Part III — Symmetries, Fields and Particles

Definitions

Based on lectures by N. Dorey

Notes taken by Dexter Chua

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This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of SU(2), the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of SU(2) and introduce the tensor product of representations. The next part of the course develops the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of SU(3). The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate SU(3) global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

Pre-requisites

Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

Contents

1 Introduction

Definition (Symmetry)**.** A *symmetry* of a physical system is a transformation of the dynamical variables which leaves the physical laws invariant.

Definition (Group)**.** A *group* is a set G of elements with a multiplication rule, obeying the axioms

(i) For all $g_1, g_2 \in G$, we have $g_1 g_2 \in G$. (closure)

(ii) There is a (necessarily unique) element $e \in G$ such that for all $g \in G$, we have $eg = ge = g$. (identity)

(iii) For every $g \in G$, there exists some (necessarily unique) $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$. (inverse)

(iv) For every $g_1, g_2, g_3 \in G$, we have $g_1(g_2 g_3) = (g_1 g_2) g_3$. (associativity)

Definition (Commutative/abelian group)**.** A group is *abelian* or *commutative* if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. A group is *non-abelian* if it is not abelian.

2 Lie groups

2.1 Definitions

Definition (Smooth map)**.** We say a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* if all partial derivatives of all orders exist.

Definition (Manifold)**.** A *manifold* (of dimension n) is a set M together with the following data:

(i) A collection U_α of subsets of M whose union is M;

(ii) A collection of bijections $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$. These are known as *charts*.

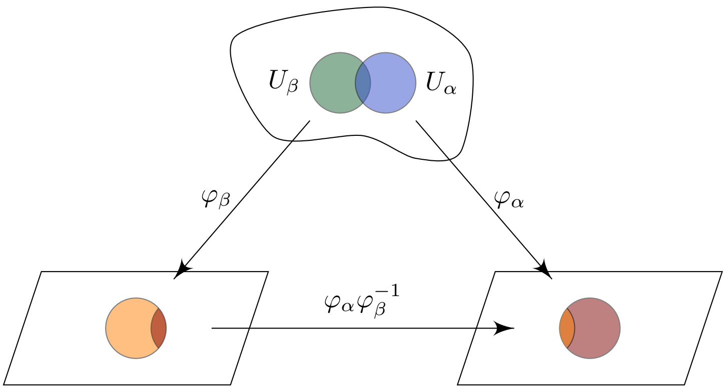
The charts have to satisfy the following compatibility condition:

– For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$, and the transition function

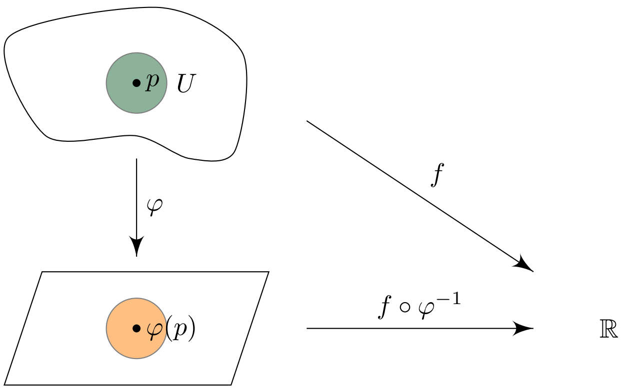
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth.

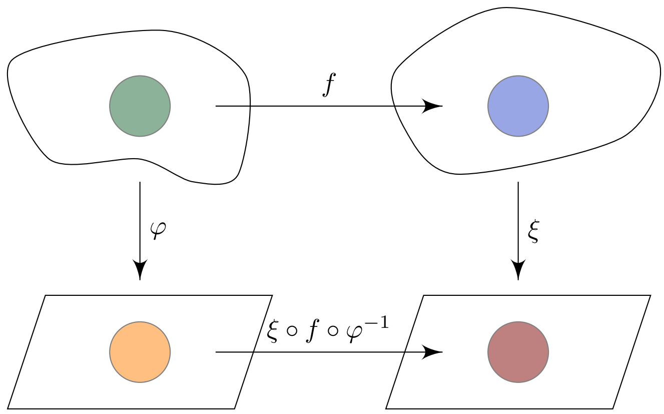
We write (M, φ_α) for the manifold if we need to specify the charts explicitly.



Definition (Smooth map)**.** Let M be a manifold. Then a map $f : M \to \mathbb{R}$ is *smooth* if it is smooth in each coordinate chart. Explicitly, for each chart $(U_\alpha, \varphi_\alpha)$, the composition $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth (in the usual sense)



More generally, if M, N are manifolds, we say $f : M \to N$ is smooth if for any chart (U, φ) of M and any chart (V, ξ) of N, the composition $\xi \circ f \circ \varphi^{-1} : \varphi(U) \to \xi(V)$ is smooth.



Definition (Lie group)**.** A *Lie group* is a group G whose underlying set is given a manifold structure, and such that the multiplication map $m : G \times G \to G$ and inverse map $i : G \to G$ are smooth maps. We sometimes write $\mathcal{M}(G)$ for the underlying manifold of G.

Definition (Dimension of Lie group)**.** The *dimension* of a Lie group G is the dimension of the underlying manifold.

Definition (Subgroup)**.** A *subgroup* H of G is a subset of G that is also a group under the same operations. We write $H \leq G$ if H is a subgroup of G.

Definition (Lie subgroup)**.** A subgroup is a *Lie subgroup* if it is also a manifold (under the induced smooth structure).

2.2 Matrix Lie groups

Definition (General linear group)**.** The *general linear group* is

$$\mathrm{GL}(n, \mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) : \det M \neq 0\}.$$

Definition (Special linear group)**.** The *special linear group* is

$$\mathrm{SL}(n, \mathbb{F}) = \{M \in \mathrm{Mat}_n(\mathbb{F}) : \det M = 1\} \leq \mathrm{GL}(n, \mathbb{F}).$$

Definition (Volume element)**.** Given a frame $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in $\mathbb{R}^n$, the *volume element* is

$$\Omega = \varepsilon_{i_1 \ldots i_n} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n}.$$

Definition (Eigenvalue)**.** A complex number λ is an *eigenvalue* of $M \in M(n)$ if there is some (possibly complex) vector $\mathbf{v}_\lambda \neq 0$ such that

$$M \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda.$$

Definition (Unitary group)**.** The *unitary group* is defined by

$$\mathrm{U}(n) = \{U \in \mathrm{GL}(n, \mathbb{C}) : U^\dagger U = I\}.$$

Definition (Special unitary group)**.** The *special unitary group* is defined by

$$\mathrm{SU}(n) = \{U \in \mathrm{U}(n) : \det U = 1\}.$$

2.3 Properties of Lie groups

Definition (Homomorphism of Lie groups). Let G, H be Lie groups. A map $J : G \to H$ is a *homomorphism* if it is smooth and for all $g_1, g_2 \in G$, we have

$$J(g_1 g_2) = J(g_1)J(g_2).$$

(the second condition says it is a homomorphism of groups)

Definition (Isomorphic Lie groups). An *isomorphism* of Lie groups is a bijective homomorphism whose inverse is also a homomorphism. Two Lie groups are *isomorphic* if there is an isomorphism between them.

Definition (Compact). A manifold (or topological space) X is *compact* if every open cover of X has a finite subcover.

If the manifold is a subspace of some $\mathbb{R}^n$, then it is compact iff it is closed and bounded.

Definition (Simply connected). A manifold M is *simply connected* if it is connected (there is a path between any two points), and every loop $l : S^1 \to M$ can be contracted to a point. Equivalently, any two paths between any two points can be continuously deformed into each other.

Definition (Fundamental group/First homotopy group). Let M be a manifold, and $x_0 \in M$ be a preferred point. We define $\pi_1(M)$ to be the equivalence classes of loops starting and ending at x_0, where two loops are considered equivalent if they can be continuously deformed into each other.

This has a group structure, with the identity given by the "loop" that stays at x_0 all the time, and composition given by doing one after the other.

3 Lie algebras

3.1 Lie algebras

Definition (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with a *bracket*

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

(i) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (antisymmetry)

(ii) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ for all $X, Y, Z \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$ ((bi)linearity)

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.(Jacobi identity)

Note that linearity in the second argument follows from linearity in the first argument and antisymmetry.

Definition (Dimension of Lie algebra)**.** The *dimension* of a Lie algebra is the dimension of the underlying vector space.

Definition (Structure constants)**.** Given a Lie algebra $\mathfrak{g}$ with a basis $\mathcal{B} = \{T^a\}$, the *structure constants* are $f^{ab}{}_c$ given by

$$[T^a, T^b] = f^{ab}{}_c T^c,$$

Definition (Homomorphism of Lie algebras)**.** A *homomorphism* of Lie algebras $\mathfrak{g}, \mathfrak{h}$ is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ such that

$$[f(X), f(Y)] = f([X, Y]).$$

Definition (Isomorphism of Lie algebras)**.** An *isomorphism* of Lie algebras is a homomorphism with an inverse that is also a homomorphism. Two Lie algebras are *isomorphic* if there is an isomorphism between them.

Definition (Subalgebra)**.** A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a vector subspace that is also a Lie algebra under the bracket.

Definition (Ideal)**.** An *ideal* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

Definition (Derived algebra)**.** The *derived algebra* of a Lie algebra $\mathfrak{g}$ is

$$\mathfrak{i} = [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}_{\mathbb{F}}\{[X, Y] : X, Y \in \mathfrak{g}\},$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ depending on the underlying field.

Definition (Center of Lie algebra)**.** The *center* of a Lie algebra $\mathfrak{g}$ is given by

$$\xi(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Definition (Abelian Lie algebra)**.** A Lie algebra $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Equivalently, if $\xi(\mathfrak{g}) = \mathfrak{g}$.

Definition (Simple Lie algebra)**.** A *simple Lie algebra* is a Lie algebra $\mathfrak{g}$ that is non-abelian and possesses no non-trivial ideals.

3.2 Differentiation

Definition (Tangent vector)**.** Let M be a manifold and write $C^\infty(M)$ for the vector space of smooth functions on M. For $p \in M$, a *tangent vector* is a linear map $v : C^\infty(M) \to \mathbb{R}$ such that for any $f, g \in C^\infty(M)$, we have

$$v(fg) = f(p)v(g) + v(f)g(p).$$

It is clear that this forms a vector space, and we write T_pM for the vector space of tangent vectors at p.

Definition (Smooth curve)**.** A *smooth curve* is a smooth map $\gamma : \mathbb{R} \to M$. More generally, a *curve* is a C^1 function $\mathbb{R} \to M$.

Definition (Derivative)**.** Let $f : M \to N$ be a map between manifolds. The *derivative* of f at $p \in M$ is the linear map

$$\mathrm{D}f_p : T_pM \to T_{f(p)}N$$

given by the formula

$$(\mathrm{D}f_p)(v)(g) = v(g \circ f)$$

for $v \in T_pM$ and $g \in C^\infty(N)$.

3.3 Lie algebras from Lie groups

Definition (Lie algebra of a Lie group)**.** Let G be a Lie group. The *Lie algebra* of G, written $\mathcal{L}(G)$ or $\mathfrak{g}$, is the tangent space T_eG under the natural Lie bracket.

3.4 The exponential map

Definition (Left and right translation)**.** For each $h \in G$, we define the *left and right translation maps*

$$\begin{aligned} L_h : G &\to G \\ g &\mapsto hg, \\ R_h : G &\to G \\ g &\mapsto gh. \end{aligned}$$

These maps are bijections, and in fact diffeomorphisms (i.e. smooth maps with smooth inverses), because they have smooth inverses $L_{h^{-1}}$ and $R_{h^{-1}}$ respectively.

Definition (Vector field)**.** A *vector field* V of G specifies a tangent vector

$$V(g) \in T_gG$$

at each point $g \in G$. Suppose we can pick coordinates $\{x_i\}$ on some subset of G, and write

$$v(g) = v^i(g)\frac{\partial}{\partial x^i} \in T_gG.$$

The vector field is *smooth* if $v^i(g) \in \mathbb{R}$ are all differentiable for any coordinate chart.

Definition (Exponential)**.** Let $M \in \mathrm{Mat}_n(\mathbb{F})$ be a matrix. The *exponential* is defined by

$$\exp(M) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} M^\ell \in \mathrm{Mat}_n(\mathbb{F}).$$

4 Representations of Lie algebras

4.1 Representations of Lie groups and algebras

Definition (Representation of group)**.** Let G be a group and V be a (finite-dimensional) vector space over a field $\mathbb{F}$. A *representation* of G on V is given by specifying invertible linear maps $D(g) : V \to V$ (i.e. $D(g) \in \mathrm{GL}(V)$) for each $g \in G$ such that

$$D(gh) = D(g)D(h)$$

for all $g, h \in G$. In the case where G is a Lie group and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we require that the map $D : G \to \mathrm{GL}(V)$ is smooth.

The space V is known as the *representation space*, and we often write the representation as the pair (V, D).

Definition (Representation of Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. A *representation* ρ of $\mathfrak{g}$ on a vector space V is a collection of linear maps

$$\rho(X) \in \mathfrak{gl}(V),$$

for each $X \in \mathfrak{g}$, i.e. $\rho(X) : V \to V$ is a linear map, not necessarily invertible. These are required to satisfy the conditions

$$[\rho(X_1), \rho(X_2)] = \rho([X_1, X_2])$$

and

$$\rho(\alpha X_1 + \beta X_2) = \alpha\rho(X_1) + \beta\rho(X_2).$$

The vector space V is known as the *representation space*. Similarly, we often write the representation as (V, ρ).

Definition (Dimension of representation)**.** The *dimension* of a representation is the dimension of the representation space.

Definition (Trivial representation)**.** Let $\mathfrak{g}$ be a Lie algebra of dimension D. The *trivial representation* is the representation $d_0 : \mathfrak{g} \to \mathbb{F}$ given by $d_0(X) = 0$ for all $X \in \mathfrak{g}$. This has dimension 1.

Definition (Fundamental representation)**.** Let $\mathfrak{g} = \mathcal{L}(G)$ for $G \subseteq \mathrm{Mat}_n(\mathbb{F})$. The *fundamental representation* is given by $d_f : \mathfrak{g} \to \mathrm{Mat}_n(\mathbb{F})$ given by

$$d_f(X) = X$$

This has $\dim(d_f) = n$.

Definition (Adjoint representation)**.** All Lie algebras come with an *adjoint representation* d_{Adj} of dimension $\dim(\mathfrak{g}) = D$. This is given by mapping $X \in \mathfrak{g}$ to the linear map

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X, Y] \end{aligned}$$

By linearity of the bracket, this is indeed a linear map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Definition (Homomorphism of representations)**.** Let $(V_1, \rho_1), (V_2, \rho_2)$ be representations of $\mathfrak{g}$. A *homomorphism* $f : (V_1, \rho_1) \to (V_2, \rho_2)$ is a linear map $f : V_1 \to V_2$ such that for all $X \in \mathfrak{g}$, we have

$$f(\rho_1(X)(v)) = \rho_2(X)(f(v))$$

for all $v \in V_1$. Alternatively, we can write this as

$$f \circ \rho_1 = \rho_2 \circ f.$$

In other words, the following diagram commutes for all $X \in \mathfrak{g}$:

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \big\downarrow{\scriptstyle \rho_1(X)} & & \big\downarrow{\scriptstyle \rho_2(X)} \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

Definition (Isomorphism of representations)**.** Two $\mathfrak{g}$-vector spaces V_1, V_2 are *isomorphic* if there is an invertible homomorphism $f : V_1 \to V_2$.

Definition (Invariant subspace)**.** Let ρ be a representation of a Lie algebra $\mathfrak{g}$ with representation space V. An *invariant subspace* is a subspace $U \subseteq V$ such that

$$\rho(X)u \in U$$

for all $X \in \mathfrak{g}$ and $u \in U$.

The *trivial subspaces* are $U = \{0\}$ and V.

Definition (Irreducible representation)**.** An *irreducible representation* is a representation with no non-trivial invariant subspaces. They are referred to as *irreps*.

4.2 Complexification and correspondence of representations

Definition (Complexification I)**.** Let V be a real vector space. We pick a basis $\{T^a\}$ of V. We define the *complexification* of V, written $V_\mathbb{C}$ as the *complex* linear span of $\{T^a\}$, i.e.

$$V_\mathbb{C} = \left\{ \sum \lambda_a T^a : \lambda_a \in \mathbb{C} \right\}.$$

There is a canonical inclusion $V \hookrightarrow V_\mathbb{C}$ given by sending $\sum \lambda_a T^a$ to $\sum \lambda_a T^a$ for $\lambda_a \in \mathbb{R}$.

Definition (Complexification II)**.** Let V be a real vector space. The *complexification* of V has underlying vector space $V_\mathbb{C} = V \oplus V$. Then the action of a complex number $\lambda = a + bi$ on (u_1, u_2) is given by

$$\lambda(u_1, u_2) = (au_1 - bu_2, au_2 + bu_1).$$

This gives $V_\mathbb{C}$ the structure of a complex vector space. We have an inclusion $V \to V_\mathbb{C}$ by inclusion into the first factor.

Definition (Complexification III)**.** Let V be a real vector space. The *complexification* of V is the tensor product $V \otimes_\mathbb{R} \mathbb{C}$, where $\mathbb{C}$ is viewed as an $(\mathbb{R}, \mathbb{C})$-bimodule.

Definition (Real form)**.** Let $\mathfrak{g}$ be a *complex* Lie algebra. A *real form* of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}_\mathbb{C} = \mathfrak{g}$.

4.3 Representations of $\mathfrak{su}(2)$

4.4 New representations from old

Definition (Conjugate representation)**.** Let ρ be a representation of a real Lie algebra $\mathfrak{g}$ on $\mathbb{C}^n$. We define the *conjugate representation* by

$$\bar{\rho}(X) = \rho(X)^*$$

for all $X \in \mathfrak{g}$.

Definition (Direct sum)**.** Let V, W be vector spaces. The *direct sum* $V \oplus W$ is given by

$$V \oplus W = \{v \oplus w : v \in V, w \in W\}$$

with operations defined by

$$\begin{aligned} (v_1 \oplus w_1) + (v_2 \oplus w_2) &= (v_1 + v_2) \oplus (w_1 + w_2) \\ \lambda(v \oplus w) &= (\lambda v) \oplus (\lambda w). \end{aligned}$$

We often suggestively write $v \oplus w$ as $v + w$. This has dimension

$$\dim(V \oplus W) = \dim V + \dim W.$$

Definition (Sum representation)**.** Suppose ρ_1 and ρ_2 are representations of $\mathfrak{g}$ with representation spaces V_1 and V_2 of dimensions d_1 and d_2. Then $V_1 \oplus V_2$ is a representation space with representation $\rho_1 \oplus \rho_2$ given by

$$(\rho_1 \oplus \rho_2)(X) \cdot (v_1 \oplus v_2) = (\rho_1(X)(v_1)) \oplus (\rho_2(X)(v_2)).$$

In coordinates, if R_i is the matrix for ρ_i, then the matrix of $\rho_1 \oplus \rho_2$ is given by

$$(R_1 \oplus R_2)(X) = \begin{pmatrix} R_1(X) & 0 \\ 0 & R_2(X) \end{pmatrix}$$

The dimension of this representation is $d_1 + d_2$.

Definition (Tensor product)**.** Let V, W be vector spaces. The *tensor product* $V \otimes W$ is spanned by elements $v \otimes w$, where $v \in V$ and $w \in W$, where we identify

$$\begin{aligned} v \otimes (\lambda_1 w_1 + \lambda_2 w_2) &= \lambda_1 (v \otimes w_1) + \lambda_2 (v \otimes w_2) \\ (\lambda_1 v_1 + \lambda_2 v_2) \otimes w &= \lambda_1 (v_1 \otimes w) + \lambda_2 (v_2 \otimes w) \end{aligned}$$

This has dimension

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

More explicitly, if $e^1, \cdots, e^n$ is a basis of V and $f^1, \cdots, f^m$ is a basis for W, then $\{e^i \otimes f^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$.

Given any two maps $F : V \to V'$ and $G : W \to W'$, we define $F \otimes G : V \otimes W \to V' \otimes W'$ by

$$(F \otimes G)(v \otimes w) = (F(v)) \otimes (G(w)),$$

and then extending linearly.

Definition (Tensor product representation)**.** Let $\mathfrak{g}$ be a Lie algebra, and ρ_1, ρ_2 be representations of $\mathfrak{g}$ with representation spaces V_1, V_2. We define the *tensor product representation* $\rho_1 \otimes \rho_2$ with representation space $V_1 \otimes V_2$ given by

$$(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes I_2 + I_1 \otimes \rho_2(X) : V_1 \otimes V_2 \to V_1 \otimes V_2,$$

where I_1 and I_2 are the identity maps on V_1 and V_2.

Definition (Completely reducible representation)**.** If (ρ, V) is a representation such that there is a basis of V in which ρ looks like

$$\rho(X) = \begin{pmatrix} \rho_1(X) & 0 & \cdots & 0 \\ 0 & \rho_2(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_n(X) \end{pmatrix},$$

then it is called *completely reducible*. In this case, we have

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n.$$

4.5 Decomposition of tensor product of $\mathfrak{su}(2)$ representations

5 Cartan classification

5.1 The Killing form

Definition (Inner product)**.** Given a vector space V over $\mathbb{F}$, an *inner product* is a symmetric bilinear map $i : V \times V \to \mathbb{F}$.

Definition (Non-degenerate inner product)**.** An inner product i is said to be *non-degenerate* if for all $v \in V$ non-zero, there is some $w \in V$ such that

$$i(v, w) \neq 0.$$

Definition (Killing form)**.** The *Killing form* of a Lie algebra $\mathfrak{g}$ is the inner product $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ given by

$$\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y),$$

where tr is the usual trace of a linear map. Since ad is linear, this is bilinear in both arguments, and the cyclicity of the trace tells us this is symmetric.

Definition (Invariant inner product)**.** An inner product κ on a Lie algebra $\mathfrak{g}$ is *invariant* if for any $X, Y, Z \in \mathfrak{g}$, we have

$$\kappa([Z, X], Y) + \kappa(X, [Z, Y]) = 0.$$

Equivalently, we have

$$\kappa(\mathrm{ad}_Z X, Y) + \kappa(X, \mathrm{ad}_Z Y) = 0.$$

Definition (Semi-simple Lie algebra)**.** A Lie algebra is *semi-simple* if it has no abelian non-trivial ideals.

Definition (Real Lie algebra of compact type)**.** We say a real Lie algebra is of *compact type* if there is a basis such that

$$\kappa^{ab} = -\kappa \delta^{ab},$$

for some $\kappa \in \mathbb{R}^+$.

5.2 The Cartan basis

Definition (ad-diagonalizable)**.** Let $\mathfrak{g}$ be a Lie algebra. We say that an element $X \in \mathfrak{g}$ is ad-*diagonalizable* if the associated map

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$$

is diagonalizable.

Definition (Cartan subalgebra)**.** A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra containing only ad-diagonalizable elements.

5.3 Things are real

Definition (String)**.** For $\alpha, \beta \in \Phi$, we define the *α-string passing through β* to be

$$S_{\alpha,\beta} = \{\beta + \rho\alpha \in \Phi : \rho \in \mathbb{Z}\}.$$

5.4 A real subalgebra

Definition (Norm of root)**.** Let $\alpha \in \Phi$ be a root. Then its *length* is

$$|\alpha| = \sqrt{(\alpha, \alpha)} > 0.$$

5.5 Simple roots

Definition (Simple root)**.** A *simple root* is a positive root that cannot be written as a sum of two positive roots. We write Φ_S for the set of simple roots.

5.6 The classification

Definition (Cartan matrix)**.** The *Cartan matrix* A^{ij} is defined as the $r \times r$ matrix

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}.$$

Definition (Dynkin diagram)**.** Given a Cartan matrix, we draw a diagram as follows:

(i) For each simple root $\alpha_{(i)} \in \Phi_S$, we draw a node

(ii) We join the nodes corresponding to $\alpha_{(i)}, \alpha_{(j)}$ with $A^{ij}A^{ji}$ many lines.

(iii) If the roots have different lengths, we draw an arrow from the longer root to the shorter root. This happens when $A^{ij} \neq A^{ji}$.

5.7 Reconstruction

6 Representation of Lie algebras

6.1 Weights

Definition (Weight of representation)**.** Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then if $v_\lambda \in V$ is an eigenvector of $\rho(H)$ for all $H \in \mathfrak{h}$, we say $\lambda \in \mathfrak{h}^*$ is a *weight* of ρ, where

$$\rho(H)v_\lambda = \lambda(H)v_\lambda$$

for all $H \in \mathfrak{h}$.

The *weight set* S_ρ of ρ is the set of all weights.

6.2 Root and weight lattices

Definition (Root lattice)**.** Let $\mathfrak{g}$ be a Lie algebra with simple roots $\alpha_{(i)}$. Then the *root lattice* is defined as

$$\mathcal{L}[\mathfrak{g}] = \mathrm{span}_\mathbb{Z}\{\alpha_{(i)} : i = 1, \cdots, r\}.$$

Definition (Simple co-root)**.** The *simple co-roots* are

$$\check{\alpha}_{(i)} = \frac{2\alpha_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})}.$$

Definition (Co-root lattice)**.** Let $\mathfrak{g}$ be a Lie algebra with simple roots $\alpha_{(i)}$. Then the *co-root lattice* is defined as

$$\check{\mathcal{L}}[\mathfrak{g}] = \mathrm{span}_\mathbb{Z}\{\check{\alpha}_{(i)} : i = 1, \cdots, r\}.$$

Definition (Weight lattice)**.** The *weight lattice* $\mathcal{L}_W[\mathfrak{g}]$ is the *dual* to the co-root lattice:

$$\mathcal{L}_W[\mathfrak{g}] = \check{\mathcal{L}}^*[\mathfrak{g}] = \{\lambda \in \mathfrak{h}^*_\mathbb{R} : (\lambda, \mu) \in \mathbb{Z} \text{ for all } \mu \in \check{\mathcal{L}}[\mathfrak{g}]\}.$$

6.3 Classification of representations

Definition (Highest weight)**.** A *highest weight* of a representation ρ is a weight $\Lambda \in S_\rho$ whose associated eigenvector v_Λ is such that

$$\rho(E^\alpha)v_\Lambda = 0$$

for all $\alpha \in \Phi_+$.

Definition (Dominant integral weight)**.** A *dominant integral weight* is a weight

$$\Lambda = \sum \Lambda^i \omega_{(i)} \in \mathcal{L}_W[\mathfrak{g}],$$

such that $\Lambda^i \geq 0$ for all i.

6.4 Decomposition of tensor products

7 Gauge theories

7.1 Electromagnetism and U(1) gauge symmetry

7.2 General case

8 Lie groups in nature

8.1 Spacetime symmetry

8.2 Possible extensions

8.3 Internal symmetries and the eightfold way