3P1a **Quantum Field Theory: Example Sheet 1** Michaelmas 2016

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk. *Starred questions may be handed in to your supervisor for feedback prior to the class* **by Part III maths students only**.

1. A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of small transverse displacements $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right]$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion of the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} q_n(t) \sin \left(\frac{n\pi x}{a} \right)$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] .$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) .$$

2. Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

3. The motion of a complex field $\psi(x)$ is governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2.$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi, \qquad \delta\psi^* = -i\alpha\psi^*.$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

4. Verify that the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$

for a triplet of real fields ϕ_a, where $a \in \{1,2,3\}$ is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \to \phi_a + \theta \epsilon_{abc} \eta_b \phi_c$$

where η_a is a unit vector. Compute the Noether current j^μ. Deduce that the three quantities

$$Q_a = \int d^3x \; \epsilon_{abc} \dot{\phi}_b \phi_c$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

5*. A Lorentz transformation $x^a \to x'^a = \Lambda^a{}_b x^b$ is such that it preserves the Minkowski metric η_{ab}, meaning that $\eta_{ab} x^a x^b = \eta_{ab} x'^a x'^b$ for all x. Show that this implies that

$$\eta_{ab} = \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b \,.$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b$$

is a Lorentz transformation when ω^{ab} is antisymmetric: i.e. $\omega^{ab} = -\omega^{ba}$.

Write down the matrix form for $\omega^a{}_b$ that corresponds to a rotation through an infinitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.

6*. Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$. Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}).$$

Using Noether's theorem, deduce the existence of the conserved current

$$j^\mu = -\omega^\rho{}_\nu [T^\mu_\rho x^\nu].$$

The three conserved charges arising from spatial rotational invariance define the *total angular momentum* of the field. Show that these charges are given by

$$Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3x \; \left(x^j T^{0k} - x^k T^{0j} \right) .$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3x \; \left(x^i T^{00} \right) = \text{constant}$$

and interpret this equation.

7. Maxwell's Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi,$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.

Using Noether's theorem, and the spacetime translational invariance of the action, to construct the energy momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^\mu = T^{\mu\nu} - F^{\rho\mu} \partial_\rho A^\nu.$$

Show that this object also defines four currents. Moreover, show that it is symmetric, gauge invariant and traceless.

8. The Lagrangian density for a massive vector field C_μ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 C_\mu C^\mu,$$

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Derive the equations of motion and show that when $m \neq 0$ they imply

$$\partial_\mu C^\mu = 0.$$

Further show that C_0 can be eliminated completely in terms of other fields by

$$\partial_i \partial^i C_0 + m^2 C_0 = \partial^i \dot{C}_i. \qquad (1)$$

Construct the canonical momenta Π_i conjugate to C_i where $i \in \{1, 2, 3\}$ and show that the canonical momentum conjugate to C_0 is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of C_0, C_i and Π_i (NB: don't be concerned that the canonical momentum for C_0 is vanishing. C_0 is non-dynamical; it is determined entirely in terms of the other fields using Eq. (1)).

9. A class of interesting theories is invariant under the scaling of all lengths by

$$x^\mu \to (x')^\mu = \lambda x^\mu \text{ and } \phi(x) \to \phi'(x) = \lambda^{-D} \phi(\lambda^{-1} x). \qquad (2)$$

Here, D is called the *scaling dimension* of the field. Consider the action for a real scalar field given by

$$S = \int d^4 x \; \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g \phi^p.$$

Find the scaling dimension D such that the derivative terms remain invariant. For what values of m and p is the scaling in Eq. (2) a symmetry of the theory? How do these conclusions change for a scalar field living in an $(n+1)-$dimensional spacetime instead of a $3+1$ dimensional spacetime?

In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^μ associated with scaling invariance.

3P1b **Quantum Field Theory: Example Sheet 2** Michaelmas 2016

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1. A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} p_n^2 + \tfrac{1}{2} \omega_n^2 \, q_n^2 \right)$$

where ω_n is the frequency of the nth mode. Compare this Hamiltonian to the Lagrangian in the previous question. The mass per unit length, σ, has now been set to unity so as to make various formulae somewhat simpler.

After quantization, q_n and p_n become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm} \, .$$

Introduce creation and annihilation operators a_n and $a_n^\dagger$,

$$a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n \, .$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm} \, .$$

Show that the Hamiltonian of the system can be written in the form

$$H = \sum_{n=1}^{\infty} \tfrac{1}{2}\, \omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n \right) \, .$$

Given the existence of a ground state $|0\rangle$ such that $a_n|0\rangle = 0$, explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n \, .$$

Show further that $[H, a_n^\dagger] = \omega_n \, a_n^\dagger$ and hence calculate the energy of the state

$$|l_1, l_2, \ldots, l_N\rangle = \left(a_1^\dagger\right)^{l_1} \left(a_2^\dagger\right)^{l_2} \ldots \left(a_N^\dagger\right)^{l_N} |0\rangle \, .$$

2. The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right], \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]. \end{aligned}$$

Show that the commutation relations

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \text{ and } [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$

imply that

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0 \text{ and } [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}).$$

3. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{1}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x\ T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \tag{2}$$

where $p^0 = E_{\vec{p}}$ in this expression. From Eq. (2), verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x).$$

4*. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x)\,.$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

5. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalised states $|p\rangle = \sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|0\rangle$ satisfy

$$\langle 0|\,\phi(x)\,|p\rangle = e^{-ip\cdot x}.$$

6*. In Example Sheet 1, you showed that the classical angular momentum of a field is given by

$$Q_i = \frac{1}{2}\epsilon_{ijk}\int d^3x\ \left(x^jT^{0k} - x^kT^{0j}\right).$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian as in Eq.(1). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = -\frac{1}{2}i\epsilon_{ijk}\int \frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p}}\left(p^j\frac{\partial}{\partial p_k} - p^k\frac{\partial}{\partial p_j}\right)a_{\vec{p}}.$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a one-particle state $|\vec{p}\rangle$ has zero angular momentum).

7. The purpose of this question is to introduce you to non-relativistic quantum field theories. This is the only place you will encounter such a thing in this course. Consider the Lagrangian for a complex scalar field ψ given by

$$\mathcal{L} = i\psi^* \partial_0 \psi - \frac{1}{2m} \nabla \psi^* \nabla \psi.$$

Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry $\psi \to e^{i\alpha}\psi$. Show that the momentum conjugate to ψ is $i\psi^*$ and compute the Hamiltonian.

We now wish to quantise this theory. We shall work in the Schrödinger picture. Explain why the correct commutation relations are

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})] = 0 \text{ and } [\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}).$$

Expand the fields in a Fourier decomposition as

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}, \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}. \end{aligned}$$

Determine the commutation relations obeyed by $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. Why do we have only a single set of creation and annihilation operators $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ even though ψ is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass m.

8. Show that the time ordered product $T(\phi(x_1)\phi(x_2))$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

9. Verify Wick's theorem for the case of three scalar fields:

$$\begin{aligned} T(\phi(x_1)\phi(x_2)\phi(x_3)) &= \ :\phi(x_1)\phi(x_2)\phi(x_3)): + \phi(x_1)\Delta_F(x_2 - x_3) \\ &\quad + \phi(x_2)\Delta_F(x_3 - x_1) + \phi(x_3)\Delta_F(x_1 - x_2). \end{aligned}$$

10. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g\psi^*\psi\phi.$$

Compute the amplitude for

(a) "Meson" decay $\phi \to \psi + \bar{\psi}$ at order g.

(b) "Nucleon-meson" scattering $\phi + \psi \to \phi + \psi$ at order g^2.

3P1c **Quantum Field Theory: Example Sheet 3** Michaelmas 2016

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1. The Weyl representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

Show that these indeed satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^a = U\gamma^a U^\dagger$, where $(\gamma')^a$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

2. Show that if $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, then

$$\left[\gamma^a\gamma^b\,,\,\gamma^c\gamma^d\right] = 2\eta^{ac}\gamma^b\gamma^d - 2\eta^{bc}\gamma^a\gamma^d - 2\eta^{bd}\gamma^c\gamma^a + 2\eta^{ad}\gamma^c\gamma^b .$$

Show further that $S^{ab} \equiv \frac{i}{4}\left[\gamma^a\,,\,\gamma^b\right] = \frac{1}{2}(\gamma^a\gamma^b - \eta^{ab})$. Use this to confirm that the matrices S^{ab} form a representation of the Lie algebra of the Lorentz group.

3. Using just the algebra $\{\gamma^a,\ \gamma^b\} = 2\eta^{ab}$ (that is to say without resorting to any particular representation of the gamma matrices), and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_a\gamma^a$ and $S^{ab} \equiv \frac{1}{4}\left[\gamma^a,\ \gamma^b\right]$, prove the following results:

   (a) $\mathrm{Tr}\gamma^a = 0$
   (b) $\mathrm{Tr}(\gamma^a\gamma^b) = 4\eta^{ab}$
   (c) $\mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = 0$
   (d) $(\gamma^5)^2 = 1$
   (e) $\mathrm{Tr}\gamma^5 = 0$
   (f) $\not{p}\not{q} = 2p\cdot q - \not{q}\not{p} = p\cdot q + 2S^{ab}p_a q_b$
   (g) $\mathrm{Tr}(\not{p}\not{q}) = 4p\cdot q$
   (h) $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$ if n is odd
   (i) $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right]$
   (j) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$
   (k) $\gamma_a\not{p}\gamma^a = -2\not{p}$
   (l) $\gamma_a\not{p}_1\not{p}_2\gamma^a = 4p_1\cdot p_2$
   (m) $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$
   (n) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\,\epsilon_{abcd}\,p_1^a\,p_2^b\,p_3^c\,p_4^d$

4*. The plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \text{ and } v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s, with $s \in \{1, 2\}$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m\delta^{rs} \end{aligned} \tag{1}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m\delta^{rs}. \end{aligned} \tag{2}$$

Show also that the orthogonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0,$$

while taking the inner product using † requires an extra minus sign

$$u^s(\vec{p})^\dagger \cdot v^r(-\vec{p}) = 0. \tag{3}$$

5. Using the same notation as Question 4, show that

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m, \tag{4}$$

$$\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m, \tag{5}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac field operator $\psi(x)$ and the conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1}^{2}\left[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right], \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1}^{2}\left[b^{s\dagger}_{\vec{p}}u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}}\right]. \end{aligned} \tag{6}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}), \\ \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} &= -(2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}), \end{aligned}$$

with all other anticommutators vanishing. Show that these imply that the field and its conjugate field satisfy the anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0, \\ \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned}$$

Note: the calculation is very similar to that for the bosonic field, but at some point you will need to make use of the identities Eqs. (4),(5).

7. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x \; \bar{\psi}(-i\gamma^i \partial_i + m)\psi$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r + c_{\vec{p}}^{r\dagger} c_{\vec{p}}^r \right] .$$

Note: again, the calculation is very similar to that of the bosonic field. This time you will need to make use of the identities in Eqs. (1), (2) and (3).

8*. The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 Dirac field as a boson. We start with the usual decomposition given in Eq. 6. This time, we choose the creation and annihilation operators to satisfy bosonic commutation relations rather than fermionic anti-commutation ones:

$$\begin{aligned} [b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}] &= 2E_p \, (2\pi)^3 \delta_{rs} \, \delta^{(3)}(\vec{p} - \vec{q}) \\ [c_{\vec{p}}^r, c_{\vec{q}}^{s\dagger}] &= -2E_p \, (2\pi)^3 \delta_{rs} \, \delta^{(3)}(\vec{p} - \vec{q}) \end{aligned}$$

with all other commutators vanishing. Note the strange minus sign for the c operators. Repeat the calculation of Question 6 to show that these are equivalent to the commutation relations

$$\begin{aligned} [\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] &= \left[\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y}) \right] = 0, \\ \left[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y}) \right] &= \delta_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y}). \end{aligned}$$

Now repeat the calculation of Question 7, to show that, after normal ordering, the Hamiltonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r - c_{\vec{p}}^{r\dagger} c_{\vec{p}}^r \right] .$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more c particles. This is the reason a theory of bosonic spin 1/2 particles is sick.

3P1d **Quantum Field Theory: Example Sheet 4** Michaelmas 2016

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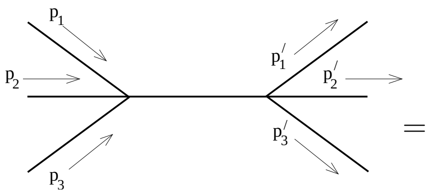
1. A real scalar field with ϕ^4 interaction has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

Use Dyson's formula and Wick's theorem to show that the leading order contribution to 3-particle→3-particle scattering is given by

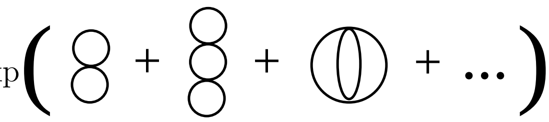


$$= (-i\lambda)^2 \frac{i}{(p_1+p_2+p_3)^2 - m^2}. \tag{2}$$

Check that this result is consistent with the Feynman rules for the theory.

2. Examine $\langle 0| S |0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different contributions arising from an application of Wick's theorem. Confirm that to order λ^2, the combinatoric factors work out so that the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$$\langle 0| S |0\rangle = \exp\Big(\quad \Big) \tag{3}$$



3*. Consider the Lagrangian density for three scalar fields ϕ_i, $i = 1, 2, 3$, given by

$$\mathcal{L} = -\sum_{i=1}^{3}\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2(\sum_{i=1}^{3}\phi_i^2) - \frac{\lambda}{8}(\sum_{i=1}^{3}\phi_i^2)^2\,. \tag{4}$$

Show that the Feynman propagator for the free field theory (i.e. $\lambda = 0$) is of the form

$$\langle 0|T\phi_i(x)\phi_j(y)|0\rangle = \delta_{ij}\Delta_F(x-y) \tag{5}$$

where $\Delta_F(x-y)$ is the usual scalar propagator. Write down the Feynman rules of the theory. Compute the amplitude for the scattering $\phi_i\phi_j \to \phi_k\phi_l$ to lowest nontrivial order in λ.

4*. The Lagrangian density for a Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar\psi(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar\psi\psi. \tag{6}$$

(a) Consider $\psi\psi \to \psi\psi$ scattering, with the initial and final states given by

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec p}E_{\vec q}}\, b_{\vec p}^{s\dagger} b_{\vec q}^{r\dagger}\,|0\rangle\,, \\ |f\rangle &= \sqrt{4E_{\vec p}E_{\vec q}}\, b_{\vec p'}^{s'\dagger} b_{\vec q'}^{r'\dagger}\,|0\rangle\,. \end{aligned} \tag{7}$$

Show using Dyson's formula and Wick's theorem that the scattering amplitude at order λ^2 is given by

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p'}) \cdot u^s(\vec{p})][\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q'})]}{(p'-p)^2 - \mu^2} - \frac{[\bar{u}^{s'}(\vec{p'}) \cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q'}) \cdot u^s(\vec{p})]}{(q'-p)^2 - \mu^2} \right). \quad (8)$$

Draw the two Feynman diagrams that correspond to these two terms.

(b) Consider now $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering, with initial and final states given by

$$|i\rangle = \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b^{s\dagger}_{\vec{p}} c^{r\dagger}_{\vec{q}} |0\rangle \quad (9)$$

$$|f\rangle = \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b^{s'\dagger}_{\vec{p'}} c^{r'\dagger}_{\vec{q'}} |0\rangle . \quad (10)$$

Show that the amplitude is given by

$$\mathcal{A} = -(-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p'}) \cdot u^s(\vec{p})][\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q'})]}{(p-p')^2 - \mu^2} - \frac{[\bar{v}^r(\vec{q}) \cdot u^s(\vec{p})][\bar{u}^{s'}(\vec{p'}) \cdot v^{r'}(\vec{q'})]}{(q+p)^2 - \mu^2} \right). \quad (11)$$

Be careful with minus signs! What are the Feynman diagrams that now contribute?

5. The Lagrangian density for a pseudoscalar Yukawa interaction is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\gamma^5\psi . \quad (12)$$

Write down the Feynman rules for this theory. Use these to write down the amplitude at order λ^2 for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering.

6. Any vector function $\mathbf{f}(\mathbf{x})$ has a decomposition into a sum of transverse (zero divergence) and longitudinal (zero curl) parts, namely

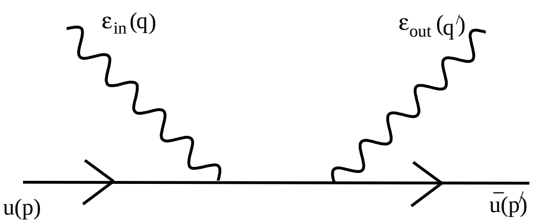
$$\mathbf{f} = \nabla \times \mathbf{g} + \nabla h \equiv \mathbf{f}^T + \mathbf{f}^L, \quad (13)$$

where $\mathbf{g}$ and h are unique if one imposes the additional constraint $\nabla \cdot \mathbf{g} = 0$ and certain vanishing conditions at infinity. By taking the divergence and curl of Eq. 13, determine $\mathbf{d}$ and h in terms of $\mathbf{f}$. Show formally that

$$\mathbf{f}^T = \mathbf{f} - \nabla(\nabla^2)^{-1}\nabla \cdot \mathbf{f}. \quad (14)$$

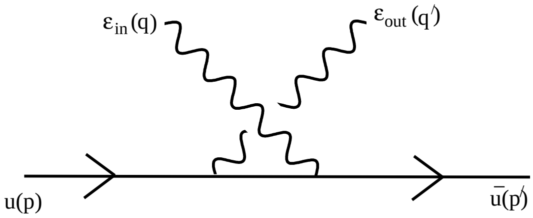
Use this result to comment on the commutation relations of the quantised electromagnetic gauge potential in Coulomb gauge.

7. Consider Compton scattering in which a photon and an electron scatter off each other. Let the incoming photon have polarisation vector $\epsilon^\mu_{\rm in}$ and the outgoing photon have polarisation $\epsilon^\mu_{\rm out}$. Use the Feynman rules to derive the following amplitude associated to the lowest order diagram,



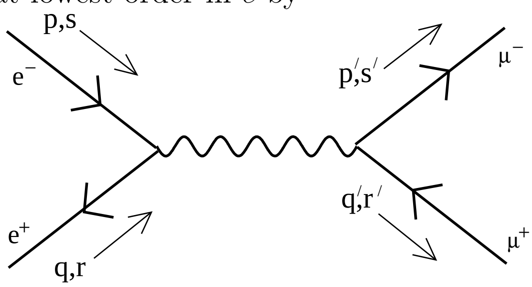
$$= i(-ie)^2 \bar{u}^{r'}(\vec{p'}) \not{\epsilon}_{\rm out} \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} \not{\epsilon}_{\rm in} u^s(\vec{p}). \quad (15)$$

Also, compute the contribution from the diagram



The complete amplitude at order e^2 is the sum of these two contributions. Show that the total amplitude vanishes if ϵ_{in} is replaced by the incoming photon momentum q then the amplitude vanishes. Check that the same holds true if ϵ_{out} is replaced by q'. *Note: it will be helpful to recall the equation* $\left(\not{p} - m \right) u(\vec{p}) = 0$ *satisfied by the spinor.*

8. Use the Feynman rules to show that the QED amplitude for $e^+e^- \to \mu^+\mu^-$ is given at lowest order in e by



$$= (-ie)^2 \frac{[\bar{v}_e^r(\vec{q})\gamma_\mu u_e^s(\vec{p})][\bar{u}_m^{s'}(\vec{p'})\gamma^\mu v_m^{r'}(\vec{q'})]}{(p+q)^2}, \quad (16)$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons, respectively.

9. Viki Weisskopf is one of the more charming characters from the history of quantum field theory. This is from his his biography

> "Pauli asked me to calculate the amplitude for pair creation of scalar particles by photons. It was only a short time after Bethe and Heitler had solved the same problem for electrons and positrons. I met Bethe in Cophenhagen at a conference and asked him to tell me how he did the calculations. I also inquired how long it would take to perform this task; he answered "It would take me three days, but you will need about three weeks." He was right, as usual; furthermore, the published result was wrong by a factor of two."

Can you do better?

10. Now you understand the role played by fields in Nature, why do you think classical physicists such as Farady and Maxwell found it useful to introduce the concept of the electric and magnetic field, but never matter fields?