Part III — Quantum Field Theory
Definitions

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Quantum Field Theory is the language in which modern particle physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics. How these fields interact with a classical electromagnetic field is described.

Interactions are described using perturbative theory and Feynman diagrams. This is first illustrated for theories with a purely scalar field interaction, and then for a couplings between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processes are computed.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

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0 Introduction

1 Classical field theory

1.1 Classical fields

Definition (Field)**.** A *field* ϕ is a physical quantity defined at every point of spacetime $(\mathbf{x}, t)$. We write the value of the field at $(\mathbf{x}, t)$ as $\phi(\mathbf{x}, t)$.

Definition (Lagrangian density)**.** Given a field $\phi(\mathbf{x}, t)$, a *Lagrangian density* is a function $\mathcal{L}(\phi, \partial_\mu \phi)$ of ϕ and its derivative.

Definition (Lagrangian)**.** Given a Lagrangian density, the *Lagrangian* is defined by

$$L = \int \mathrm{d}^3\mathbf{x}\ \mathcal{L}(\phi, \partial_\mu \phi).$$

Definition (Action)**.** Given a Lagrangian and a time interval $[t_1, t_2]$, the *action* is defined by

$$S = \int_{t_1}^{t_2} \mathrm{d}t\ L(t) = \int \mathrm{d}^4x\ \mathcal{L}.$$

Definition (Principle of least action)**.** The equation of motion of a Lagrangian system is given by the *principle of least action* — we vary the field slightly, keeping values at the boundary fixed, and require the first-order change $\delta S = 0$.

1.2 Lorentz invariance

1.3 Symmetries and Noether's theorem for field theories

1.4 Hamiltonian mechanics

Definition (Conjugate momentum)**.** Given a Lagrangian system for a field ϕ, we define the *conjugate momentum* by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$

Definition (Hamiltonian density)**.** The *Hamiltonian density* is given by

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(x),$$

where we replace all occurrences of $\dot{\phi}(x)$ by expressing it in terms of $\pi(x)$.

Definition (Hamiltonian)**.** The *Hamiltonian* of a Hamiltonian system is

$$H = \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}.$$

Definition (Hamilton's equations)**.** Hamilton's equations are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi}, \quad \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \phi}.$$

These give us the equations of motion of ϕ.

2 Free field theory

2.1 Review of simple harmonic oscillator

2.2 The quantum field

Definition (Real scalar quantum field)**.** A *(real, scalar) quantum field* is an operator-valued function of space ϕ, with conjugate momentum π, satisfying the commutation relations

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})]$$

and

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}).$$

In case where we have many fields labelled by $a \in I$, the commutation relations are

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 = [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})]$$

and

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_a^b.$$

Definition (Schrödinger equation)**.** The *Schrödinger equation* says

$$i\frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle = H |\psi\rangle .$$

2.3 Real scalar fields

Definition (Normal order)**.** Given a string of operators

$$\phi_1(\mathbf{x}_1) \cdots \phi_n(\mathbf{x}_n),$$

the *normal order* is what you obtain when you put all the annihilation operators to the right of (i.e. acting before) all the creation operators. This is written as

$$:\phi_1(\mathbf{x}_1) \cdots \phi_n(\mathbf{x}_n): .$$

2.4 Complex scalar fields

2.5 The Heisenberg picture

Definition (Causal theory)**.** A theory is *causal* if for any space-like separated points x, y, and any two fields ϕ, ψ, we have

$$[\phi(x), \psi(y)] = 0.$$

2.6 Propagators

Definition (Propagator)**.** The *propagator* of a real scalar field ϕ is defined to be

$$D(x - y) = \langle 0| \phi(x)\phi(y) |0\rangle .$$

Definition (Feynman propagator)**.** The *Feynman propagator* is

$$\Delta_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \begin{cases} \langle 0| \phi(x)\phi(y) |0\rangle & x^0 > y^0 \\ \langle 0| \phi(y)\phi(x) |0\rangle & y^0 > x^0 \end{cases}$$

Definition (Retarded Green's function)**.** The *retarded Green's function* is given by

$$\Delta_R(x-y) = \begin{cases} [\phi(x), \phi(y)] & x^0 > y^0 \\ 0 & y^0 > x^0 \end{cases}$$

3 Interacting fields

3.1 Interaction Lagrangians

3.2 Interaction picture

Definition (S-matrix)**.** The *S-matrix* is defined as

$$S = U(\infty, -\infty).$$

3.3 Wick's theorem

Definition (Contraction)**.** The *contraction* of two fields ϕ, ψ is defined to be

$$\contraction{}{\phi}{}{\psi}\phi\psi = T(\phi\psi) - {:}\phi\psi{:}\,.$$

More generally, if we have a string of operators, we can contract just some of them:

$$\cdots \contraction{}{\phi}{(x_1)\cdots}{\phi}\phi(x_1)\cdots\phi(x_2)\cdots,$$

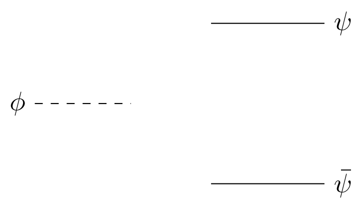
by replacing the two fields with the contraction.

In general, the contraction will be a c-function, i.e. a number. So where we decide to put the result of the contraction wouldn't matter.

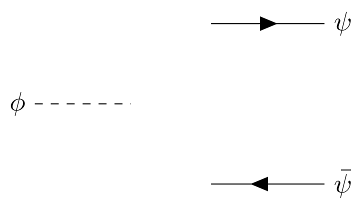
3.4 Feynman diagrams

Definition (Feynman diagram)**.** In the scalar Yukawa theory, given an initial state $|i\rangle$ and final state $|f\rangle$, a *Feynman diagram* consists of:

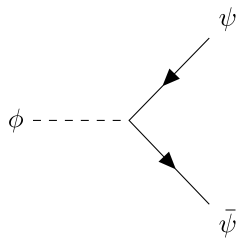
- An external line for all particles in the initial and final states. A dashed line is used for a ϕ-particle, and solid lines are used for $\psi/\bar{\psi}$-particles.



- Each ψ-particle comes with an arrow. An initial ψ-particle has an incoming arrow, and a final ψ-particle has an outgoing arrow. The reverse is done for $\bar{\psi}$-particles.

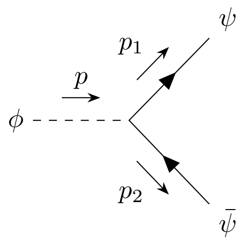


– We join the lines together with more lines and vertices so that the only loose ends are the initial and final states. The possible vertices correspond to the interaction terms in the Lagrangian. For example, the only interaction term in the Lagrangian here is $\psi^\dagger\psi\phi$, so the only possible vertex is one that joins a ϕ line with two ψ lines that point in opposite directions:



Each such vertex represents an interaction, and the fact that the arrows match up in all possible interactions ensures that charge is conserved.

– Assign a directed momentum p to each line, i.e. an arrow into or out of the diagram to each line.



The initial and final particles already have momentum specified in the initial and final state, and the internal lines are given "dummy" momenta k_i (which we will later integrate over).

Definition (Feynman rules)**.** To each Feynman diagram in the interaction, we write down a term like this:

(i) To each vertex in the Feynman diagram, we write a factor of

$$(-ig)(2\pi)^4\delta^4\left(\sum_i k_i\right),$$

where the sum goes through all lines going into the vertex (and put a negative sign for those going out).

(ii) For each internal line with momentum k, we integrate the product of all factors above by

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} D(k^2),$$

where

$$D(k^2) = \frac{i}{k^2 - m^2 + i\varepsilon} \text{ for } \phi$$
$$D(k^2) = \frac{i}{k^2 - \mu^2 + i\varepsilon} \text{ for } \psi$$

3.5 Amplitudes

Definition (Amplitude)**.** The *amplitude* $\mathcal{A}_{f,i}$ of a scattering process from $|i\rangle$ to $|f\rangle$ is defined by

$$\langle f| S - \mathbf{1} |i\rangle = i\mathcal{A}_{f,i}(2\pi)^4\delta^4(p_F - p_I).$$

where p_F is the sum of final state 4-momenta, and p_I is the sum of initial state 4-momenta. The factor of i sticking out is by convention, to match with non-relativistic quantum mechanics.

3.6 Correlation functions and vacuum bubbles

Definition (Correlation/Green's function)**.** The *correlation* or *Green's function* is defined as

$$G^{(n)}(x_1, \cdots, x_n) = \langle\Omega| T\phi_H(x_1)\cdots\phi_H(x_n) |\Omega\rangle,$$

where ϕ_H denotes the operators in the Heisenberg picture.

4 Spinors

4.1 The Lorentz group and the Lorentz algebra

Definition (Lorentz group)**.** The *Lorentz group*, denoted $\mathrm{O}(1,3)$, is the group of all Lorentz transformations. Explicitly, it is given by

$$\mathrm{O}(1,3) = \{\Lambda \in M_{4\times 4} : \Lambda^T \eta \Lambda = \eta\},$$

where

$$\eta = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the *Minkowski metric*. Alternatively, $\mathrm{O}(1,3)$ is the group of all matrices Λ such that

$$\langle \Lambda x, \Lambda y\rangle = \langle x, y\rangle,$$

for all $x, y \in \mathbb{R}^{1+3}$, where $\langle x, y\rangle$ denotes the inner product given by the Minkowski metric

$$\langle x, y\rangle = x^T \eta y.$$

Definition (Representation of the Lorentz group)**.** A *representation of the Lorentz group* is a vector space V and a linear map $D(\Lambda) : V \to V$ for each $\Lambda \in \mathrm{O}(1,3)$ such that

$$D(\mathbf{1}) = \mathbf{1}, \quad D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$$

for any $\Lambda_1, \Lambda_2 \in \mathrm{O}(1,3)$.

The space V is called the *representation space*.

Definition (Lorentz algebra)**.** The *Lorentz algebra* is

$$\mathfrak{o}(1,3) = \{\omega \in M_{4\times 4} : \omega^{\mu\nu} + \omega^{\nu\mu} = 0\}.$$

Definition (Representation of Lorentz algebra)**.** A *representation of the Lorentz algebra* is a collection of matrices that satisfy the same commutation relations as the Lorentz algebra.

Formally, this is given by a vector space V and a linear map $R(\omega) : V \to V$ for each $\omega \in \mathfrak{o}(1,3)$ such that

$$R(a\omega + b\omega') = aR(\omega) + bR(\omega'), \quad R([\omega, \omega']) = [R(\omega), R(\omega')]$$

for all $\omega, \omega' \in \mathfrak{o}(1,3)$ and $a, b \in \mathbb{R}$.

Definition (Restricted Lorentz group)**.** The *restricted Lorentz group* consists of the elements in the Lorentz group that preserve orientation and direction of time.

4.2 The Clifford algebra and the spin representation

Notation (Anticommutator)**.** We write

$$\{A, B\} = AB + BA$$

for the *anticommutator* of two matrices/linear maps.

Definition (Clifford algebra)**.** The *Clifford algebra* is the algebra generated by $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ subject to the relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}.$$

More explicitly, we have

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{for} \quad \mu \neq \nu.$$

and

$$(\gamma^0)^2 = \mathbf{1}, \quad (\gamma^i)^2 = -\mathbf{1}.$$

A *representation* of the Clifford algebra is then a collection of matrices (or linear maps) satisfying the relations listed above.

Definition (Spin group)**.** The *spin group* $\mathrm{Spin}(1,3)$ is the universal cover of $\mathrm{SO}^+(1,3)$. This comes with a canonical surjection to $\mathrm{SO}^+(1,3)$, sending "Λ and a path to Λ" to Λ.

4.3 Properties of the spin representation

Definition (Dirac spinor)**.** A *Dirac spinor* is a vector in the representation space of the spin representation. It may also refer to such a vector for each point in space.

Definition (Cospinor)**.** A *cospinor* is an element in the dual space to space of spinors, i.e. a cospinor X is a linear map that takes in a spinor ψ as an argument and returns a number $X\psi$. A cospinor can be represented as a "row vector" and transforms under Λ as

$$X \mapsto XS[\Lambda]^{-1}.$$

Definition (Dirac adjoint)**.** For any Dirac spinor ψ, its *Dirac adjoint* is given by

$$\bar{\psi} = \psi^\dagger\gamma^0.$$

4.4 The Dirac equation

Definition (Dirac Lagrangian)**.** The *Dirac Lagrangian* is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$$

Definition (Dirac equation)**.** The *Dirac equation* is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Notation (Slash notation)**.** We write

$$A_\mu\gamma^\mu \equiv \not{A}.$$

4.5 Chiral/Weyl spinors and γ^5

Definition (Weyl/chiral spinor). A *left (right)-handed chiral spinor* is a 2-component complex vector U_+ and U_- respectively that transform under the action of the Lorentz/spin group as follows:

Under a rotation with rotation parameters $\boldsymbol{\phi}$, both of them transform as

$$U_\pm \mapsto e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} U_\pm,$$

Under a boost $\boldsymbol{\chi}$, they transform as

$$U_\pm \mapsto e^{\pm\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} U_\pm.$$

So these are two two-dimensional representations of the spin group.

Definition (Weyl equation). The *Weyl equation* is

$$i\bar{\sigma}^\mu \partial_\mu U_+ = 0.$$

Definition (γ^5).

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3.$$

4.6 Parity operator

Definition (Pseudoscalar). A *pseudoscalar* is a number that does not change under Lorentz boosts and rotations, but changes sign under a parity operator.

Definition (Axial vector). An *axial vector* is a quantity that transforms as vectors under rotations and boosts, but gain an additional sign when transforming under parity.

4.7 Solutions to Dirac's equation

Definition (Helicity operator). The helicity operator is a projection of angular momentum along the direction of motion

$$h = \hat{\mathbf{p}} \cdot \mathbf{J} = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & \sigma^i \end{pmatrix}.$$

4.8 Symmetries and currents

5 Quantizing the Dirac field

5.1 Fermion quantization

Definition.
$$iS_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \bar{\psi}_\beta(y)\}.$$

Definition (Feynman propagator)**.** The *Feynman propagator* of a spinor field is the time-ordered product

$$S_F(x-y) = \langle 0| T\psi_\alpha(x)\bar{\psi}_\beta(y) |0\rangle = \begin{cases} \langle 0| \psi_\alpha(x)\bar{\psi}_\beta(y) |0\rangle & x^0 > y^0 \\ -\langle 0| \bar{\psi}_\beta(y)\psi_\alpha(x) |0\rangle & y^0 > x^0 \end{cases}$$

5.2 Yukawa theory

5.3 Feynman rules

6 Quantum electrodynamics

6.1 Classical electrodynamics

Definition (Lorenz gauge)**.** The *Lorenz gauge* is specified by

$$\partial_\mu A^\mu = 0$$

Definition (Coulomb gauge)**.** The *Coulomb gauge* requires

$$\nabla \cdot \mathbf{A} = 0.$$

6.2 Quantization of the electromagnetic field

6.3 Coupling to matter in classical field theory

6.4 Quantization of interactions

6.5 Computations and diagrams