ and $f(y) = x$.

4. Use Zorn's Lemma to show that every partial order on a set may be extended to a total order.

5. Give a direct proof of Zorn's Lemma (not using ordinals and not using the Axiom of Choice) for countable posets.

6. Show that the statement 'for any sets X and Y, either X injects into Y or Y injects into X' is equivalent to the Axiom of Choice (in the presence of the other rules for building sets). [Hint for one direction: Hartogs' Lemma.]

7. What is yellow and equivalent to the Axiom of Choice?

8. Formulate sets of axioms in suitable languages (to be specified) for the following theories.
(i) The theory of fields of characteristic 2
(ii) The theory of posets having no maximal element
(iii) The theory of bipartite graphs
(iv) The theory of algebraically closed fields
(v) The theory of groups of order 60
(vi) The theory of simple groups of order 60
(vii) The theory of real vector spaces

9. Write down axioms (in the language of partial orders) for the theory of total orders that are dense (between any two elements is a third) and have no greatest or least element. Show that every countable model of this theory is isomorphic to $\mathbb{Q}$.

10. Show that the theory of fields of positive characteristic is not axiomatisable (in the language of fields), and that the theory of fields of characteristic zero is axiomatisable but not finitely axiomatisable.

11. Is every countable model of Peano Arithmetic isomorphic to $\mathbb{N}$?

12. Write down axioms, in a suitable language, for the theory of groups that have an element of infinite order. Can this be done in the language of groups?

13. Let L be the language consisting of a single function symbol f, of arity 1. Write down a theory T that asserts that f is a bijection with no finite orbits, and describe the countable models of T. Prove that T is a complete theory.

14. Show that the following theories are not axiomatisable.
(i) The theory of connected graphs (in the language of graphs)
(ii) The theory of simple groups (in the language of groups)
$^+$(iii) The theory of non-abelian simple groups (in the language of groups)

1. Show that the Empty-Set Axiom is deducible from the Axioms of Infinity and Separation (or, if you prefer, just from the Axiom of Infinity), and that the Axiom of Separation is deducible from the Axiom of Replacement.

2. Show that the Pair-Set Axiom is deducible from the Axioms of Empty-Set, Power-Set and Replacement.

3. Write down sentences (in the language of ZF) to express the assertions that, for any two sets x and y, the product $x \times y$ and the set of all functions from x to y exist. Indicate how to deduce these sentences from the axioms of ZF.

4. Is it true that if x is a transitive set then the relation $\in$ on x is a transitive relation? Does the converse hold?

5. Let F be a function-class that is an automorphism of $(V, \in)$. Use $\in$-induction to show that F must be the identity.

6. What is the rank of $\{2, 3, 6\}$? What is the rank of $\{\{2, 3\}, \{6\}\}$? Work out the ranks of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, using your favourite constructions of these objects from ω.

7. A set x is called *hereditarily finite* if each member of $TC(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with V_ω. Which of the axioms of ZF are satisfied in the structure HF (i.e. the set HF, with the relation $\in |HF$)?

8. Which of the axioms of ZF are satisfied in the structure $V_{\omega+\omega}$?

9. What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

10. Is there an ordinal α such that $\omega_\alpha = \alpha$?

11. Explain why, for each $n \in \omega$, there is no surjection from $\aleph_n$ to $\aleph_{n+1}$. Use this fact to show that there is no surjection from $\aleph_\omega$ to ${\aleph_\omega}^{\aleph_0}$, and deduce that $2^{\aleph_0} \neq \aleph_\omega$.

12. If ZF is consistent then, by Downward Löwenheim-Skolem, it has a countable model. Doesn't this contradict the fact that, for example, the power-set of ω is uncountable?

13. Prove (in ZF) that a countable union of countable sets cannot have cardinality $\aleph_2$.

14. The function-classes $x + y = x \triangle y$ and $xy = x \cap y$ 'make V into a ring', in the sense that all of the axioms for a ring hold in this structure. Is it possible to make V into a ring with 1?

$^+$15. Show that the function $f(n) = 2^n$ is definable in the language of PA – in other words, find a formula $p(x, y)$ in the language of PA such that, in the natural numbers, $p(n, m)$ holds if and only if $m = 2^n$.