Symmetries, Fields and Particles. Examples 1.

1. O(n) consists of $n \times n$ real matrices M satisfying $M^T M = I$. Check that O(n) is a group. U(n) consists of $n \times n$ complex matrices U satisfying $U^\dagger U = I$. Check similarly that U(n) is a group.

Verify that O(n) and SO(n) are the subgroups of real matrices in, respectively, U(n) and SU(n). By considering how U(n) matrices act on vectors in $\mathbb{C}^n$, and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, show that U(n) is a subgroup of SO($2n$).

2. Show that for matrices $M \in$ O(n), the first column of M is an arbitrary unit vector, the second is a unit vector orthogonal to the first, ..., the kth column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of O(n). By similar reasoning, determine the dimension of U(n).

Show that any column of a unitary matrix U is not in the (complex) linear span of the remaining columns.

3. Consider the real 3×3 matrix,

$$R(\mathbf{n}, \theta)_{ij} = \cos\theta\, \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector in $\mathbb{R}^3$. Verify that $\mathbf{n}$ is an eigenvector of $R(\mathbf{n}, \theta)$ with eigenvalue one. Now choose an orthonormal basis for $\mathbb{R}^3$ with basis vectors $\{\mathbf{n}, \mathbf{m}, \tilde{\mathbf{m}}\}$ satisfying,

$$\mathbf{m} \cdot \mathbf{m} = 1, \qquad \mathbf{m} \cdot \mathbf{n} = 0, \qquad \tilde{\mathbf{m}} = \mathbf{n} \times \mathbf{m}.$$

By considering the action of $R(\mathbf{n}, \theta)$ on these basis vectors show that this matrix corresponds to a rotation through an angle θ about an axis parallel to $\mathbf{n}$ and check that it is an element of $SO(3)$.

4. Show that the set of matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ forms a group. How would you check that it is a Lie group? Assuming that it is a Lie group, determine its dimension. By splitting α and β into real and imaginary parts, consider the group manifold as a subset of $\mathbb{R}^4$ and show that it is non-compact. You may use the fact that a compact subset S of $\mathbb{R}^n$ is necessarily bounded; in other words there exists $B > 0$ such that $|\mathbf{x}| < B$ for all $\mathbf{x} \in S$.

5. Show that any SU(2) matrix U can be expressed in the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Deduce that an alternative form for an SU(2) matrix is

$$U = a_0 I + i\mathbf{a} \cdot \boldsymbol{\sigma}$$

with $(a_0, \mathbf{a})$ real, $\boldsymbol{\sigma}$ the Pauli matrices, and $a_0^2 + \mathbf{a} \cdot \mathbf{a} = 1$. Using the second form, calculate the product of two SU(2) matrices.

6. Consider a real vector space V with product $* : V \times V \to V$. The product is bilinear and associative. In other words, for all elements X,Y,$Z \in V$ and scalars α, $\beta \in \mathbb{R}$, we have

$$(\alpha X + \beta Y) * Z = \alpha X * Z + \beta Y * Z, \qquad Z * (\alpha X + \beta Y) = \alpha Z * X + \beta Z * Y$$

and also $(X * Y) * Z = X * (Y * Z)$. Define the bracket of two vectors X and $Y \in V$ as the commutator,

$$[X, Y] = X * Y - Y * X$$

Show that, equipped with this bracket, V becomes a Lie algebra.

7. Verify that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

forms a matrix Lie group, G. What is the underlying manifold of G? Is the group abelian? Find the Lie algebra, L(G), and calculate the bracket of two general elements of it. Is the Lie algebra simple?

8. A useful basis for the Lie algebra of GL(n) consists of the n^2 matrices T^{ij} $(1 \leq i, j \leq n)$, where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Find the structure constants in this basis.

9. Let $\exp iH = U$. Show that if H is hermitian then U is unitary. Show also, that if H is traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \to \exp X$ sends L(G), the Lie algebra of G, to G?

Symmetries, Fields and Particles. Examples 2.

1. Let X be an element of the Lie algebra of a matrix Lie group G. Consider the curve in G defined by $g(t) = \mathrm{Exp}(tX)$ where t is a real parameter. Show that,

$$g(t_1)g(t_2) = g(t_2)g(t_1) = g(t_1 + t_2)$$

for all values of t_1 and t_2. Assuming there is no non-zero value of t for which $g(t)$ is equal to the identity, show that the curve defines a Lie subgroup of G which is isomorphic to $(\mathbb{R}, +)$ (ie the real line with addition as the group multiplication law).

2. Verify the Baker–Campbell–Hausdorff (BCH) formula

$$\exp X \cdot \exp Y = \exp\Big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + \dots\Big)$$

to the order shown.

3. Let $g(t) = \mathrm{Exp}\ it\sigma_1$. By evaluating $g(t)$ as a matrix, show that $\{g(t) : 0 \le t \le 2\pi\}$ is a 1-parameter subgroup of SU(2). Describe geometrically how this subgroup sits inside the group manifold of SU(2).

4. For each $A \in SU(2)$, we define a 3×3 matrix with entries,

$$R(A)_{ij} = \tfrac{1}{2}\mathrm{tr}_2\left(\sigma_i A \sigma_j A^\dagger\right) \qquad - (1)$$

for $i, j = 1, 2, 3$. Using the Pauli matrix identity,

$$\sum_{j=1}^{3} (\sigma_j)_{\alpha\beta}(\sigma_j)_{\delta\gamma} = 2\delta_{\alpha\gamma}\delta_{\delta\beta} - \delta_{\alpha\beta}\delta_{\delta\gamma}$$

show that $R(A)$ is an element of $SO(3)$. Hint: You may appeal to the fact that $SU(2) \simeq S^3$ is connected to check that $\mathrm{det}R = 1$.

*(Harder) Check that we may invert Eqn (1) to express $A \in SU(2)$ in terms of $R \in SO(3)$ by setting,

$$A = \pm\frac{(I_2 + \sigma_i R_{ij}\sigma_j)}{2\sqrt{1 + \mathrm{Tr}_3 R}}$$

where I_2 is the 2×2 unit matrix and summation over repeated indices is implied.*

This map provides an isomorphism between $SO(3)$ and $SU(2)/\mathbb{Z}_2$.

5. Let G be a matrix Lie group and d a representation of its Lie algebra $L(G)$. For group elements $g = \mathrm{Exp}\,X$ with $X \in L(G)$ we define $D(g) = \mathrm{Exp}(d(X))$. Using the BCH formula show that

$$D(g_1 g_2) = D(g_1)D(g_2)$$

for all group elements $g_1 = \mathrm{Exp}\, X_1$ and $g_2 = \mathrm{Exp}\, X_2$ with $X_1,\ X_2 \in L(G)$. Can we conclude that D is a representation of G? Explain your answer.

6. (a) Let L be a real Lie algebra (i.e. there is a basis $T^a : a = 1, \ldots, n = \mathrm{Dim}\, L$ with real structure constants f^{ab}_{c}). Suppose R is a representation of L. Write down the algebraic equations that the matrices $R(T^a)$ must satisfy. Show that the complex conjugate matrices $\bar{R}(T^a) = R(T^a)^*$ also define a representation of L.

(b) Show that the fundamental representation $R(T^a) = -\frac{1}{2} i\sigma_a$, with $a = 1, 2, 3$ and its complex conjugate $\bar{R}(T^a) = \frac{1}{2} i(\sigma_a)^*$ are *equivalent* representations of L(SU(2)).

Show that the *weights* of the $L(SU(2))$ representations R and $\bar{R}$ are the same.

7. Let R_1 and R_2 be two representations of a Lie algebra L with representation spaces V_1 and V_2 respectively. The tensor product of R_1 and R_2 is defined by the formula,

$$R_1 \otimes R_2(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X)$$

where I_1 and I_2 are the identity maps on V_1 and V_2 respectively. Show that $R_1 \otimes R_2$ is a representation of L with representation space $V_1 \otimes V_2$.

8. Find the multiplicity of each weight of the tensor product $R_N \otimes R_M$ where N and M are nonegative integers. Here R_Λ denotes the irreducible representation of $L(SU(2))$ with highest weight Λ defined in the lectures. Deduce the Clebsch-Gordon decomposition,

$$R_N \otimes R_M = R_{|N-M|} \oplus R_{|N-M|+2} \oplus \cdots \oplus R_{N+M}\,.$$

Verify that the dimensions of the reducible representation defined on the two sides of the equation are the same.

9 a) A Lie algebra is semi-simple if it has no abelian ideals. A semi-simple Lie algebra has a non-degenerate Killing form.

Consider the Lie algebra defined in Sheet 1, Question 8. Is it semi-simple? Find its Killing form explicitly and determine whether it is degenerate.

b) A finite-dimensional real Lie algebra is of compact type if it has a basis $\{T^a\}$ in which the Killing form has components, $\kappa^{ab} = -\kappa\, \delta^{ab}$ for some positive constant κ. Let L be a real Lie algebra of compact type and let I be an ideal of L. Let $\mathrm{I}_\perp$ denote the orthogonal complement of I with respect to the Killing form κ. ($Y \in \mathrm{I}_\perp$ if and only if $\kappa(X, Y) = 0$ for all $X \in \mathrm{I}$.) By considering $\kappa(X, [Y, Z])$, where $X \in \mathrm{I}$, $Y \in \mathrm{L}$ and $Z \in \mathrm{I}_\perp$, show that $\mathrm{I}_\perp$ is an ideal and that

$$\mathrm{L} = \mathrm{I} \oplus \mathrm{I}_\perp$$

where the summands mutually commute.

Deduce that any semi-simple complex Lie algebra M of finite dimension is the direct sum of a finite number of simple Lie algebras. You may use the fact stated in the lectures that any such M has a real form of compact type.

Symmetries, Fields and Particles. Examples 3

1. In this question and the following ones, $\mathrm{L}(G)$ denotes the Lie algebra of a Lie group G and $\mathrm{L}_{\mathbb{C}}(G)$ denotes its *complexification*.

i) Show that

$$\mathrm{L}_{\mathbb{C}}\left(SU(2)\right) \simeq \mathrm{L}\left(SL(2,\mathbb{C})\right)$$

where the RHS is considered as a complex Lie algebra.

ii) By considering the subalgebra of real matrices show that $\mathrm{L}_{\mathbb{C}}\left(SU(2)\right)$ has two inequivalent real forms.

2. Write down a basis for a Cartan subalgebra of $\mathrm{L}_{\mathbb{C}}(SO(2n))$ and for a Cartan subalgebra of $\mathrm{L}_{\mathbb{C}}(SO(2n+1))$ for arbitrary integer n. Hence find the roots of the Lie algebras $\mathrm{L}_{\mathbb{C}}(SO(3))$ and $\mathrm{L}_{\mathbb{C}}(SO(4))$ and write down the corresponding step operators in your chosen basis.

3. If α is a root of a simple complex Lie algebra of finite dimension show that the only values of $k \in \mathbb{C}$ for which $k\alpha$ is a root are $k = \pm 1$

4. Starting from the constraint,

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \qquad\qquad (*)$$

on the inner product of any two roots α and β, show that any complex simple Lie algebra of finite dimension has roots of at most two different lengths.

5.

i) Using the constraint (*) of question 3, show that the off-diagonal elements of the Cartan matrix A of a finite-dimensional complex simple Lie algebra of rank r obey

$$A_{ij}A_{ji} < 4 \quad \text{ for all } \quad i \neq j = 1, 2, \ldots, r.$$

ii) Consider possible 3×3 Cartan matrices of the form,

$$\begin{pmatrix} 2 & l & m \\ l' & 2 & n \\ m' & n' & 2 \end{pmatrix}$$

Using the constraints on the Cartan matrix derived in the lectures together with the result of part i), show that l, m and n cannot all be non-zero, and find all the allowed Cartan matrices with $m = 0$. Show that your solutions exhaust all the rank 3 simple Lie algebras in the Cartan classification. Why do no additional solutions with $m \neq 0$ arise?

6. Find a set of simple roots for the matrix Lie algebra $\mathrm{L}_{\mathbb{C}}(SU(3))$ and determine the corresponding Cartan matrix showing that it coincides with the Cartan matrix of the Lie algebra A_2 in the Cartan classification.

7. Starting from its Dynkin diagram, construct the root system of the simple Lie algebra B_2.

8. The simple Lie algebra A_2 has simple roots α, β and a single additional positive root $\theta = \alpha + \beta$. Choose a basis for the the Lie algebra consisting of the generators $\{h^\alpha, h^\beta, e^{\pm\alpha}, e^{\pm\beta}, e^{\pm\theta}\}$. Write down all brackets between Cartan generators and step operators and also the brackets between step operators including normalisation constants for any brackets which are potentially non-zero. Determine the constraints on the normalisation constants which follow from the Jacobi identity. How could you fix any remaining ambiguities?

9. Determine the weights of the A_2 representation with Dynkin labels $(2, 0)$.

Symmetries, Fields and Particles. Examples 4

1. Starting from the corresponding Cartan matrix construct the root and weight lattices of the Lie algebra $B_2 = \mathrm{L}_{\mathbb{C}}(SO(5))$. Using the algorithm introduced in the lectures, find the weights of the fundamental representation and those of the adjoint. Determine the degeneracy of the weight zero in the adjoint.

2. Show that the isospin I and hypercharge Y of the lightest mesons are correctly determined by the relations $I = H^1/2$ and $Y = (H^1 + 2H^2)/3$ where H^1 and H^2 are the standard basis for the Cartan subalgebra of A_2 and these generators act on states in the adjoint representation.

3. Show that the ten-dimensional representation $R_{3,0}$ of A_2 corresponds to a reducible representation of the $\mathrm{L}_{\mathbb{C}}(SU(2))$ subalgebra corresponding to any root. Find the irreducible components of this representation. Does your answer depend on the particular root chosen?

4. Decompose the following tensor products of $A_2 = \mathrm{L}_{\mathbb{C}}(SU(3))$ representations into irreducible components; i) $\mathbf{3} \otimes \bar{\mathbf{3}}$ and ii) $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$.

5. Find the non-trivial B_2 representation of smallest dimension and decompose the tensor product of two copies into irreducibles, giving the dimension of each component.

6. Consider a gauge theory whose gauge group, G is a matrix Lie group. The corresponding gauge field,

$$A_\mu : \ \mathbb{R}^{3,1} \to \mathrm{L}(G)$$

transforms as

$$A_\mu \to A'_\mu = g A_\mu g^{-1} - (\partial_\mu g) g^{-1}$$

under a gauge transformation,

$$g : \ \mathbb{R}^{3,1} \to G \qquad (*)$$

For the case $G = SU(N)$, check that $A'_\mu(x)$ takes values in the Lie algebra $\mathrm{L}(G)$. Explain why this is true for any matrix Lie group G. Writing $g = \exp(\epsilon X)$ with $\epsilon << 1$, show that the corresponding infinitessimal gauge transformation coincides with the one defined in the lectures.

7. Let $G \subset Mat_N(\mathbb{C})$ be a compact matrix Lie group. A scalar field in the fundamental representation of G, corresponds to an N-component vector $\phi_{\mathrm{F}}(x) \in \mathbb{C}^N$ defined at each spacetime point $x \in \mathbb{R}^{3,1}$ which transforms as,

$$\phi_{\mathrm{F}} \to \phi'_{\mathrm{F}} = g\phi_{\mathrm{F}}$$

under the gauge transformation (*) defined above.
A scalar field in the adjoint representation of G, corresponds to an $N \times N$ matrix $\phi_{\mathrm{A}}(x) \in \mathrm{Mat}_N(\mathbb{C})$ defined at each spacetime point $x \in \mathbb{R}^{3,1}$ which transforms as,

$$\phi_{\mathrm{A}} \to \phi'_{\mathrm{A}} = g\phi_{\mathrm{A}} g^{-1}$$

Find explicit formulae for covariant derivatives $D^{(\mathrm{F})}_\mu$, $D^{(\mathrm{A})}_\mu$, such that $D^{(\mathrm{F})}_\mu \phi_{\mathrm{F}}$ and $D^{(\mathrm{A})}_\mu \phi_{\mathrm{A}}$ transform in the fundamental and adjoint of G respectively. Hence write down gauge-invariant Lagrangians describing the coupling of these scalar fields to the the gauge field A_μ. You may assume that the finite-dimensional representations of a compact Lie group are unitary.

8. Show that the field-strength tensor $F_{\mu\nu}$ for a matrix Lie group G can be written as,

$$F_{\mu\nu} = [D^{(\mathrm{A})}_\mu, D^{(\mathrm{A})}_\nu]$$

Hence show that $F_{\mu\nu}$ transforms in the adjoint representation of G ie,

$$F_{\mu\nu} \to F'_{\mu\nu} = gF_{\mu\nu}g^{-1}$$

under the gauge transformation (*). Thus show that the Lagrangian,

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{Tr}_N\left[F_{\mu\nu}F^{\mu\nu}\right]$$

is gauge invariant.

9. Using the fact that the Killing form,

$$\kappa\,(X, Y) = \mathrm{Tr}\,[\,\mathrm{ad}_X \circ \mathrm{ad}_Y] \qquad \forall X, Y \in \mathrm{L}(G)$$

is the unique invariant inner product any simple Lie algebra $\mathrm{L}(G)$ up to scalar multiplication, deduce that, when $\mathrm{L}(G)$ is simple, the gauge-field Lagrangian defined in the previous question is proportional to the one given in the lectures. Determine the constant of proportionality in the case $G = SU(N)$.