Part III — Symmetries, Fields and Particles
Theorems with proof

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This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of SU(2), the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of SU(2) and introduce the tensor product of representations. The next part of the course develops the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of SU(3). The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate SU(3) global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

Pre-requisites

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

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1 Introduction

2 Lie groups

2.1 Definitions

2.2 Matrix Lie groups

Lemma. The *general linear group*:

$$\mathrm{GL}(n, \mathbb{R}) = \{M \in \mathrm{Mat}_n(\mathbb{R}) : \det M \neq 0\}$$

and *orthogonal group*:

$$\mathrm{O}(n) = \{M \in \mathrm{GL}(n, \mathbb{R}) : M^T M = I\}$$

are Lie groups.

Lemma. The *special orthogonal group* $\mathrm{SO}(n)$:

$$\mathrm{SO}(n) = \{M \in \mathrm{O}(n) : \det M = 1\}$$

is a Lie group.

Theorem. Let M be a real matrix. Then λ is an eigenvalue iff λ^* is an eigenvalue. Moreover, if M is orthogonal, then $|\lambda|^2 = 1$.

Proof. Suppose $M\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda$. Then applying the complex conjugate gives $M\mathbf{v}_\lambda^* = \lambda^* \mathbf{v}_\lambda^*$.

Now suppose M is orthogonal. Then $M\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda$ for some non-zero $\mathbf{v}_\lambda$. We take the norm to obtain $|M\mathbf{v}_\lambda| = |\lambda||\mathbf{v}_\lambda|$. Using the fact that $|\mathbf{v}_\lambda| = |M\mathbf{v}_\lambda|$, we have $|\lambda| = 1$. So done. □

2.3 Properties of Lie groups

3 Lie algebras

3.1 Lie algebras

Proposition.
$$f^{ba}{}_c = -f^{ab}{}_c.$$

Proposition.
$$f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0.$$

3.2 Differentiation

Proposition. Let M be a manifold with local coordinates $\{x^i\}_{i=1,\cdots,D}$ for some region $U \subseteq M$ containing p. Then T_pM has basis

$$\left\{\frac{\partial}{\partial x^j}\right\}_{j=1,\cdots,D}.$$

In particular, $\dim T_pM = \dim M$.

3.3 Lie algebras from Lie groups

Theorem. The tangent space of a Lie group G at the identity naturally admits a Lie bracket
$$[\,\cdot\,,\,\cdot\,] : T_eG \times T_eG \to T_eG$$
such that
$$\mathcal{L}(G) = (T_eG, [\,\cdot\,,\,\cdot\,])$$
is a Lie algebra.

Proof. We will only prove it for the case of a matrix Lie group $G \subseteq \mathrm{Mat}_n(\mathbb{F})$. Then T_IG can be naturally identified as a subspace of $\mathrm{Mat}_n(\mathbb{F})$. There is then an obvious candidate for the Lie bracket — the actual commutator:

$$[X, Y] = XY - YX.$$

The basic axioms of a Lie algebra can be easily (but painfully) checked.

However, we are not yet done. We have to check that if we take the bracket of two elements in $T_I(G)$, then it still stays within $T_I(G)$. This will be done by producing a curve in G whose derivative at 0 is the commutator $[X, Y]$.

In general, let γ be a smooth curve in G with $\gamma(0) = I$. Then we can Taylor expand
$$\gamma(t) = I + \dot\gamma(0)t + \ddot\gamma(0)t^2 + O(t^3),$$
Now given $X, Y \in T_eG$, we take curves γ_1, γ_2 such that $\dot\gamma_1(0) = X$ and $\dot\gamma_2(0) = Y$. Consider the curve given by

$$\gamma(t) = \gamma_1^{-1}(t)\gamma_2^{-1}(t)\gamma_1(t)\gamma_2(t) \in G.$$

We can Taylor expand this to find that

$$\gamma(t) = I + [X, Y]t^2 + O(t^3).$$

This isn't too helpful, as $[X, Y]$ is not the coefficient of t. We now do the slightly dodgy step, where we consider the curve

$$\tilde{\gamma}(t) = \gamma(\sqrt{t}) = I + [X, Y]t + O(t^{3/2}).$$

Now this is only defined for $t \geq 0$, but it is good enough, and we see that its derivative at $t = 0$ is $[X, Y]$. So the commutator is in $T_I(G)$. So we know that $\mathcal{L}(G)$ is a Lie algebra. □

3.4 The exponential map

Proposition. Let G be a Lie group of dimension > 0. Then G has a nowhere-vanishing vector field.

Theorem (Poincare-Hopf theorem)**.** Let M be a compact manifold. If M has non-zero Euler characteristic, then any vector field on M has a zero.

Theorem (Hairy ball theorem)**.** Any smooth vector field on S^2 has a zero. More generally, any smooth vector field on S^{2n} has a zero.

Theorem. For any matrix Lie group G, the map exp restricts to a map $\mathcal{L}(G) \to G$.

Proof. We will not prove this, but on the first example sheet, we will prove this manually for $G = \mathrm{SU}(n)$. □

Theorem (Baker–Campbell–Hausdorff formula)**.** We have

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \cdots\right).$$

Proposition. Let G be a Lie group, and $\mathfrak{g}$ be its Lie algebra. Then the image of $\mathfrak{g}$ under exp is the connected component of e.

4 Representations of Lie algebras

4.1 Representations of Lie groups and algebras

Proposition. Let D be a representation of a group G. Then $D(e) = I$ and $D(g^{-1}) = D(g)^{-1}$.

Proof. We have
$$D(e) = D(ee) = D(e)D(e).$$
Since $D(e)$ is invertible, multiplying by the inverse gives
$$D(e) = I.$$
Similarly, we have
$$D(g)D(g^{-1}) = D(gg^{-1}) = D(e) = I.$$
So it follows that $D(g)^{-1} = D(g^{-1})$. □

Proposition. The adjoint representation is a representation.

Proof. Since the bracket is linear in both components, we know the adjoint representation is a linear map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. It remains to show that
$$[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}.$$
But the Jacobi identity says
$$[\mathrm{ad}_X, \mathrm{ad}_Y](Z) = [X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z] = \mathrm{ad}_{[X,Y]}(Z).$$ □

4.2 Complexification and correspondence of representations

Lemma. Given a representation $D : G \to \mathrm{GL}(V)$, the induced representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation.

Proof. We will again only prove this in the case of a matrix Lie group, so that we can use the construction we had for the Lie bracket.

We have to check that the bracket is preserved. We take curves $\gamma_1, \gamma_2 : \mathbb{R} \to G$ passing through I at 0 such that $\dot{\gamma}_i(0) = X_i$ for $i = 1, 2$. We write
$$\gamma(t) = \gamma_1^{-1}(t)\gamma_2^{-1}(t)\gamma_1(t)\gamma_2(t) \in G.$$
We can again Taylor expand this to obtain
$$\gamma(t) = I + t^2[X_1, X_2] + O(t^3).$$
Essentially by the definition of the derivative, applying D to this gives
$$D(\gamma(t)) = I + t^2\rho([X_1, X_2]) + O(t^3).$$
On the other hand, we can apply D to $(*)$ before Taylor expanding. We get
$$D(\gamma) = D(\gamma_1^{-1})D(\gamma_2^{-2})D(\gamma_1)D(\gamma_2).$$

So as before, since

$$D(\gamma_i) = I + t\rho(X_i) + O(t^2),$$

it follows that

$$D(\gamma)(t) = I + t^2[\rho(X_1), \rho(X_2)] + O(t^3).$$

So we must have

$$\rho([X_1, X_2]) = [\rho(X_1), \rho(X_2)]. \qquad \square$$

Theorem. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then there is a unique representation $D : G \to \mathrm{GL}(V)$ of G that induces ρ.

Theorem. Let $\mathfrak{g}$ be a real Lie algebra. Then the complex representations of $\mathfrak{g}$ are exactly the (complex) representations of $\mathfrak{g}_{\mathbb{C}}$.

Explicitly, if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation, then we can extend it to $\mathfrak{g}_{\mathbb{C}}$ by declaring

$$\rho(X + iY) = \rho(X) + i\rho(Y).$$

Conversely, if $\rho_{\mathbb{C}} : \mathfrak{g}_C \to \mathfrak{gl}(V)$, restricting it to $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$ gives a representation of $\mathfrak{g}$.

Proof. Just stare at it and see that the formula works. $\square$

4.3 Representations of $\mathfrak{su}(2)$

Proposition. The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are labelled by $\Lambda \in \mathbb{Z}_{\geq 0}$, which we call ρ_Λ, with weights given by

$$\{-\Lambda, -\Lambda + 2, \cdots, \Lambda - 2, \Lambda\}.$$

The weights are all non-degenerate, i.e. each only has one eigenvector. We have $\dim(\rho_\Lambda) = \Lambda + 1$.

Proof. We've done most of the work. Given any irrep, we can pick any eigenvector of $\rho(H)$ and keep applying E_+ to get a highest weight vector v_Λ, then the above computations show that

$$\{v_\Lambda, v_{\Lambda-2}, \cdots, v_{-\Lambda}\}$$

is a subspace of the irrep closed under the action of ρ. By irreducibility, this must be the whole of the representation space. $\square$

4.4 New representations from old

Theorem. If ρ_i for $i = 1, \cdots, m$ are finite-dimensional irreps of a simple Lie algebra $\mathfrak{g}$, then $\rho_1 \otimes \cdots \otimes \rho_m$ is completely reducible to irreps, i.e. we can find $\tilde{\rho}_1, \cdots, \tilde{\rho}_k$ such that

$$\rho_1 \otimes \cdots \otimes \rho_m = \tilde{\rho}_1 \oplus \tilde{\rho}_2 \oplus \cdots \oplus \tilde{\rho}_k.$$

4.5 Decomposition of tensor product of $\mathfrak{su}(2)$ representations

Proposition.

$$\rho_M \otimes \rho_N = \rho_{|N-M|} \oplus \rho_{|N-M|+2} \oplus \cdots \oplus \rho_{N+M}.$$

5 Cartan classification

5.1 The Killing form

Proposition. The Killing form is invariant.

Proof. We have

$$\begin{aligned}\kappa([Z,X],Y) &= \mathrm{tr}(\mathrm{ad}_{[Z,X]} \circ \mathrm{ad}_Y)\\ &= \mathrm{tr}([\mathrm{ad}_Z, \mathrm{ad}_X] \circ \mathrm{ad}_Y)\\ &= \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y)\\ &= \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X \circ \mathrm{ad}_Y) - \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y)\end{aligned}$$

Similarly, we have

$$\kappa(X,[Z,Y]) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Z \circ \mathrm{ad}_Y) - \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z).$$

Adding them together, we obtain

$$\kappa([Z,X],Y) + \kappa(X,[Z,Y]) = \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X \circ \mathrm{ad}_Y) - \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{ad}_Z).$$

By the cyclicity of tr, this vanishes. □

Theorem (Cartan)**.** The Killing form of a Lie algebra $\mathfrak{g}$ is non-degenerate iff $\mathfrak{g}$ is semi-simple.

Proof. We are only going to prove one direction — if κ is non-degenerate, then $\mathfrak{g}$ is semi-simple.

Suppose we had an abelian ideal $\mathfrak{a} \subseteq \mathfrak{g}$. We want to show that $\kappa(A,X) = 0$ for all $A \in \mathfrak{a}$ and $X \in \mathfrak{g}$. Indeed, we pick a basis of $\mathfrak{a}$, and extend it to a basis of $\mathfrak{g}$. Then since $[X,A] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$ and $A \in \mathfrak{a}$, we know the matrix of ad_X must look like

$$\mathrm{ad}_X = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Also, if $A \in \mathfrak{a}$, then since $\mathfrak{a}$ is an abelian ideal, ad_A kills everything in $\mathfrak{a}$, and $\mathrm{ad}_A(X) \in \mathfrak{a}$ for all $X \in \mathfrak{g}$. So the matrix must look something like

$$\mathrm{ad}_A = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

So we know

$$\mathrm{ad}_A \circ \mathrm{ad}_X = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

and the trace vanishes. So $\kappa(A,X) = 0$ for all $X \in \mathfrak{g}$ and $A \in \mathfrak{a}$. So $A = 0$. So $\mathfrak{a}$ is trivial. □

Theorem. Every complex semi-simple Lie algebra (of finite dimension) has a real form of compact type.

5.2 The Cartan basis

Proposition. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $X \in \mathfrak{g}$. If $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

Proof. Omitted. □

Lemma. Let $H \in \mathfrak{h}$ and $\alpha \in \Phi$. Then

$$\kappa(H, E^\alpha) = 0.$$

Proof. Let $H' \in \mathfrak{h}$. Then

$$\begin{aligned}\alpha(H')\kappa(H, E^\alpha) &= \kappa(H, \alpha(H')E^\alpha)\\ &= \kappa(H, [H', E^\alpha])\\ &= -\kappa([H', H], E^\alpha)\\ &= -\kappa(0, E^\alpha)\\ &= 0\end{aligned}$$

But since $\alpha \neq 0$, we know that there is some H' such that $\alpha(H') \neq 0$. □

Lemma. For any roots $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, we have

$$\kappa(E^\alpha, E^\beta) = 0.$$

Proof. Again let $H \in \mathfrak{h}$. Then we have

$$\begin{aligned}(\alpha(H) + \beta(H))\kappa(E^\alpha, E^\beta) &= \kappa([H, E^\alpha], E^\beta) + \kappa(E^\alpha, [H, E^\beta]),\\ &= 0\end{aligned}$$

where the final line comes from the invariance of the Killing form. Since $\alpha + \beta$ does not vanish by assumption, we must have $\kappa(E^\alpha, E^\beta) = 0$. □

Lemma. If $H \in \mathfrak{h}$, then there is some $H' \in \mathfrak{h}$ such that $\kappa(H, H') \neq 0$.

Proof. Given an H, since κ is non-degenerate, there is some $X \in \mathfrak{g}$ such that $\kappa(H, X) \neq 0$. Write $X = H' + E$, where $H' \in \mathfrak{h}$ and E is in the span of the E^α.

$$0 \neq \kappa(H, X) = \kappa(H, H') + \kappa(H, E) = \kappa(H, H').$$ □

Lemma. Let $\alpha \in \Phi$. Then $-\alpha \in \Phi$. Moreover,

$$\kappa(E^\alpha, E^{-\alpha}) \neq 0$$

Proof. We know that

$$\kappa(E^\alpha, E^\beta) = \kappa(E^\alpha, H^i) = 0$$

for all $\beta \neq -\alpha$ and all i. But κ is non-degenerate, and $\{E^\beta, H^i\}$ span $\mathfrak{g}$. So there must be some $E^{-\alpha}$ in the basis set, and

$$\kappa(E^\alpha, E^{-\alpha}) \neq 0.$$ □

Theorem.

$$\begin{aligned}[][H^i, H^j] &= 0\\ [H^i, E^\alpha] &= \alpha^i E^\alpha\\ [E^\alpha, E^\beta] &= \begin{cases} N_{\alpha,\beta}E^{\alpha+\beta} & \alpha + \beta \in \Phi\\ \kappa(E^\alpha, E^\beta)H^\alpha & \alpha + \beta = 0\\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

5.3 Things are real

Theorem.

$$(\alpha, \beta) \in \mathbb{R}$$

for all $\alpha, \beta \in \Phi$.

Proposition. For any $\alpha, \beta \in \Phi$, we have

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Proof. For $\rho = n_\pm$, we have

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2n_- = -\Lambda \quad \text{and} \quad \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2n_+ = \Lambda.$$

Adding the two equations yields

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = -(n_+ + n_-) \in \mathbb{Z}. \qquad \square$$

Lemma. We have

$$(\alpha, \beta) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha, \delta)(\delta, \beta),$$

where $\mathcal{N}$ is the normalization factor appearing in the Killing form

$$\kappa(X, Y) = \frac{1}{\mathcal{N}} \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y).$$

Proof. We pick the Cartan-Weyl basis, with

$$[H^i, E^\delta] = \delta^i E^\delta$$

for all $i = 1, \cdots, r$ and $\delta \in \Phi$. Then the inner product is defined by

$$\kappa^{ij} = \kappa(H^i, H^j) = \frac{1}{\mathcal{N}} \operatorname{tr}[\operatorname{ad}_{H^i} \circ \operatorname{ad}_{H^j}].$$

But we know that these matrices ad_{H^i} are diagonal in the Cartan-Weyl basis, and the non-zero diagonal entries are exactly the δ^i. So we can write this as

$$\kappa^{ij} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \delta^i \delta^j.$$

Now recall that our inner product was defined by

$$(\alpha, \beta) = \alpha^i \beta^j (\kappa^{-1})_{ij} = \kappa^{ij} \alpha_i \beta_j,$$

where we define

$$\beta_j = (\kappa^{-1})_{jk} \beta^k.$$

Putting in our explicit formula for the κ^{ij}, this is

$$(\alpha, \beta) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \alpha_i \delta^i \delta^j \beta_j = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha, \delta)(\delta, \beta). \qquad \square$$

Corollary.

$$(\alpha, \beta) \in \mathbb{R}$$

for all $\alpha, \beta \in \Phi$.

Proof. We write

$$R_{\alpha,\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Then the previous formula tells us

$$\frac{2}{(\beta, \beta)} R_{\alpha,\beta} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta}$$

We know that $R_{\alpha,\beta}$ are all integers, and in particular real. So (β, β) must be real as well. So it follows that (α, β) is real since $R_{\alpha,\beta}$ is an integer. □

5.4 A real subalgebra

Proposition. The roots Φ span $\mathfrak{h}^*$. In particular, we know

$$|\Phi| \geq \dim \mathfrak{h}^*.$$

Proof. Suppose the roots do not span $\mathfrak{h}^*$. Then the space spanned by the roots would have a non-trivial orthogonal complement. So we can find $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha) = 0$ for all $\alpha \in \Phi$. We now define

$$H_\lambda = \lambda_i H^i \in \mathfrak{h}.$$

Then as usual we have

$$[H_\lambda, H] = 0 \text{ for all } H \in \mathfrak{h}.$$

Also, we know

$$[H_\lambda, E^\alpha] = (\lambda, \alpha) E^\alpha = 0.$$

for all roots $\alpha \in \Phi$ by assumption. So H_λ commutes with everything in the Lie algebra. This would make $\langle H_\lambda \rangle$ a non-trivial ideal, which is a contradiction since $\mathfrak{g}$ is simple. □

Proposition. $\mathfrak{h}^*_{\mathbb{R}}$ contains all roots.

Proof. We know that $\mathfrak{h}^*$ is spanned by the $\alpha_{(i)}$ as a complex vector space. So given an $\beta \in \mathfrak{h}^*$, we can find some $\beta^i \in \mathbb{C}$ such that

$$\beta = \sum_{i=1}^{r} \beta^i \alpha_{(i)}.$$

Taking the inner product with $\alpha_{(j)}$, we know

$$(\beta, \alpha_{(j)}) = \sum_{i=1}^{r} \beta^i (\alpha_{(i)}, \alpha_{(j)}).$$

We now use the fact that the inner products are all real! So β^i is the solution to a set of real linear equations, and the equations are non-degenerate since the $\alpha_{(i)}$ form a basis and the Killing form is non-degenerate. So β^i must be real. So $\beta \in \mathfrak{h}^*_{\mathbb{R}}$. □

Proposition. The Killing form induces a positive-definite inner product on $\mathfrak{h}_{\mathbb{R}}^*$.

Proof. It remains to show that $(\lambda, \lambda) \geq 0$ for all λ, with equality iff $\lambda = 0$. We can write

$$(\lambda, \lambda) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\lambda, \delta)^2 \geq 0.$$

If this vanishes, then $(\lambda, \delta) = 0$ for all $\delta \in \Phi$. But the roots span, so this implies that λ kills everything, and is thus 0 by non-degeneracy. □

5.5 Simple roots

Proposition. Any positive root can be written as a linear combination of simple roots with positive integer coefficients. So every root can be written as a linear combination of simple roots.

Proof. Given any positive root, if it cannot be decomposed into a positive sum of other roots, then it is simple. Otherwise, do so, and further decompose the constituents. This will have to stop because there are only finitely many roots, and then you are done. □

Corollary. The simple roots span $\mathfrak{h}_{\mathbb{R}}^*$.

Proposition. If $\alpha, \beta \in \Phi$ are simple, then $\alpha - \beta$ is *not* a root.

Proof. Suppose $\alpha - \beta$ were a root. By swapping α and β if necessary, we may wlog assume that $\alpha - \beta$ is a positive root. Then

$$\alpha = \beta + (\alpha - \beta)$$

is a sum of two positive roots, which is a contradiction. □

Proposition. If $\alpha, \beta \in \Phi_S$, then the α-string through β, namely

$$S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi\},$$

has length

$$\ell_{\alpha\beta} = 1 - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{N}.$$

Proof. Recall that there exists $n_\pm$ such that

$$S_{\alpha,\beta} = \{\beta + n\alpha : n_- \leq n \leq n_+\},$$

We have shown before that

$$n_+ + n_- = -\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

In the case where α, β are simple roots, we know that $\beta - \alpha$ is not a root. So $n_- \geq 0$. But we know β is a root. So we know that $n_- = 0$ and hence

$$n_+ = -\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{N}.$$

So there are

$$n_+ + 1 = 1 - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

things in the string. □

Corollary. For any distinct simple roots α, β, we have

$$(\alpha, \beta) \leq 0.$$

Proposition. Simple roots are linearly independent.

Proof. Suppose we have a non-trivial linear combination λ of the simple roots. We write

$$\lambda = \lambda_+ - \lambda_- = \sum_{i \in I_+} c_i \alpha_{(i)} - \sum_{j \in I_-} b_j \alpha_{(j)},$$

where $c_i, b_j \geq 0$ and I_+, I_- are disjoint. If I_- is empty, then the sum is a positive root, and in particular is non-zero. Similarly, if I_+ is empty, then the sum is negative. So it suffices to focus on the case $c_i, b_j > 0$.

Then we have

$$\begin{aligned}(\lambda, \lambda) &= (\lambda_+, \lambda_+) + (\lambda_-, \lambda_-) - 2(\lambda_+, \lambda_-)\\ &> -2(\lambda_+, \lambda_-)\\ &= -2 \sum_{i \in I_+} \sum_{j \in I_-} c_i b_j (\alpha_{(i)}, \alpha_{(j)})\\ &\geq 0,\end{aligned}$$

since $(\alpha_{(i)}, \alpha_{(j)}) \leq 0$ for all simple roots $\alpha_{(i)}, \alpha_{(j)}$. So in particular λ is non-zero. ☐

Corollary. There are exactly $r = \operatorname{rank} \mathfrak{g}$ simple roots roots, i.e.

$$|\Phi_S| = r.$$

5.6 The classification

Theorem (Cartan)**.** Any finite-dimensional, simple, complex Lie algebra is uniquely determined by its Cartan matrix.

Proposition. We always have $A^{ii} = 2$ for $i = 1, \cdots, r$.

Proposition. $A^{ij} = 0$ if and only if $A^{ji} = 0$.

Proposition. $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.

Proposition. We have $\det A > 0$.

Proof. Recall that we defined our inner product as

$$(\alpha, \beta) = \alpha^T \kappa^{-1} \beta,$$

and we know this is positive definite. So we know $\det \kappa^{-1} > 0$. We now write the Cartan matrix as

$$A = \kappa^{-1} D,$$

where we have

$$D_k^j = \frac{2}{(\alpha_{(j)}, \alpha_{(j)})} \delta_k^j.$$

Then we have

$$\det D = \prod_j \frac{2}{(\alpha_{(j)}, \alpha_{(j)})} > 0.$$

So it follows that

$$\det A = \det \kappa^{-1} \det D > 0.$$ □

Proposition. The Cartan matrix A is irreducible.

Proposition.

(i) $A^{ii} = 2$ for all i.

(ii) $A^{ij} = 0$ if and only if $A^{ji} = 0$.

(iii) $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.

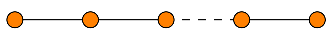
(iv) $\det A > 0$.

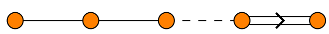
(v) A is irreducible.

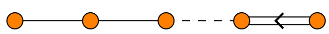
Proposition. A simple Lie algebra has roots of at most 2 distinct lengths.

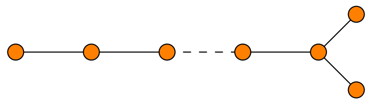
Proof. See example sheet. □

Theorem (Cartan classification)**.** The possible Dynkin diagrams include the following infinite families (where n is the number of vertices):

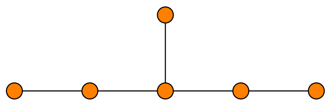
A_n : 

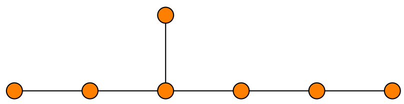
B_n : 

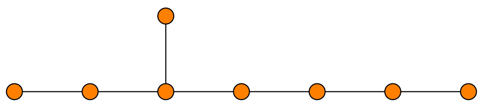
C_n : 

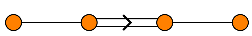
D_n : 

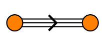
And there are also five exceptional cases:

E_6 : 

E_7 : 

E_8 : 

F_4 : 

G_2 : 

5.7 Reconstruction

6 Representation of Lie algebras

6.1 Weights

6.2 Root and weight lattices

6.3 Classification of representations

Theorem. For any finite-dimensional representation of $\mathfrak{g}$, if

$$\lambda = \sum_{i=1}^{r} \lambda^i \omega_{(i)} \in S_\rho,$$

then we know

$$\lambda - m_{(i)}\alpha_{(i)} \in S_\rho,$$

for all $m_{(i)} \in \mathbb{Z}$ and $0 \leq m_{(i)} \leq \lambda^i$.

If we know further that ρ is irreducible, then we can in fact obtain all weights by starting at the highest weight and applying this procedure.

Moreover, for any

$$\Lambda = \sum \Lambda^i \omega_{(i)} \in \mathcal{L}_W[\mathfrak{g}],$$

this is the highest weight of some irreducible representation if and only if $\Lambda^i \geq 0$ for all i.

6.4 Decomposition of tensor products

7 Gauge theories

7.1 Electromagnetism and U(1) gauge symmetry

7.2 General case

Proposition. We have

$$\delta_X(\mathrm{D}_\mu\phi) = \rho(X)\mathrm{D}_\mu\phi.$$

Proof. We have

$$\begin{aligned}
\delta_X(\mathrm{D}_\mu\phi) &= \delta_X(\partial_\mu\phi + \rho(A_\mu)\phi)\\
&= \partial_\mu(\delta_X\phi) + \rho(A_\mu)\delta_X\phi + \rho(\delta_X A_\mu)\phi\\
&= \partial_\mu(\rho(X)\phi) + \rho(A_\mu)\rho(X)\phi - \rho(\partial_\mu X)\phi + \rho([X, A_\mu])\phi\\
&= \rho(\partial_\mu X)\phi + \rho(X)\partial_\mu\phi + \rho(X)\rho(A_\mu)\phi\\
&\quad\quad + [\rho(A_\mu), \rho(X)]\phi - \rho(\partial_\mu X)\phi + \rho([X, A_\mu])\phi\\
&= \rho(X)(\partial_\mu\phi + \rho(A_\mu)\phi)\\
&= \rho(X)\mathrm{D}_\mu\phi,
\end{aligned}$$

as required. □

Lemma. We have

$$\delta_X(F_{\mu\nu}) = [X, F_{\mu\nu}] \in \mathcal{L}(G).$$

Proof. We have

$$\begin{aligned}
\delta_X(F_{\mu\nu}) &= \partial_\mu(\delta_X A_\nu) - \partial_\nu(\delta_X A_\mu) + [\delta_X A_\mu, A_\nu] + [A_\mu, \delta_X A_\nu]\\
&= \partial_\mu\partial_\nu X + \partial_\mu([X, A_\nu]) - \partial_\nu\partial_\mu X - \partial_\nu([X, A_\mu]) - [\partial_\mu X, A_\nu]\\
&\quad\quad - [A_\mu, \partial_\nu X] + [[X, A_\mu], A_\nu] + [A_\mu, [X, A_\nu]]\\
&= [X, \partial_\mu A_\nu] - [X, \partial_\nu A_\mu] + ([X, [A_\mu, A_n]]\\
&= [X, F_{\mu\nu}].
\end{aligned}$$

where we used the Jacobi identity in the last part. □

8 Lie groups in nature

8.1 Spacetime symmetry

8.2 Possible extensions

Theorem (Coleman-Mondula)**.** In an interactive quantum field theory (satisfying a few sensible conditions), the largest possible symmetry group is the Poincaré group times some internal symmetry that commutes with the Poincaré group.

8.3 Internal symmetries and the eightfold way