Part III — Quantum Field Theory
Theorems

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Quantum Field Theory is the language in which modern particle physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics. How these fields interact with a classical electromagnetic field is described.

Interactions are described using perturbative theory and Feynman diagrams. This is first illustrated for theories with a purely scalar field interaction, and then for a couplings between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Contents

0 Introduction

1 Classical field theory

1.1 Classical fields

Proposition (Euler-Lagrange equations)**.** The equations of motion for a field are given by the *Euler-Lagrange equations*:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

1.2 Lorentz invariance

1.3 Symmetries and Noether's theorem for field theories

Theorem (Noether's theorem)**.** Every continuous symmetry of $\mathcal{L}$ gives rise to a *conserved current* $j^\mu(x)$ such that the equation of motion implies that

$$\partial_\mu j^\mu = 0.$$

More explicitly, this gives

$$\partial_0 j^0 + \nabla \cdot \mathbf{j} = 0.$$

A conserved current gives rise to a *conserved charge*

$$Q = \int_{\mathbb{R}^3} j^0 \mathrm{d}^3\mathbf{x},$$

since

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \frac{\mathrm{d}j^0}{\mathrm{d}t}\,\mathrm{d}^3\mathbf{x}\\ &= -\int_{\mathbb{R}^3} \nabla \cdot \mathbf{j}\,\mathrm{d}^3\mathbf{x}\\ &= 0,\end{aligned}$$

assuming that $j^i \to 0$ as $|\mathbf{x}| \to \infty$.

1.4 Hamiltonian mechanics

2 Free field theory

2.1 Review of simple harmonic oscillator

2.2 The quantum field

2.3 Real scalar fields

Proposition. We have

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x}).$$

Proposition. The canonical commutation relations of ϕ, π, namely

$$\begin{aligned}[\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0\\ [\pi(\mathbf{x}), \pi(\mathbf{y})] &= 0\\ [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y})\end{aligned}$$

are equivalent to

$$\begin{aligned}[a_{\mathbf{p}}, a_{\mathbf{q}}] &= 0\\ [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] &= 0\\ [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}).\end{aligned}$$

Proposition. The expression

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}}$$

is Lorentz-invariant, where

$$E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$$

for some fixed m.

Proposition. The expression

$$2E_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{q})$$

is Lorentz invariant.

2.4 Complex scalar fields

2.5 The Heisenberg picture

Proposition. Let A and B be operators. Then

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots.$$

In particular, if $[A, B] = cB$ for some constant c, then we have

$$e^A B e^{-A} = e^c B.$$

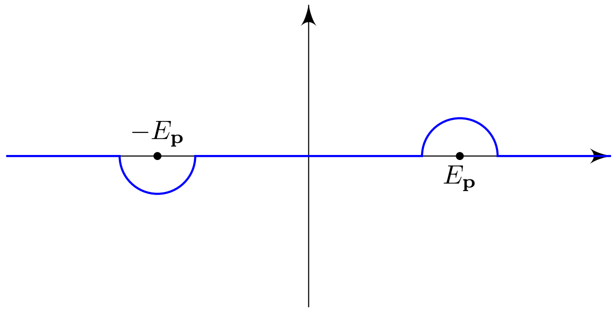
2.6 Propagators

Proposition.
$$D(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}.$$

Proposition. We have
$$\Delta(x-y) = D(x-y) - D(y-x).$$

Proposition. We have
$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}.$$

This expression is *a priori* ill-defined since for each $\mathbf{p}$, the integrand over p^0 has a pole whenever $(p^0)^2 = \mathbf{p}^2 + m^2$. So we need a prescription for avoiding this. We replace this with a complex contour integral with contour given by



3 Interacting fields

3.1 Interaction Lagrangians

3.2 Interaction picture

Proposition. In the interaction picture, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}O_I = i[H_0, O_I]$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle_I = H_I\left|\psi(t)\right\rangle_I,$$

where H_I is defined by

$$H_I = (H_{\mathrm{int}})_I = e^{iH_0t}(H_{\mathrm{int}})_S e^{-iH_0t}.$$

Proposition (Dyson's formula)**.** The solution to the Schrödinger equation in the interaction picture is given by

$$U(t, t_0) = T\exp\left(-i\int_{t_0}^t H_I(t')dt'\right),$$

where T stands for *time ordering*: operators evaluated at earlier times appear to the right of operators evaluated at leter times when we write out the power series. More explicitly,

$$T\{O_1(t_1)O_2(t_2)\} = \begin{cases} O_1(t_1)O_2(t_2) & t_1 > t_2\\ O_2(t_2)O_1(t_1) & t_2 > t_1\end{cases}.$$

We do not specify what happens when $t_1 = t_2$, but it doesn't matter in our case since the operators are then equal.

Thus, we have

$$\begin{aligned}U(t, t_0) = \mathbf{1} - i\int_{t_0}^t \mathrm{d}t'\; H_I(t') + \frac{(-i)^2}{2}\bigg\{&\int_{t_0}^t \mathrm{d}t'\int_{t'}^t\; \mathrm{d}t'' H_I(t'')H_I(t')\\ &+\int_{t_0}^t \mathrm{d}t'\int_{t_0}^{t'} \mathrm{d}t''\; H_I(t')H_I(t'')\bigg\} + \cdots.\end{aligned}$$

We now notice that

$$\int_{t_0}^t \mathrm{d}t'\int_{t'}^t\; \mathrm{d}t''\; H_I(t'')H_I(t') = \int_{t_0}^t \mathrm{d}t''\int_{t_0}^{t''}\; \mathrm{d}t'\; H_I(t'')H_I(t'),$$

since we are integrating over all $t_0 \leq t' \leq t'' \leq t$ on both sides. Swapping t' and t'' shows that the two second-order terms are indeed the same, so we have

$$U(t, t_0) = \mathbf{1} - i\int_{t_0}^t \mathrm{d}t'\; H_I(t') + (-i)^2\int_{t_0}^t \mathrm{d}t'\int_{t_0}^{t'}\; \mathrm{d}t''\; H_I(t'')H_I(t') + \cdots.$$

3.3 Wick's theorem

Proposition. Let ϕ be a real scalar field. Then

$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2) = \Delta_F(x_1 - x_2).$$

Proposition. For a complex scalar field, we have

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi^\dagger(y) = \Delta_F(x - y) = \contraction{}{\psi}{^\dagger(y)}{\psi}\psi^\dagger(y)\psi(x),$$

whereas

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = 0 = \contraction{}{\psi}{^\dagger(x)}{\psi}\psi^\dagger(x)\psi^\dagger(y).$$

Theorem (Wick's theorem)**.** For any collection of fields $\phi_1 = \phi_1(x_1), \phi_2 = \phi_2(x_2), \cdots$, we have

$$T(\phi_1\phi_2\cdots\phi_n) = {:}\phi_1\phi_2\cdots\phi_n{:} + \text{all possible contractions}$$

3.4 Feynman diagrams

3.5 Amplitudes

3.6 Correlation functions and vacuum bubbles

Lemma. The free vacuum and interacting vacuum are related via

$$|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} U_I(t, -\infty)\,|0\rangle = \frac{1}{\langle\Omega|0\rangle} U_S(t, -\infty)\,|0\rangle\,.$$

Similarly, we have

$$\langle\Omega| = \frac{1}{\langle\Omega|0\rangle}\,\langle 0|\,U(\infty, t).$$

Proposition.

$$G^{(n)}(x_1, \cdots, x_n) = \frac{\langle 0|\,T\phi_I(x_1)\cdots\phi_I(x_n)S\,|0\rangle}{\langle 0|\,S\,|0\rangle}.$$

4 Spinors

4.1 The Lorentz group and the Lorentz algebra

Proposition.

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}.$$

4.2 The Clifford algebra and the spin representation

Proposition. Suppose γ^μ is a representation of the Clifford algebra. Then the matrices given by

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

define a representation of the Lorentz algebra.

Lemma.

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}.$$

Proposition. Let $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, and define

$$\Omega_{ij} = -\varepsilon_{ijk}\phi_k.$$

Then in the chiral representation of S, writing $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, we have

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} & \mathbf{0} \\ \mathbf{0} & e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

Proposition. Write $\boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3)$. Then if

$$\Omega_{0i} = -\Omega_{i0} = -\chi_i,$$

then

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

is the boost in the $\boldsymbol{\chi}$ direction, and

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

4.3 Properties of the spin representation

Proposition. We have

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger.$$

Proposition.

$$S[\Lambda]^{-1} = \gamma^0 S[\Lambda]^\dagger \gamma^0,$$

where $S[\Lambda]^\dagger$ denotes the Hermitian conjugate as a matrix (under the usual basis).

Proposition. If ψ is a Dirac spinor, then

$$\bar{\psi} = \psi^\dagger \gamma^0$$

is a cospinor.

Corollary. For any spinor ψ, the quantity $\bar{\psi}\psi$ is a scalar, i.e. it doesn't transform under a Lorentz transformation.

Proposition. We have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu \gamma^\nu.$$

Corollary. The object $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector, and $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor.

4.4 The Dirac equation

4.5 Chiral/Weyl spinors and γ^5

Proposition. A Dirac spinor is the direct sum of a left-handed chiral spinor and a right-handed one.

Proposition. We have

$$\{\gamma^\mu, \gamma^5\} = 0, \quad (\gamma^5)^2 = \mathbf{1}$$

for all γ^μ, and

$$[S^{\mu\nu}, \gamma^5] = 0.$$

Proposition.

$$P_\pm^2 = P_\pm, \quad P_+ P_- = P_- P_+ = 0.$$

4.6 Parity operator

Axiom. The parity operator P acts on the spinors as γ^0.

Proposition.

$$P : \psi \mapsto \gamma^0 \psi, \quad P : \bar{\psi} \mapsto \bar{\psi}\gamma^0.$$

Proposition. We have

$$P : \gamma^5 \mapsto -\gamma^5.$$

Proposition. We have

$$P : P_\pm \mapsto P_\mp.$$

In particular, we have

$$P\psi_\pm = \psi_\mp.$$

4.7 Solutions to Dirac's equation

Proposition. We have a solution

$$u_{\mathbf{p}} = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}$$

for any 2-component spinor ξ normalized such that $\xi^\dagger \xi = 1$.

4.8 Symmetries and currents

5 Quantizing the Dirac field

5.1 Fermion quantization

Axiom. The spinor field operators satisfy

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = \{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = 0,$$

and

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\} = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}).$$

Proposition. The anti-commutation relations above are equivalent to

$$\{c_{\mathbf{p}}^r, c_{\mathbf{q}}^{s\dagger}\} = \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3\delta^{rs}\delta^3(\mathbf{p} - \mathbf{q}),$$

and all other anti-commutators vanishing.

Proposition.

$$H = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s + c_{\mathbf{p}}^{s\dagger} c_{\mathbf{p}}^s\right).$$

Proposition.

$$\contraction{}{\psi}{(x)}{\bar\psi}\psi(x)\bar{\psi}(y) = T(\psi(x)\bar{\psi}(y)) - {:}\psi(x)\bar{\psi}(y){:} = S_F(x - y).$$

5.2 Yukawa theory

5.3 Feynman rules

6 Quantum electrodynamics

6.1 Classical electrodynamics

6.2 Quantization of the electromagnetic field

Theorem.
$$[a_{\mathbf{p}}^{\lambda}, a_{\mathbf{q}}^{\lambda'}] = [a_{\mathbf{p}}^{\lambda\dagger}, a_{\mathbf{q}}^{\lambda'\dagger}] = 0$$
and
$$[a_{\mathbf{p}}^{\lambda}, a_{\mathbf{q}}^{\lambda'\dagger}] = -\eta^{\lambda\lambda'}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}).$$

Theorem. The Feynman propagator for the electromagnetic field, under a general gauge α, is
$$\langle 0| TA_\mu(x)A_\nu(y) |0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{-i}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} + (\alpha - 1)\frac{p_\mu p_\nu}{p^2}\right)e^{-ip\cdot(x-y)}.$$

6.3 Coupling to matter in classical field theory

6.4 Quantization of interactions

6.5 Computations and diagrams