Part II — Logic and Set Theory
Definitions

Based on lectures by I. B. Leader
Notes taken by Dexter Chua

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

No specific prerequisites.

Ordinals and cardinals
Well-orderings and order-types. Examples of countable ordinals. Uncountable ordinals and Hartogs' lemma. Induction and recursion for ordinals. Ordinal arithmetic. Cardinals; the hierarchy of alephs. Cardinal arithmetic. [5]

Posets and Zorn's lemma
Partially ordered sets; Hasse diagrams, chains, maximal elements. Lattices and Boolean algebras. Complete and chain-complete posets; fixed-point theorems. The axiom of choice and Zorn's lemma. Applications of Zorn's lemma in mathematics. The well-ordering principle. [5]

Propositional logic
The propositional calculus. Semantic and syntactic entailment. The deduction and completeness theorems. Applications: compactness and decidability. [3]

Predicate logic
The predicate calculus with equality. Examples of first-order languages and theories. Statement of the completeness theorem; *sketch of proof*. The compactness theorem and the Lowenheim-Skolem theorems. Limitations of first-order logic. Model theory. [5]

Set theory
Set theory as a first-order theory; the axioms of ZF set theory. Transitive closures, epsilon-induction and epsilon-recursion. Well-founded relations. Mostowski's collapsing theorem. The rank function and the von Neumann hierarchy. [5]

Consistency
Problems of consistency and independence [1]

Contents

0 Introduction

1 Propositional calculus

1.1 Propositions

Definition (Propositions)**.** Let P be a set of *primitive propositions*. These are a bunch of (meaningless) symbols (e.g. p, q, r), which are used as the basic building blocks of our more interesting propositions. These are usually interpreted to take a truth value. Usually, any symbol (composed of alphabets and subscripts) is in the set of primitive propositions.

The set of *propositions*, written as L or $L(P)$, is defined inductively by

(i) If $p \in P$, then $p \in L$.

(ii) $\perp \in L$, where $\perp$ is read as "false" (also a meaningless symbol).

(iii) If $p, q \in L$, then $(p \Rightarrow q) \in L$.

Definition (Logical symbols)**.**

$\neg p$	("not p")	is an abbreviation for	$(p \Rightarrow \perp)$
$p \wedge q$	("p and q")	is an abbreviation for	$\neg(p \Rightarrow (\neg q))$
$p \vee q$	("p or q")	is an abbreviation for	$(\neg p) \Rightarrow q$

1.2 Semantic entailment

Definition (Valuation)**.** A *valuation* on L is a function $v : L \to \{0, 1\}$ such that:

– $v(\perp) = 0$,

– $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1, v(q) = 0, \\ 1 & \text{otherwise} \end{cases}$

We interpret $v(p)$ to be the truth value of p, with 0 denoting "false" and 1 denoting "true".

Note that we do not impose any restriction of $v(p)$ when p is a primitive proposition.

Definition (Tautology)**.** t is a *tautology*, written as $\models t$, if $v(t) = 1$ for all valuations v.

Definition (Semantic entailment)**.** For $S \subseteq L$, $t \in L$, we say S *entails* t, S *semantically implies* t or $S \models t$ if, for any v such that $v(s) = 1$ for all $s \in S$, we have $v(t) = 1$.

Definition (Truth and model)**.** If $v(t) = 1$, then we say that t is *true* in v, or v is a *model* of t. For $S \subseteq L$, a valuation v is a *model* of S if $v(s) = 1$ for all $s \in S$.

1.3 Syntactic implication

Definition (Proof and syntactic entailment)**.** For any $S \subseteq L$, a *proof* of t from S is a finite sequence $t_1, t_2, \cdots t_n$ of propositions, with $t_n = t$, such that each t_i is one of the following:

(i) An axiom

(ii) A member of S

(iii) A proposition t_i such that there exist $j, k < i$ with t_j being $t_k \Rightarrow t_i$.

If there is a proof of t from S, we say that S *proves* or *syntactically entails* t, written $S \vdash t$.

If $\emptyset \vdash t$, we say t is a *theorem* and write $\vdash t$.

In a proof of t from S, t is the *conclusion* and S is the set of *hypothesis* or *premises*.

Definition (Consistent)**.** S is *inconsistent* if $S \vdash \perp$. S is *consistent* if it is not inconsistent.

2 Well-orderings and ordinals

2.1 Well-orderings

Definition ((Strict) total order)**.** A *(strict) total order* or *linear order* is a pair $(X, <)$, where X is a set and $<$ is a relation on X that satisfies

(i) $x \not< x$ for all x (irreflexivity)

(ii) If $x < y$, $y < z$, then $x < z$ (transitivity)

(iii) $x < y$ or $x = y$ or $y < x$ (trichotomy)

Definition ((Non-strict) total order)**.** A *(non-strict) total order* is a pair $(X, \leq)$, where X is a set and $\leq$ is a relation on X that satisfies

(i) $x \leq x$ (reflexivity)

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity)

(iii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry)

(iv) $x \leq y$ or $y \leq x$ (trichotomy)

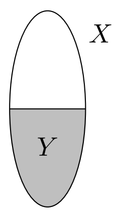
Definition (Well order)**.** A total order $(X, <)$ is a *well-ordering* if every (non-empty) subset has a least element, i.e.

$$(\forall S \subseteq X)[S \neq \emptyset \Rightarrow (\exists x \in S)(\forall y \in S)\, y \geq x].$$

Definition (Order isomorphism)**.** Say the total orders X, Y are *isomorphic* if there exists a bijection $f : X \to Y$ that is order-preserving, i.e. $x < y \Rightarrow f(x) < f(y)$.

Definition (Initial segment)**.** A subset Y of a totally ordered X is an *initial segment* if

$$x \in Y, y < x \Rightarrow y \in Y,$$



Definition (Restriction of function)**.** For $f : A \to B$ and $C \subseteq A$, the *restriction* of f to C is

$$f|_C = \{(x, f(x)) : x \in C\}.$$

Notation. Write $X \leq Y$ if X is isomorphic to an initial segment of Y.

We write $X < Y$ if $X \leq Y$ but X is not isomorphic to Y, i.e. X is isomorphic to a proper initial segment of Y.

2.2 New well-orderings from old

Definition (Successor)**.** Given X, choose some $x \notin X$ and define a well-ordering on $X \cup \{x\}$ by setting $y < x$ for all $y \in X$. This is the *successor* of X, written X^+.

Definition (Extension)**.** For well-orderings $(X, <_X)$ and $(Y, <_Y)$, we say Y *extends* X if X is a proper initial segment of Y and $<_X$ and $<_Y$ agree when defined.

Definition (Nested family)**.** We say well-orderings $\{X_i : i \in I\}$ form a *nested* family if for any $i, j \in I$, either X_i extends X_j, or X_j extends X_i.

2.3 Ordinals

Definition (Ordinal)**.** An *ordinal* is a well-ordered set, with two regarded as the same if they are isomorphic. We write ordinals as Greek letters α, β etc.

Definition (Order type)**.** If a well-ordering X has corresponding ordinal α, we say X has *order type* α, and write $\operatorname{otp}(X) = \alpha$.

Notation. For each $k \in \mathbb{N}$, we write k for the order type of the (unique) well-ordering of size k. We write ω for the order type of $\mathbb{N}$.

Notation. For ordinals α, β, write $\alpha \leq \beta$ if $X \leq Y$ for some X of order type α, Y of order type β. This does not depend on the choice of X and Y (since any two choices must be isomorphic).

Notation. Write $I_\alpha = \{\beta : \beta < \alpha\}$.

Notation. Write $\gamma(X)$ for the least ordinal that does not inject into X. e.g. $\gamma(\omega) = \omega_1$.

2.4 Successors and limits

Definition (Successor ordinal)**.** An ordinal α is a *successor ordinal* if there is a greatest element β below it. Then $\alpha = \beta^+$.

Definition (Limit ordinal)**.** An ordinal α is a limit if it has no greatest element below it. We usually write λ for limit ordinals.

2.5 Ordinal arithmetic

Definition (Ordinal addition (inductive))**.** Define $\alpha + \beta$ by recursion on β (α is fixed):

- $\alpha + 0 = \alpha$.
- $\alpha + \beta^+ = (\alpha + \beta)^+$.
- $\alpha + \lambda = \sup\{\alpha + \gamma : \gamma < \lambda\}$ for non-zero limit λ.

Definition (Ordinal addition (synthetic))**.** $\alpha + \beta$ is the order type of $\alpha \sqcup \beta$ (α disjoint union β, e.g. $\alpha \times \{0\} \cup \beta \times \{1\}$), with all α before all of β

$$\alpha + \beta = \underbrace{\qquad\alpha\qquad}\underbrace{\quad\beta\quad}$$

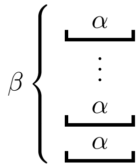
Definition (Ordinal multiplication (inductive))**.** We define $\alpha \cdot \beta$ by induction on β by:

(i) $\alpha \cdot 0 = 0$.

(ii) $\alpha \cdot (\beta^+) = \alpha \cdot \beta + \alpha$.

(iii) $\alpha \cdot \lambda = \sup\{\alpha \cdot \gamma : \gamma < \lambda\}$ for λ a non-zero limit.

Definition (Ordinal multiplication (synthetic))**.**



Formally, $\alpha \cdot \beta$ is the order type of $\alpha \times \beta$, with $(x, y) < (x', y')$ if $y < y'$ or ($y = y'$ and $x < x'$).

Definition (Ordinal exponentiation (inductive))**.** α^β is defined as

(i) $\alpha^0 = 1$

(ii) $\alpha^{\beta^+} = \alpha^\beta \cdot \alpha$

(iii) $\alpha^\lambda = \sup\{\alpha^\gamma : \gamma < \lambda\}$.

2.6 Normal functions*

Definition (Normal function)**.** A function $f : \mathrm{On} \to \mathrm{On}$ is *normal* if

(i) For any ordinal α, we have $f(\alpha) < f(\alpha^+)$.

(ii) If λ is a non-zero limit ordinal, then $f(\lambda) = \sup\{f(\gamma) : \gamma < \lambda\}$.

3 Posets and Zorn's lemma

3.1 Partial orders

Definition (Partial ordering (poset)). A *partially ordered set* or *poset* is a pair $(X, \leq)$, where X is a set and $\leq$ is a relation on X that satisfies

(i) $x \leq x$ for all $x \in X$ (reflexivity)

(ii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)

(iii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)

We write $x < y$ to mean $x \leq y$ and $x \neq y$. We can also define posets in terms of $<$:

(i) $x \not< x$ for all $x \in X$ (irreflexive)

(ii) $x < y$ and $y < z \Rightarrow x < z$ (transitive)

Definition (Hasse diagram). A *Hasse diagram* for a poset X consists of a drawing of the points of X in the plane with an upwards line from x to y if y *covers* x:

Definition (Cover). In a poset, y *covers* x if $y > x$ and no z has $y > z > x$.

Definition (Chain and antichain). In a poset, a subset S is a *chain* if it is totally ordered, i.e. for all x, y, $x \leq y$ or $y \leq x$. An *antichain* is a subset in which no two things are related.

Definition (Upper bound and supremum). For $S \subset X$, an *upper bound* for S is an $x \in X$ such that $\forall y \in S : x \geq y$.

$x \in X$ is a *least upper bound*, *supremum* or *join* of S, written $x = \sup S$ or $x = \bigvee S$, if x is an upper bound for S, and for all $y \in X$, if y is an upper bound, then $y \geq x$.

Definition (Complete poset). A poset X is *complete* if every $S \subseteq X$ has a supremum. In particular, it has a greatest element (i.e. x such that $\forall y : x \geq y$), namely $\sup X$, and least element (i.e. x such that $\forall y : x \leq y$), namely $\sup \emptyset$.

Definition (Fixed point). A *fixed point* of a function $f : X \to X$ is an x such that $f(x) = x$.

Definition (Order-preserving function). For a poset X, $f : X \to X$ is *order-preserving* of $x \leq y \Rightarrow f(x) \leq f(y)$.

Definition (Maximal element). In a poset X, $x \in X$ is maximal if no $y \in X$ has $y > x$.

3.2 Zorn's lemma and axiom of choice

3.3 Bourbaki-Witt theorem*

Definition (Chain-complete poset). We say a poset X is *chain-complete* if $X \neq \emptyset$ and every non-empty chain has a supremum.

Definition (Inflationary function). A function $f : X \to X$ is *inflationary* if $f(x) \geq x$ for all x.

4 Predicate logic

4.1 Language of predicate logic

Definition (Language)**.** Let Ω (function symbols) and Π (relation symbols) be disjoint sets, and $\alpha : \Omega \cup \Pi \to \mathbb{N}$ a function ('arity').

The *language* $L = L(\Omega, \Pi, \alpha)$ is the set of formulae, defined as follows:

- *Variables*: we have some variables $x_1, x_2, \cdots$. Sometimes (i.e. always), we write $x, y, z, \cdots$ instead.
- *Terms*: these are defined inductively by
  - (i) Every variable is a term
  - (ii) If $f \in \Omega$, $\alpha(f) = n$, and $t_1, \cdots, t_n$ are terms, then $ft_1 \cdots t_n$ is a term. We often write $f(t_1, \cdots, t_n)$ instead.

  Example. In the language of groups $\Omega = \{m, i, e\}$, $\Pi = \emptyset$, and $\alpha(m) = 2, \alpha(i) = 1, \alpha(e) = 0$. Then $e, x_1, m(x_1, x_2), i(m(x_1, x_1))$ are terms.

- *Atomic formulae*: there are three sorts:
  - (i) $\perp$
  - (ii) $(s = t)$ for any terms s, t.
  - (iii) $(\phi t_1 \cdots t_n)$ for any $\phi \in \Pi$ with $\alpha(\phi) = n$ and $t_1, \cdots, t_n$ terms.

  Example. In the language of posets, $\Omega = \emptyset$, $\Pi = \{\leq\}$ and $\alpha(\leq) = 2$. Then $(x_1 = x_1)$, $x_1 \leq x_2$ (really means $(\leq x_1 x_2)$) are atomic formulae.

- *Formulae*: defined inductively by
  - (i) Atomic formulae are formulae
  - (ii) $(p \Rightarrow q)$ is a formula for any formulae p, q.
  - (iii) $(\forall x)p$ is a formula for any formula p and variable x.

  Example. In the language of groups, $e = e$, $x_1 = e$, $m(e, e) = e$, $(\forall x) m(x, i(x)) = e$, $(\forall x)(m(x, x) = e \Rightarrow (\exists y)(m(y, y) = x))$ are formulae.

Definition (Closed term)**.** A term is *closed* if it has no variables.

Definition (Free and bound variables)**.** An occurrence of a variable x in a formula p is *bound* if it is inside brackets of a $(\forall x)$ quantifier. It is *free* otherwise.

Definition (Sentence)**.** A *sentence* is a formula with no free variables.

Definition (Substitution)**.** For a formula p, a variable x and a term t, the *substitution* $p[t/x]$ is obtained by replacing each free occurrence of x with t.

4.2 Semantic entailment

Definition (Structure). An *L-structure* is a non-empty set A with a function $f_A : A^n \to A$ for each $f \in \Omega, \alpha(f) = n$, and a relation $\phi_A \subseteq A^n$, for each $\phi \in \Pi$, $\alpha(\phi) = n$.

Definition (Interpretation). To define the *interpretation* $p_A \in 0, 1$ for each sentence p and L-structure A, we define inductively:

(i) Closed terms: define $t_A \in A$ for each closed term t by

$$(ft_1, \cdots, t_n)_A = f_A(t_{1_A}, t_{2_A} \cdots, t_{n_A})$$

for any $f \in \Omega$, $\alpha(f) = n$, and closed terms $t_1, \cdots, t_n$.

Example. $(m(m(e, e), e))_A = m_A(m_A(e_A, e_A), e_A)$.

(ii) Atomic formulae:

$$\begin{aligned} \perp_A &= 0 \\ (s = t)_A &= \begin{cases} 1 & s_A = t_A \\ 0 & s_A \neq t_A \end{cases} \\ (\phi t_1 \cdots t_n)_A &= \begin{cases} 1 & (t_{1_A}, \cdots, t_{n_A}) \in \phi_A \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

(iii) Sentences:

$$\begin{aligned} (p \Rightarrow q)_A &= \begin{cases} 0 & p_A = 1, q_A = 0 \\ 1 & \text{otherwise} \end{cases} \\ ((\forall x)p)_A &= \begin{cases} 1 & p[\bar{a}/x]_{\bar{A}} \text{ for all } a \in A \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where for any $a \in A$, we define a new language L' by adding a constant $\bar{a}$ and make A into an L' structure $\bar{A}$ by setting $\bar{a}_{\bar{A}} = a$.

Definition (Theory). A *theory* is a set of sentences.

Definition (Model). If a sentence p has $p_A = 1$, we say that p *holds* in A, or p is *true* in A, or A is a model of p.

For a theory S, a *model* of S is a structure that is a model for each $s \in S$.

Definition (Semantic entailment). For a theory S and a sentence t, S *entails* t, written as $S \models t$, if every model of S is a model of t.

"Whenever S is true, t is also true".

Definition (Tautology). t is a *tautology*, written $\models t$, if $\emptyset \models t$, i.e. it is true everywhere.

4.3 Syntactic implication

Definition (Axioms of predicate logic)**.** The *axioms of predicate logic* consists of the 3 usual axioms, 2 to explain how = works, and 2 to explain how $\forall$ works. They are

1. $p \Rightarrow (q \Rightarrow p)$ for any formulae p, q.
2. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (q \Rightarrow r)]$ for any formulae p, q, r.
3. $(\neg\neg p \Rightarrow p)$ for any formula p.
4. $(\forall x)(x = x)$ for any variable x.
5. $(\forall x)(\forall y)\big((x = y) \Rightarrow (p \Rightarrow p[y/x])\big)$ for any variable x, y and formula p, with y not occurring bound in p.
6. $[(\forall x)p] \Rightarrow p[t/x]$ for any formula p, variable x, term t with no free variable of t occurring bound in p.
7. $[(\forall x)(p \Rightarrow q)] \Rightarrow [p \Rightarrow (\forall x)q]$ for any formulae p, q with variable x not occurring free in p.

The *deduction rules* are

1. Modus ponens: From p and $p \Rightarrow q$, we can deduce q.
2. Generalization: From r, we can deduce $(\forall x)r$, provided that no premise used in the proof so far had x as a free variable.

Definition (Proof)**.** A *proof* of p from S is a sequence of statements, in which each statement is either an axiom, a statement in S, or obtained via modus ponens or generalization.

Definition (Syntactic implication)**.** If there exists a proof a formula p form a set of formulae S, we write $S \vdash p$ "S proves t".

Definition (Theorem)**.** If $S \vdash p$, we say p is a *theorem* of S. (e.g. a theorem of group theory)

4.4 Peano Arithmetic

Definition (Peano's axioms)**.** The axioms of *Peano's arithmetic* (PA) are

(i) $(\forall x)\neg(s(x) = 0)$.

(ii) $(\forall x)(\forall y)((s(x) = s(y)) \Rightarrow (x = y))$.

(iii) $(\forall y_1) \cdots (\forall y_n)[p[0/x] \wedge (\forall x)(p \Rightarrow p[s(x)/x])] \Rightarrow (\forall x)p$. This is actually infinitely many axioms — one for each formula p, free variables $y_1, \cdots, y_n, x$, i.e. it is an axiom scheme.

(iv) $(\forall x)(x + 0 = x)$.

(v) $(\forall x)(\forall y)(x + s(y) = s(x + y))$.

(vi) $(\forall x)(x \times 0 = 0)$.

(vii) $(\forall x)(\forall y)(x \times s(y) = (x + y) + x)$.

Note that our third axiom looks rather funny with all the $(\forall y_i)$ in front. Our first guess at writing it would be

$$[p[0/x] \wedge (\forall x)(p \Rightarrow p[s(x)/x])] \Rightarrow (\forall x)p.$$

However, this is in fact not sufficient. Suppose we want to prove that for all x and y, $x + y = y + x$. The natural thing to do would be to fix a y and induct on x (or the other way round). We want to be able to fix *any* y to do so. So we need a $(\forall y)$ in front of our induction axiom, so that we can prove it for all values of y all at once, instead of proving it once for $y = 0$, once for $y = 1$, once for $y = 1 + 1$ etc. This is important, since we might have an uncountable model of PA, and we cannot name all y. When we actually think about it, we can just forget about the $(\forall y_i)$s. But just remember that formally we need them.

4.5 Completeness and categoricity*

Definition (Complete theory)**.** A theory T is *complete* if for all propositions p in the language, either $T \vdash p$ or $T \vdash \neg p$.

Definition (κ-categorical)**.** Let κ be an infinite cardinal. Then a theory T is *κ-categorical* if there is a unique model of the theory of cardinality κ up to isomorphism.

5 Set theory

5.1 Axioms of set theory

Definition (Zermelo-Fraenkel set theory)**.** *Zermelo-Fraenkel set theory* (ZF) has language $\Omega = \emptyset$, $\Pi = \{\in\}$, with arity 2.

Definition (Ordered pair)**.** An *ordered pair* (x, y) is $\{\{x\}, \{x, y\}\}$.
We define "x is an ordered pair" to mean $(\exists y)(\exists z)(x = (y, z))$.

Definition (Function)**.** We define "f is a *function*" to mean

$$(\forall x)(x \in f \Rightarrow x \text{ is an ordered pair})\wedge$$
$$(\forall x)(\forall y)(\forall z)[(x, y) \in f \wedge (x, z) \in f] \Rightarrow y = z.$$

We define $x = \operatorname{dom} f$ to mean f is a function and $(\forall y)(y \in x \Leftrightarrow (\exists z)((y, z) \in f))$.
We define $f : x \to y$ to mean f is a function and $\operatorname{dom} f = x$ and

$$(\forall z)[(\exists t)((t, z) \in f) \Rightarrow z \in y].$$

Definition (Class)**.** Let $(V, \in)$ be an L-structure. A *class* is a collection C of points of V such that, for some formula p with free variable x (and maybe more funny parameters), we have

$$x \in C \Leftrightarrow p \text{ holds.}$$

Intuitively, everything of the form $\{x \in V : p(x)\}$ is a class.

Definition (Proper class)**.** We say C is a *proper class* if C is not a set (in V), ie

$$\neg(\exists y)(\forall x)(x \in y \Leftrightarrow p).$$

Definition (Function-class)**.** A *function-class* F is a collection of ordered pairs such that there is a formula p with free variables x, y (and maybe more) such that

$$(x, y) \in F \Leftrightarrow p \text{ holds, and } (x, y) \in F \wedge (x, z) \in F \Rightarrow y = z.$$

Definition (ZFC)**.** ZFC is the axioms ZF + AC, where AC is the axiom of choice, "every family of non-empty sets has a choice function".

$$\begin{aligned}&(\forall f)[(\forall x)(x \in \operatorname{dom} f \Rightarrow f(x) \neq \emptyset) \Rightarrow\\ &\qquad(\exists g)(\operatorname{dom} g = \operatorname{dom} f) \wedge (\forall x)(x \in \operatorname{dom} g \Rightarrow g(x) \in f(x))]\end{aligned}$$

Here we define a family of sets $\{A_i : i \in I\}$ to be a function $f : I \to V$ such that $i \mapsto A_i$.

5.2 Properties of ZF

Definition (Transitive set)**.** A set x is *transitive* if every member of a member of x is a member of x:

$$(\forall y)((\exists z)(y \in z \wedge z \in x) \Rightarrow y \in x).$$

This can be more concisely written as $\bigcup x \subseteq x$, but is very confusing and impossible to understand!

Definition (Well-founded relation)**.** A relation-class R is *well-founded* if every set has a R-minimal element.

Definition (Local relation)**.** A relation-class R is *local* if $\{x : p(x, y)\}$ is a set for each y.

Definition (Extensional relation)**.** We say a relation r on set a is *extensional* if

$$(\forall x \in a)(\forall y \in a)((\forall z \in a)(z\, r\, x \Leftrightarrow z\, r\, y) \Rightarrow x = y).$$

i.e. it obeys the axiom of extension.

Definition (Ordinal)**.** An *ordinal* is a transitive set, totally ordered by $\in$.

5.3 Picture of the universe

Definition (von Neumann hierarchy)**.** Define sets V_α for $\alpha \in \mathrm{On}$ (where On is the class of ordinals) by $\in$-recursion:

(i) $V_0 = \emptyset$.

(ii) $V_{\alpha+1} = \mathbb{P}(V_\alpha)$.

(iii) $V_\lambda = \bigcup\{V_\gamma : \gamma < \lambda\}$ for λ a non-zero limit ordinal.

Definition (Rank)**.** The *rank* of a set x is defined recursively by

$$\mathrm{rank}(x) = \sup\{(\mathrm{rank}\, y)^+ : y \in x\}.$$

6 Cardinals

Notation. Write $x \leftrightarrow y$ for $\exists f : f$ is a bijection from x to y.

6.1 Definitions

Definition (Cardinality)**.** The *cardinality* of a set x, written $\mathrm{card}(x)$, is the least ordinal α such that $x \leftrightarrow \alpha$.

Definition (Initial ordinal)**.** We say an ordinal α is *initial* if

$$(\forall \beta < \alpha)(\neg \beta \leftrightarrow \alpha),$$

i.e. it is the smallest ordinal of that cardinality.

Definition (Omega ordinals)**.** We define ω_α for each $\alpha \in \mathrm{On}$ by

(i) $\omega_0 = \omega$;

(ii) $\omega_{\alpha+1} = \gamma(\omega_\alpha)$;

(iii) $\omega_\lambda = \sup\{\omega_\alpha : \alpha < \lambda\}$ for non-zero limit λ.

Definition (Aleph number)**.** Write $\aleph_\alpha$ ("aleph-α") for $\mathrm{card}(\omega_\alpha)$.

Definition (Cardinal (in)equality)**.** For cardinals n and m, write $m \leq n$ if M injects into N, where $\mathrm{card}\, M = m, \mathrm{card}\, N = n$. This clearly does not depend on M and N.

So $m \leq n$ and $n \leq m$ implies $n = m$ by Schröder-Bernstein. Write $m < n$ if $m \leq n$ by $m \neq n$.

6.2 Cardinal arithmetic

Definition (Cardinal addition, multiplication and exponentiation)**.** For cardinals m, n, write $m + n$ for $\mathrm{card}(M \sqcup N)$; mn for $\mathrm{card}(M \times N)$; and m^n for $\mathrm{card}(M^N)$, where $M^N = \{f : f \text{ is a function } N \to M\}$. Note that this coincides with our usual definition of X^n for finite n.

7 Incompleteness*

Definition (Definability)**.** A subset $S \subseteq \mathbb{N}$ is *definable* if there is a formula p with one free variable such that

$$\forall m \in \mathbb{N} : \; m \in S \Leftrightarrow p(m) \text{ holds.}$$

Similarly, $f : \mathbb{N} \to \mathbb{N}$ is *definable* if there exists a formula $p(x, y)$ such that

$$\forall m, n \in \mathbb{N} : \; f(m) = n \Leftrightarrow p(m, n) \text{ holds.}$$