Part III — Symmetries, Fields and Particles
Theorems

Based on lectures by N. Dorey
Notes taken by Dexter Chua

Michaelmas 2016

This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of SU(2), the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of SU(2) and introduce the tensor product of representations. The next part of the course develops the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of SU(3). The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate SU(3) global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

Pre-requisites

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

Contents

1 Introduction

2 Lie groups

2.1 Definitions

2.2 Matrix Lie groups

Lemma. The *general linear group*:

$$\mathrm{GL}(n, \mathbb{R}) = \{M \in \mathrm{Mat}_n(\mathbb{R}) : \det M \neq 0\}$$

and *orthogonal group*:

$$\mathrm{O}(n) = \{M \in \mathrm{GL}(n, \mathbb{R}) : M^T M = I\}$$

are Lie groups.

Lemma. The *special orthogonal group* $\mathrm{SO}(n)$:

$$\mathrm{SO}(n) = \{M \in \mathrm{O}(n) : \det M = 1\}$$

is a Lie group.

Theorem. Let M be a real matrix. Then λ is an eigenvalue iff λ^* is an eigenvalue. Moreover, if M is orthogonal, then $|\lambda|^2 = 1$.

2.3 Properties of Lie groups

3 Lie algebras

3.1 Lie algebras

Proposition.

$$f^{ba}{}_c = -f^{ab}{}_c.$$

Proposition.

$$f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0.$$

3.2 Differentiation

Proposition. Let M be a manifold with local coordinates $\{x^i\}_{i=1,\cdots,D}$ for some region $U \subseteq M$ containing p. Then T_pM has basis

$$\left\{\frac{\partial}{\partial x^j}\right\}_{j=1,\cdots,D}.$$

In particular, $\dim T_pM = \dim M$.

3.3 Lie algebras from Lie groups

Theorem. The tangent space of a Lie group G at the identity naturally admits a Lie bracket

$$[\,\cdot\,,\,\cdot\,] : T_eG \times T_eG \to T_eG$$

such that

$$\mathcal{L}(G) = (T_eG, [\,\cdot\,,\,\cdot\,])$$

is a Lie algebra.

3.4 The exponential map

Proposition. Let G be a Lie group of dimension > 0. Then G has a nowhere-vanishing vector field.

Theorem (Poincare-Hopf theorem)**.** Let M be a compact manifold. If M has non-zero Euler characteristic, then any vector field on M has a zero.

Theorem (Hairy ball theorem)**.** Any smooth vector field on S^2 has a zero. More generally, any smooth vector field on S^{2n} has a zero.

Theorem. For any matrix Lie group G, the map exp restricts to a map $\mathcal{L}(G) \to G$.

Theorem (Baker–Campbell–Hausdorff formula)**.** We have

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) + \cdots\right).$$

Proposition. Let G be a Lie group, and $\mathfrak{g}$ be its Lie algebra. Then the image of $\mathfrak{g}$ under exp is the connected component of e.

4 Representations of Lie algebras

4.1 Representations of Lie groups and algebras

Proposition. Let D be a representation of a group G. Then $D(e) = I$ and $D(g^{-1}) = D(g)^{-1}$.

Proposition. The adjoint representation is a representation.

4.2 Complexification and correspondence of representations

Lemma. Given a representation $D : G \to \mathrm{GL}(V)$, the induced representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation.

Theorem. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then there is a unique representation $D : G \to \mathrm{GL}(V)$ of G that induces ρ.

Theorem. Let $\mathfrak{g}$ be a real Lie algebra. Then the complex representations of $\mathfrak{g}$ are exactly the (complex) representations of $\mathfrak{g}_{\mathbb{C}}$.

Explicitly, if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation, then we can extend it to $\mathfrak{g}_{\mathbb{C}}$ by declaring

$$\rho(X + iY) = \rho(X) + i\rho(Y).$$

Conversely, if $\rho_{\mathbb{C}} : \mathfrak{g}_C \to \mathfrak{gl}(V)$, restricting it to $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$ gives a representation of $\mathfrak{g}$.

4.3 Representations of $\mathfrak{su}(2)$

Proposition. The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are labelled by $\Lambda \in \mathbb{Z}_{\geq 0}$, which we call ρ_Λ, with weights given by

$$\{-\Lambda, -\Lambda + 2, \cdots, \Lambda - 2, \Lambda\}.$$

The weights are all non-degenerate, i.e. each only has one eigenvector. We have $\dim(\rho_\Lambda) = \Lambda + 1$.

4.4 New representations from old

Theorem. If ρ_i for $i = 1, \cdots, m$ are finite-dimensional irreps of a simple Lie algebra $\mathfrak{g}$, then $\rho_1 \otimes \cdots \otimes \rho_m$ is completely reducible to irreps, i.e. we can find $\tilde{\rho}_1, \cdots, \tilde{\rho}_k$ such that

$$\rho_1 \otimes \cdots \otimes \rho_m = \tilde{\rho}_1 \oplus \tilde{\rho}_2 \oplus \cdots \oplus \tilde{\rho}_k.$$

4.5 Decomposition of tensor product of $\mathfrak{su}(2)$ representations

Proposition.

$$\rho_M \otimes \rho_N = \rho_{|N-M|} \oplus \rho_{|N-M|+2} \oplus \cdots \oplus \rho_{N+M}.$$

5 Cartan classification

5.1 The Killing form

Proposition. The Killing form is invariant.

Theorem (Cartan)**.** The Killing form of a Lie algebra $\mathfrak{g}$ is non-degenerate iff $\mathfrak{g}$ is semi-simple.

Theorem. Every complex semi-simple Lie algebra (of finite dimension) has a real form of compact type.

5.2 The Cartan basis

Proposition. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $X \in \mathfrak{g}$. If $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

Lemma. Let $H \in \mathfrak{h}$ and $\alpha \in \Phi$. Then

$$\kappa(H, E^\alpha) = 0.$$

Lemma. For any roots $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, we have

$$\kappa(E^\alpha, E^\beta) = 0.$$

Lemma. If $H \in \mathfrak{h}$, then there is some $H' \in \mathfrak{h}$ such that $\kappa(H, H') \neq 0$.

Lemma. Let $\alpha \in \Phi$. Then $-\alpha \in \Phi$. Moreover,

$$\kappa(E^\alpha, E^{-\alpha}) \neq 0$$

Theorem.

$$\begin{aligned}
[H^i, H^j] &= 0\\
[H^i, E^\alpha] &= \alpha^i E^\alpha\\
[E^\alpha, E^\beta] &= \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \alpha + \beta \in \Phi \\ \kappa(E^\alpha, E^\beta) H^\alpha & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

5.3 Things are real

Theorem.

$$(\alpha, \beta) \in \mathbb{R}$$

for all $\alpha, \beta \in \Phi$.

Proposition. For any $\alpha, \beta \in \Phi$, we have

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Lemma. We have
$$(\alpha, \beta) = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha, \delta)(\delta, \beta),$$
where $\mathcal{N}$ is the normalization factor appearing in the Killing form
$$\kappa(X, Y) = \frac{1}{\mathcal{N}} \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

Corollary.
$$(\alpha, \beta) \in \mathbb{R}$$
for all $\alpha, \beta \in \Phi$.

5.4 A real subalgebra

Proposition. The roots Φ span $\mathfrak{h}^*$. In particular, we know
$$|\Phi| \geq \dim \mathfrak{h}^*.$$

Proposition. $\mathfrak{h}^*_{\mathbb{R}}$ contains all roots.

Proposition. The Killing form induces a positive-definite inner product on $\mathfrak{h}^*_{\mathbb{R}}$.

5.5 Simple roots

Proposition. Any positive root can be written as a linear combination of simple roots with positive integer coefficients. So every root can be written as a linear combination of simple roots.

Corollary. The simple roots span $\mathfrak{h}^*_{\mathbb{R}}$.

Proposition. If $\alpha, \beta \in \Phi$ are simple, then $\alpha - \beta$ is *not* a root.

Proposition. If $\alpha, \beta \in \Phi_S$, then the α-string through β, namely
$$S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi\},$$
has length
$$\ell_{\alpha\beta} = 1 - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{N}.$$

Corollary. For any distinct simple roots α, β, we have
$$(\alpha, \beta) \leq 0.$$

Proposition. Simple roots are linearly independent.

Corollary. There are exactly $r = \operatorname{rank} \mathfrak{g}$ simple roots roots, i.e.
$$|\Phi_S| = r.$$

5.6 The classification

Theorem (Cartan)**.** Any finite-dimensional, simple, complex Lie algebra is uniquely determined by its Cartan matrix.

Proposition. We always have $A^{ii} = 2$ for $i = 1, \cdots, r$.

Proposition. $A^{ij} = 0$ if and only if $A^{ji} = 0$.

Proposition. $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.

Proposition. We have $\det A > 0$.

Proposition. The Cartan matrix A is irreducible.

Proposition.

(i) $A^{ii} = 2$ for all i.

(ii) $A^{ij} = 0$ if and only if $A^{ji} = 0$.

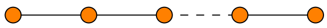
(iii) $A^{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.

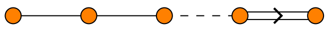
(iv) $\det A > 0$.

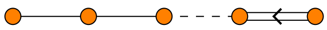
(v) A is irreducible.

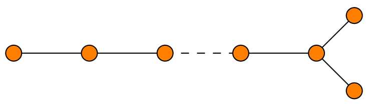
Proposition. A simple Lie algebra has roots of at most 2 distinct lengths.

Theorem (Cartan classification)**.** The possible Dynkin diagrams include the following infinite families (where n is the number of vertices):

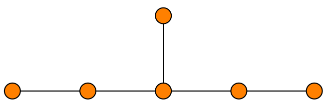
A_n :


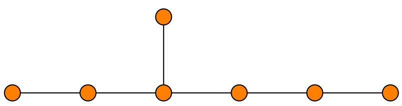
B_n :


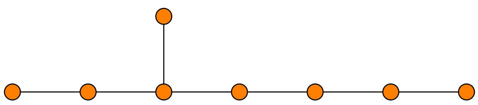
C_n :


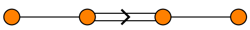
D_n :


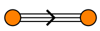
And there are also five exceptional cases:

E_6 :


E_7 :


E_8 :


F_4 :


G_2 :


5.7 Reconstruction

6 Representation of Lie algebras

6.1 Weights

6.2 Root and weight lattices

6.3 Classification of representations

Theorem. For any finite-dimensional representation of $\mathfrak{g}$, if

$$\lambda = \sum_{i=1}^{r} \lambda^i \omega_{(i)} \in S_\rho,$$

then we know

$$\lambda - m_{(i)}\alpha_{(i)} \in S_\rho,$$

for all $m_{(i)} \in \mathbb{Z}$ and $0 \leq m_{(i)} \leq \lambda^i$.

If we know further that ρ is irreducible, then we can in fact obtain all weights by starting at the highest weight and applying this procedure.

Moreover, for any

$$\Lambda = \sum \Lambda^i \omega_{(i)} \in \mathcal{L}_W[\mathfrak{g}],$$

this is the highest weight of some irreducible representation if and only if $\Lambda^i \geq 0$ for all i.

6.4 Decomposition of tensor products

7 Gauge theories

7.1 Electromagnetism and U(1) gauge symmetry

7.2 General case

Proposition. We have

$$\delta_X(\mathrm{D}_\mu \phi) = \rho(X)\mathrm{D}_\mu \phi.$$

Lemma. We have

$$\delta_X(F_{\mu\nu}) = [X, F_{\mu\nu}] \in \mathcal{L}(G).$$

8 Lie groups in nature

8.1 Spacetime symmetry

8.2 Possible extensions

Theorem (Coleman-Mondula)**.** In an interactive quantum field theory (satisfying a few sensible conditions), the largest possible symmetry group is the Poincaré group times some internal symmetry that commutes with the Poincaré group.

8.3 Internal symmetries and the eightfold way