# Part III — Quantum Field Theory
## Theorems with proof

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Michaelmas 2016



Quantum Field Theory is the language in which modern particle physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics. How these fields interact with a classical electromagnetic field is described.

Interactions are described using perturbative theory and Feynman diagrams. This is first illustrated for theories with a purely scalar field interaction, and then for a couplings between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

### Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

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# 0 Introduction

# 1 Classical field theory

## 1.1 Classical fields

**Proposition** (Euler-Lagrange equations)**.** The equations of motion for a field are given by the *Euler-Lagrange equations*:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

## 1.2 Lorentz invariance

## 1.3 Symmetries and Noether's theorem for field theories

**Theorem** (Noether's theorem)**.** Every continuous symmetry of $\mathcal{L}$ gives rise to a *conserved current* $j^\mu(x)$ such that the equation of motion implies that

$$\partial_\mu j^\mu = 0.$$

More explicitly, this gives

$$\partial_0 j^0 + \nabla \cdot \mathbf{j} = 0.$$

A conserved current gives rise to a *conserved charge*

$$Q = \int_{\mathbb{R}^3} j^0 \mathrm{d}^3\mathbf{x},$$

since

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \frac{\mathrm{d}j^0}{\mathrm{d}t}\,\mathrm{d}^3\mathbf{x} \\ &= -\int_{\mathbb{R}^3} \nabla \cdot \mathbf{j}\,\mathrm{d}^3\mathbf{x} \\ &= 0, \end{aligned}$$

assuming that $j^i \to 0$ as $|\mathbf{x}| \to \infty$.

*Proof.* Consider making an arbitrary transformation of the field $\phi_a \mapsto \phi_a + \delta\phi_a$. We then have

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu\phi_a) \\ &= \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right]\delta\phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right). \end{aligned}$$

When the equations of motion are satisfied, we know the first term always vanishes. So we are left with

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right).$$

If the specific transformation $\delta\phi_a = X_a$ we are considering is a *symmetry*, then $\delta\mathcal{L} = 0$ (this is the definition of a symmetry). In this case, we can define a conserved current by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a,$$

and by the equations above, this is actually conserved. □

## 1.4 Hamiltonian mechanics

# 2 Free field theory

## 2.1 Review of simple harmonic oscillator

## 2.2 The quantum field

## 2.3 Real scalar fields

**Proposition.** We have

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x}).$$

**Proposition.** The canonical commutation relations of $\phi, \pi$, namely

$$\begin{aligned}
[\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0\\
[\pi(\mathbf{x}), \pi(\mathbf{y})] &= 0\\
[\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^3(\mathbf{x}-\mathbf{y})
\end{aligned}$$

are equivalent to

$$\begin{aligned}
[a_\mathbf{p}, a_\mathbf{q}] &= 0\\
[a_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] &= 0\\
[a_\mathbf{p}, a_\mathbf{q}^\dagger] &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).
\end{aligned}$$

*Proof.* We will only prove one small part of the equivalence, as the others are similar tedious and boring computations, and you are not going to read it anyway. We will use the commutation relations for the $a_\mathbf{p}$ to obtain the commutation relations for $\phi$ and $\pi$. We can compute

$$\begin{aligned}
&[\phi(\mathbf{x}), \pi(\mathbf{y})]\\
&= \int \frac{\mathrm{d}^3\mathbf{p}\;\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{(-i)}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}}\left(-[a_\mathbf{p}, a_\mathbf{q}^\dagger]e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} + [a_\mathbf{p}^\dagger, a_\mathbf{q}]e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}}\right)\\
&= \int \frac{\mathrm{d}^3\mathbf{p}\;\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{(-i)}{2}\sqrt{\frac{\omega_\mathbf{q}}{\omega_\mathbf{p}}}(2\pi)^3\left(-\delta^3(\mathbf{p}-\mathbf{q})e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} - \delta^3(\mathbf{q}-\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}}\right)\\
&= \frac{(-i)}{2}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(-e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\right)\\
&= i\delta^3(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

Note that to prove the inverse direction, we have to invert the relation between $\phi(\mathbf{x}), \pi(\mathbf{x})$ and $a_\mathbf{p}, a_\mathbf{p}^\dagger$ and express $a_\mathbf{p}$ and $a_\mathbf{p}^\dagger$ in terms of $\phi$ and $\pi$ by using

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\;\phi(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}} &= \frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a_{-\mathbf{p}} + a_\mathbf{p}^\dagger\right)\\
\int \mathrm{d}^3\mathbf{x}\;\pi(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}} &= (-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(a_{-\mathbf{p}} - a_\mathbf{p}^\dagger\right).
\end{aligned}$$

□

**Proposition.** The expression

$$\int\frac{\mathrm{d}^3\mathbf{p}}{2E_\mathbf{p}}$$

is Lorentz-invariant, where

$$E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$$

for some fixed $m$.

*Proof.* We know $\int \mathrm{d}^4 p$ certainly is Lorentz invariant, and

$$m^2 = p_\mu p^\mu = p^2 = p_0^2 - \mathbf{p}^2$$

is also a Lorentz-invariant quantity. So for any $m$, the expression

$$\int \mathrm{d}^4 p \; \delta(p_0^2 - \mathbf{p}^2 - m^2)$$

is also Lorentz invariant. Writing

$$E_{\mathbf{p}}^2 = p_0^2 = \mathbf{p}^2 + m^2,$$

integrating over $p_0$ in the integral gives

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2p_0} = \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}},$$

and this is Lorentz invariant. □

**Proposition.** The expression

$$2E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{q})$$

is Lorentz invariant.

*Proof.* We have

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}} \cdot (2E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{q})) = 1.$$

Since the RHS is Lorentz invariant, and the measure is Lorentz invariant, we know $2E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{q})$ must be Lorentz invariant. □

## 2.4 Complex scalar fields

## 2.5 The Heisenberg picture

**Proposition.** Let $A$ and $B$ be operators. Then

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots.$$

In particular, if $[A, B] = cB$ for some constant $c$, then we have

$$e^A B e^{-A} = e^c B.$$

*Proof.* For $\lambda$ a real variable, note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}(e^{\lambda A}Be^{-\lambda A}) &= \lim_{\varepsilon\to 0}\frac{e^{(\lambda+\varepsilon)A}Be^{-(\lambda+\varepsilon)A} - e^{\lambda A}Be^{-\lambda A}}{\varepsilon}\\ &= \lim_{\varepsilon\to 0} e^{\lambda A}\frac{e^{\varepsilon A}Be^{-\varepsilon A} - B}{\varepsilon}e^{-\lambda A}\\ &= \lim_{\varepsilon\to 0} e^{\lambda A}\frac{(1+\varepsilon A)B(1-\varepsilon A) - B + o(\varepsilon)}{\varepsilon}e^{-\lambda A}\\ &= \lim_{\varepsilon\to 0} e^{\lambda A}\frac{(\varepsilon(AB - BA) + o(\varepsilon))}{\varepsilon}e^{-\lambda A}\\ &= e^{\lambda A}[A, B]e^{-\lambda A}.\end{aligned}$$

So by induction, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(e^{\lambda A}Be^{-\lambda A}) = e^{\lambda A}[A, [A, \cdots [A, B]\cdots]]e^{-\lambda A}.$$

Evaluating these at $\lambda = 0$, we obtain a power series representation

$$e^{\lambda A}Be^{-\lambda A} = B + \lambda[A, B] + \frac{\lambda^2}{2}[A, [A, B]] + \cdots .$$

Putting $\lambda = 1$ then gives the desired result. □

## 2.6 Propagators

**Proposition.**

$$D(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}.$$

**Proposition.** We have

$$\Delta(x-y) = D(x-y) - D(y-x).$$

*Proof.*

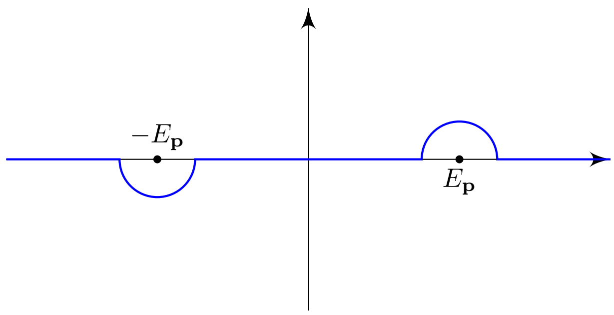
$$\Delta(x-y) = [\phi(x), \phi(y)] = \langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = D(x-y) - D(y-x),$$

where the second equality follows as $[\phi(x), \phi(y)]$ is just an ordinary function. □

**Proposition.** We have

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{i}{p^2 - m^2}e^{-ip\cdot(x-y)}.$$

This expression is *a priori* ill-defined since for each $\mathbf{p}$, the integrand over $p^0$ has a pole whenever $(p^0)^2 = \mathbf{p}^2 + m^2$. So we need a prescription for avoiding this. We replace this with a complex contour integral with contour given by



*Proof.* To compare with our previous computations of $D(x-y)$, we evaluate the $p^0$ integral for each $\mathbf{p}$. Writing

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_{\mathbf{p}}^2} = \frac{1}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})},$$

we see that the residue of the pole at $p^0 = \pm E_{\mathbf{p}}$ is $\pm \frac{1}{2E_{\mathbf{p}}}$.

When $x^0 > y^0$, we close the contour in the lower plane $p^0 \to -i\infty$, so $e^{-p^0(x^0 - t^0)} \to e^{-\infty} = 0$. Then $\int \mathrm{d}p^0$ picks up the residue at $p^0 = E_{\mathbf{p}}$. So the Feynman propagator is

$$\begin{aligned}\Delta_F(x-y) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^4} \frac{(-2\pi i)}{2E_{\mathbf{p}}} i e^{-iE_{\mathbf{p}}(x^0 - y^0) + i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi^3)} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}\\ &= D(x-y)\end{aligned}$$

When $x^0 < y^0$, we close the contour in the upper-half plane. Then we have

$$\begin{aligned}\Delta_F(x-y) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^4} \frac{2\pi i}{-2E_{\mathbf{p}}} i e^{iE_{\mathbf{p}}(x^0 - y^0) + i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(y^0 - x^0) - i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\end{aligned}$$

We again use the trick of flipping the sign of $\mathbf{p}$ to obtain

$$\begin{aligned}&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(y-x)}\\ &= D(y-x).\end{aligned}$$

□

# 3 Interacting fields

## 3.1 Interaction Lagrangians

## 3.2 Interaction picture

**Proposition.** In the interaction picture, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t} O_I = i[H_0, O_I]$$
$$\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle_I = H_I |\psi(t)\rangle_I,$$

where $H_I$ is defined by

$$H_I = (H_{\text{int}})_I = e^{iH_0 t}(H_{\text{int}})_S e^{-iH_0 t}.$$

*Proof.* The first part we've essentially done before, but we will write it out again for completeness.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(e^{iH_0 t} O_S e^{-iH_0 t}) &= \lim_{\varepsilon \to 0} \frac{e^{iH_0(t+\varepsilon)} O_S e^{-iH_0(t+\varepsilon)} - e^{iH_0 t} O_S e^{-iH_0 t}}{\varepsilon}\\
&= \lim_{\varepsilon \to 0} e^{iH_0 t} \frac{e^{iH_0 \varepsilon} O_S e^{-iH_0 \varepsilon} - O_S}{\varepsilon} e^{-iH_0 t}\\
&= \lim_{\varepsilon \to 0} e^{iH_0 t} \frac{(1 + iH_0 \varepsilon) O_S (1 - iH_0 \varepsilon) - O_S + o(\varepsilon)}{\varepsilon} e^{-iH_0 t}\\
&= \lim_{\varepsilon \to 0} e^{iH_0 t} \frac{i\varepsilon (H_0 O_S - O_S H_0) + o(\varepsilon)}{\varepsilon} e^{-iH_0 t}\\
&= e^{iH_0 t} i[H_0, O_S] e^{-iH_0 t}\\
&= i[H_0, O_I].
\end{aligned}$$

For the second part, we start with Schrödinger's equation

$$i \frac{\mathrm{d}\, |\psi(t)\rangle_S}{\mathrm{d}t} = H_S |\psi(t)\rangle_S.$$

Putting in our definitions, we have

$$i \frac{\mathrm{d}}{\mathrm{d}t}(e^{-iH_0 t} |\psi(t)\rangle_I) = (H_0 + H_{\text{int}})_S e^{-iH_0 t} |\psi(t)\rangle_I.$$

Using the product rule and chain rule, we obtain

$$H_0 e^{-iH_0 t} |\psi(t)\rangle_I + i e^{-iH_0 t} \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle_I = (H_0 + H_{\text{int}})_S e^{-iH_0 t} |\psi(t)\rangle_I.$$

Rearranging gives us

$$i \frac{\mathrm{d}\, |\psi(t)\rangle_I}{\mathrm{d}t} = e^{iH_0 t}(H_{\text{int}})_S e^{-iH_0 t} |\psi(t)\rangle_I. \qquad \square$$

**Proposition** (Dyson's formula)**.** The solution to the Schrödinger equation in the interaction picture is given by

$$U(t, t_0) = T \exp\left(-i \int_{t_0}^{t} H_I(t')\, dt'\right),$$

where $T$ stands for *time ordering*: operators evaluated at earlier times appear to the right of operators evaluated at leter times when we write out the power series. More explicitly,

$$T\{O_1(t_1)O_2(t_2)\} = \begin{cases} O_1(t_1)O_2(t_2) & t_1 > t_2 \\ O_2(t_2)O_1(t_1) & t_2 > t_1 \end{cases}.$$

We do not specify what happens when $t_1 = t_2$, but it doesn't matter in our case since the operators are then equal.

Thus, we have

$$\begin{aligned} U(t, t_0) = \mathbf{1} - i\int_{t_0}^t \mathrm{d}t'\ H_I(t') + \frac{(-i)^2}{2}\bigg\{\int_{t_0}^t \mathrm{d}t' \int_{t'}^t \ \mathrm{d}t'' H_I(t'')H_I(t') \\ + \int_{t_0}^t \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\ H_I(t')H_I(t'')\bigg\} + \cdots. \end{aligned}$$

We now notice that

$$\int_{t_0}^t \mathrm{d}t' \int_{t'}^t \ \mathrm{d}t''\ H_I(t'')H_I(t') = \int_{t_0}^t \mathrm{d}t'' \int_{t_0}^{t''} \ \mathrm{d}t'\ H_I(t'')H_I(t'),$$

since we are integrating over all $t_0 \leq t' \leq t'' \leq t$ on both sides. Swapping $t'$ and $t''$ shows that the two second-order terms are indeed the same, so we have

$$U(t, t_0) = \mathbf{1} - i\int_{t_0}^t \mathrm{d}t'\ H_I(t') + (-i)^2 \int_{t_0}^t \mathrm{d}t' \int_{t_0}^{t'} \ \mathrm{d}t''\ H_I(t'')H_I(t') + \cdots.$$

*Proof.* This from the last presentation above, using the fundamental theorem of calculus. □

## 3.3 Wick's theorem

**Proposition.** Let $\phi$ be a real scalar field. Then

$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2) = \Delta_F(x_1 - x_2).$$

*Proof.* We write $\phi = \phi^+ + \phi^-$, where

$$\phi^+(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} a_\mathbf{p} e^{-ip\cdot x},\quad \phi^-(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} a_\mathbf{p}^\dagger e^{ip\cdot x}.$$

Then we get normal order if all the $\phi^-$ appear before the $\phi^+$.

When $x^0 > y^0$, we have

$$\begin{aligned} T\phi(x)\phi(y) &= \phi(x)\phi(y) \\ &= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + [\phi^+(x), \phi^-(y)] \\ &\quad\quad + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y). \end{aligned}$$

So we have

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(x-y).$$

By symmetry, for $y^0 > x^0$, we have

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x).$$

Recalling the definition of the Feynman propagator, we have

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y). \qquad \square$$

**Proposition.** For a complex scalar field, we have

$$\overbracket{\psi(x)\psi^\dagger}(y) = \Delta_F(x-y) = \overbracket{\psi^\dagger(y)\psi}(x),$$

whereas

$$\overbracket{\psi(x)\psi}(y) = 0 = \overbracket{\psi^\dagger(x)\psi^\dagger}(y).$$

**Theorem** (Wick's theorem)**.** For any collection of fields $\phi_1 = \phi_1(x_1), \phi_2 = \phi_2(x_2), \cdots$, we have

$$T(\phi_1\phi_2\cdots\phi_n) = :\phi_1\phi_2\cdots\phi_n: + \text{ all possible contractions}$$

*Proof sketch.* We will provide a rough proof sketch.

By definition this is true for $n = 2$. Suppose this is true for $\phi_2\cdots\phi_n$, and now add $\phi_1$ with

$$x_1^0 > x_k^0 \text{ for all } k \in \{2, \cdots, n\}.$$

Then we have

$$T(\phi_1\cdots\phi_n) = (\phi_1^+ + \phi_1^-)(:\phi_2\cdots\phi_n: + \text{ other contractions}).$$

The $\phi^-$ term stays where it is, as that already gives normal ordering, and the $\phi_1^+$ term has to make its way past the $\phi_k^-$ operators. So we can write the RHS as a normal-ordered product. Each time we move past the $\phi_k^-$, we pick up a contraction $\overbracket{\phi_1\phi_k}$. $\square$

## 3.4 Feynman diagrams

## 3.5 Amplitudes

## 3.6 Correlation functions and vacuum bubbles

**Lemma.** The free vacuum and interacting vacuum are related via

$$|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} U_I(t, -\infty)\,|0\rangle = \frac{1}{\langle\Omega|0\rangle} U_S(t, -\infty)\,|0\rangle\,.$$

Similarly, we have

$$\langle\Omega| = \frac{1}{\langle\Omega|0\rangle}\,\langle 0|\, U(\infty, t).$$

*Proof.* Note that we have the last equality because $U_I(t, -\infty)$ and $U_S(t, -\infty)$ differs by factors of $e^{-iHt_0}$ which acts as the identity on $|0\rangle$.

Consider an arbitrary state $|\Psi\rangle$. We want to show that

$$\langle\Psi| U(t, -\infty) |0\rangle = \langle\Psi|\Omega\rangle \langle\Omega|0\rangle .$$

The trick is to consider a complete set of eigenstates $\{\Omega, |x\rangle\}$ for $H$ with energies $E_x$. Then we have

$$U(t, t_0) |x\rangle = e^{iE_x(t_0 - t)} |x\rangle .$$

Also, we know that

$$\mathbf{1} = |\Omega\rangle \langle\Omega| + \int \mathrm{d}x \ |x\rangle \langle x| .$$

So we have

$$\begin{aligned}\langle\Psi| U(t, -\infty) |0\rangle &= \langle\Psi| U(t, -\infty) \left(|\Omega\rangle \langle\Omega| + \int \mathrm{d}x\, |x\rangle \langle x|\right) |0\rangle \\ &= \langle\Psi|\Omega\rangle \langle\Omega|0\rangle + \lim_{t' \to -\infty} \int \mathrm{d}x\ e^{iE_x(t'-t)} \langle\Psi|x\rangle \langle x|0\rangle .\end{aligned}$$

We now claim that the second term vanishes in the limit. As in most of the things in the course, we do not have a rigorous justification for this, but it is not too far-stretched to say so since for any well-behaved (genuine) function $f(x)$, the *Riemann-Lebesgue lemma* tells us that for any fixed $a, b \in \mathbb{R}$, we have

$$\lim_{\mu \to \infty} \int_a^b f(x) e^{i\mu x}\ \mathrm{d}x = 0.$$

If this were to hold, then

$$\langle\Psi| U(t, -\infty) |0\rangle = \langle\Psi|\Omega\rangle \langle\Omega|0\rangle .$$

So the result follows. The other direction follows similarly. □

**Proposition.**

$$G^{(n)}(x_1, \cdots, x_n) = \frac{\langle 0| T\phi_I(x_1) \cdots \phi_I(x_n) S |0\rangle}{\langle 0| S |0\rangle}.$$

*Proof.* We can wlog consider the specific example $t_1 > t_2 > \cdots > t_n$. We start by working on the denominator. We have

$$\langle 0| S |0\rangle = \langle 0| U(\infty, t) U(t, -\infty) |0\rangle = \langle 0|\Omega\rangle \langle\Omega|\Omega\rangle \langle\Omega|0\rangle .$$

For the numerator, we note that $S$ can be written as

$$S = U_I(\infty, t_1) U_I(t_1, t_2) \cdots U_I(t_n, -\infty).$$

So after time-ordering, we know the numerator of the right hand side is

$$\langle 0| U_I(\infty, t_1) \phi_I(x_1) U_I(t_1, t_2) \phi_I(x_2) \cdots \phi_I(n) U_I(t_n, -\infty) |0\rangle$$

$$\begin{aligned}&\langle 0| U_I(\infty, t_1) \phi_I(x_1) U_I(t_1, t_2) \phi_I(x_2) \cdots \phi_I(n) U_I(t_n, -\infty) |0\rangle \\ &= \langle 0| U_I(\infty, t_1) \phi_H(x_1) \cdots \phi_H(x_n) U_I(t_n, -\infty) |0\rangle \\ &= \langle 0|\Omega\rangle \langle\Omega| T\phi_H(x_1) \cdots \phi_H(x_n) |\Omega\rangle \langle\Omega|0\rangle .\end{aligned}$$

So the result follows. □

# 4 Spinors

## 4.1 The Lorentz group and the Lorentz algebra

**Proposition.**

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}.$$

## 4.2 The Clifford algebra and the spin representation

**Proposition.** Suppose $\gamma^\mu$ is a representation of the Clifford algebra. Then the matrices given by

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

define a representation of the Lorentz algebra.

**Lemma.**

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}.$$

*Proof.*

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \left[\frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\eta^{\mu\nu}, \gamma^\rho\right] \\ &= \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu \\ &= \gamma^\mu(\eta^{\nu\rho} - \gamma^\rho\gamma^\nu) - (\eta^{\mu\rho} - \gamma^\mu\gamma^\rho)\gamma^\nu \\ &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}. \end{aligned}$$

□

*Proof of proposition.* We have to show that

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho} S^{\mu\sigma} - \eta^{\mu\rho} S^{\nu\sigma} + \eta^{\mu\sigma} S^{\nu\rho} - \eta^{\nu\sigma} S^{\mu\rho}.$$

The proof involves, again, writing everything out. Using the fact that $\eta^{\rho\sigma}$ commutes with everything, we know

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \frac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] \\ &= \frac{1}{2}\big([S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \gamma^\rho[S^{\mu\nu}, \gamma^\sigma]\big) \\ &= \frac{1}{2}\big(\gamma^\mu\eta^{\nu\rho}\gamma^\sigma - \gamma^\nu\eta^{\mu\rho}\gamma^\sigma + \gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \gamma^\rho\gamma^\nu\eta^{\mu\sigma}\big). \end{aligned}$$

Then the result follows form the fact that

$$\gamma^\mu\gamma^\sigma = 2S^{\mu\sigma} + \eta^{\mu\sigma}.$$

□

**Proposition.** Let $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, and define

$$\Omega_{ij} = -\varepsilon_{ijk}\phi_k.$$

Then in the chiral representation of $S$, writing $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, we have

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} & \mathbf{0} \\ \mathbf{0} & e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

**Proposition.** Write $\boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3)$. Then if

$$\Omega_{0i} = -\Omega_{i0} = -\chi_i,$$

then

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

is the boost in the $\boldsymbol{\chi}$ direction, and

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

## 4.3 Properties of the spin representation

**Proposition.** We have

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger.$$

*Proof.* This is true by checking all possible $\mu$. □

**Proposition.**

$$S[\Lambda]^{-1} = \gamma^0 S[\Lambda]^\dagger \gamma^0,$$

where $S[\Lambda]^\dagger$ denotes the Hermitian conjugate as a matrix (under the usual basis).

*Proof.* We note that

$$(S^{\mu\nu})^\dagger = \frac{1}{4}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = -\gamma^0\left(\frac{1}{4}[\gamma^\mu, \gamma^\nu]\right)\gamma^0 = -\gamma^0 S^{\mu\nu}\gamma^0.$$

So we have

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\mu\nu}(S^{\mu\nu})^\dagger\right) = \exp\left(-\frac{1}{2}\gamma^0\Omega_{\mu\nu}S^{\mu\nu}\gamma^0\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0,$$

using the fact that $(\gamma^0)^2 = \mathbf{1}$ and $\exp(-A) = (\exp A)^{-1}$. Multiplying both sides on both sides by $\gamma^0$ gives the desired formula. □

**Proposition.** If $\psi$ is a Dirac spinor, then

$$\bar\psi = \psi^\dagger\gamma^0$$

is a cospinor.

*Proof.* $\bar\psi$ transforms as

$$\bar\psi \mapsto \psi^\dagger S[\Lambda]^\dagger\gamma^0 = \psi^\dagger\gamma^0(\gamma^0 S[\Lambda]^\dagger\gamma^0) = \bar\psi S[\Lambda]^{-1}.$$ □

**Corollary.** For any spinor $\psi$, the quantity $\bar\psi\psi$ is a scalar, i.e. it doesn't transform under a Lorentz transformation.

**Proposition.** We have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu.$$

*Proof.* We work infinitesimally. So this reduces to

$$\left(\mathbf{1} - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(\mathbf{1} + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \left(\mathbf{1} + \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu.$$

This becomes

$$[S^{\rho\sigma}, \gamma^\mu] = -(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu.$$

But we can use the explicit formula for $M$ to compute

$$-(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu = (\eta^{\sigma\mu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\sigma{}_\nu)\gamma^\nu = \gamma^\rho\eta^{\sigma\mu} - \gamma^\sigma\eta^{\rho\mu},$$

and we have previously shown this is equal to $[S^{\rho\sigma}, \gamma^\mu]$. □

**Corollary.** The object $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector, and $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor.

## 4.4 The Dirac equation

## 4.5 Chiral/Weyl spinors and $\gamma^5$

**Proposition.** A Dirac spinor is the direct sum of a left-handed chiral spinor and a right-handed one.

**Proposition.** We have

$$\{\gamma^\mu, \gamma^5\} = 0, \quad (\gamma^5)^2 = \mathbf{1}$$

for all $\gamma^\mu$, and

$$[S^{\mu\nu}, \gamma^5] = 0.$$

**Proposition.**

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0.$$

*Proof.* We have

$$P_\pm^2 = \frac{1}{4}(\mathbf{1} \pm \gamma^5)^2 = \frac{1}{4}(\mathbf{1} + (\gamma^5)^2 \pm 2\gamma^5) = \frac{1}{2}(\mathbf{1} \pm \gamma^5),$$

and

$$P_+P_- = \frac{1}{4}(\mathbf{1} + \gamma^5)(\mathbf{1} - \gamma^5) = \frac{1}{4}(\mathbf{1} - (\gamma^5)^2) = 0.$$ □

## 4.6 Parity operator

**Axiom.** The parity operator $P$ acts on the spinors as $\gamma^0$.

**Proposition.**

$$P : \psi \mapsto \gamma^0\psi, \quad P : \bar{\psi} \mapsto \bar{\psi}\gamma^0.$$

**Proposition.** We have

$$P : \gamma^5 \mapsto -\gamma^5.$$

*Proof.* The $\gamma^1, \gamma^2, \gamma^3$ each pick up a negative sign, while $\gamma^0$ does not change. □

**Proposition.** We have

$$P : P_\pm \mapsto P_\mp.$$

In particular, we have

$$P\psi_\pm = \psi_\mp.$$

## 4.7 Solutions to Dirac's equation

**Proposition.** We have a solution

$$u_{\mathbf{p}} = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}$$

for any 2-component spinor $\xi$ normalized such that $\xi^\dagger \xi = 1$.

*Proof.* We write

$$u_{\mathbf{p}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then our equation gives

$$\begin{aligned} (p \cdot \sigma) u_2 &= m u_1 \\ (p \cdot \bar{\sigma}) u_1 &= m u_2 \end{aligned}$$

Either of these equations can be derived from the other, since

$$\begin{aligned} (p \cdot \sigma)(p \cdot \bar{\sigma}) &= p_0^2 - p_i p_j \sigma^i \sigma^j \\ &= p_0^2 - p_i p_j \delta^{ij} \\ &= p_\mu p^\mu \\ &= m^2. \end{aligned}$$

We try the ansatz

$$u_1 = (p \cdot \sigma)\xi'$$

for a spinor $\xi'$. Then our equation above gives

$$u_2 = \frac{1}{m}(p \cdot \bar{\sigma})(p \cdot \sigma)\xi' = m\xi'.$$

So any vector of the form

$$u_{\mathbf{p}} = A \begin{pmatrix} (p \cdot \sigma)\xi' \\ m\xi' \end{pmatrix}$$

is a solution, for $A$ a constant. To make this look more symmetric, we choose

$$A = \frac{1}{m}, \quad \xi' = \sqrt{p \cdot \bar{\sigma}}\xi,$$

with $\xi$ another spinor. Then we have

$$u_1 = \frac{1}{m}(p \cdot \sigma)\sqrt{p \cdot \bar{\sigma}}\xi = \sqrt{p \cdot \sigma}\xi.$$

This gives our claimed solution. □

## 4.8 Symmetries and currents

# 5 Quantizing the Dirac field

## 5.1 Fermion quantization

**Axiom.** The spinor field operators satisfy

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = 0,$$

and

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}).$$

**Proposition.** The anti-commutation relations above are equivalent to

$$\{c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\dagger}\} = \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3\delta^{rs}\delta^3(\mathbf{p} - \mathbf{q}),$$

and all other anti-commutators vanishing.

**Proposition.**

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s\right).$$

**Proposition.**

$$\overbrace{\psi(x)\bar{\psi}}(y) = T(\psi(x)\bar{\psi}(y)) - {:}\psi(x)\bar{\psi}(y){:} = S_F(x - y).$$

## 5.2 Yukawa theory

## 5.3 Feynman rules

# 6 Quantum electrodynamics

## 6.1 Classical electrodynamics

## 6.2 Quantization of the electromagnetic field

**Theorem.**

$$[a_{\mathbf{p}}^{\lambda}, a_{\mathbf{q}}^{\lambda'}] = [a_{\mathbf{p}}^{\lambda\dagger}, a_{\mathbf{q}}^{\lambda'\dagger}] = 0$$

and

$$[a_{\mathbf{p}}^{\lambda}, a_{\mathbf{q}}^{\lambda'\dagger}] = -\eta^{\lambda\lambda'}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}).$$

**Theorem.** The Feynman propagator for the electromagnetic field, under a general gauge $\alpha$, is

$$\langle 0| TA_\mu(x)A_\nu(y) |0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{-i}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right)e^{-ip\cdot(x-y)}.$$

## 6.3 Coupling to matter in classical field theory

## 6.4 Quantization of interactions

## 6.5 Computations and diagrams