Part III — Percolation and Random Walks on Graphs

Theorems with proof

Based on lectures by P. Sousi

Notes taken by Dexter Chua

Michaelmas 2017

A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Contents

0 Introduction

1 Percolation

1.1 The critical probability

Lemma. θ is an increasing function of p.

Proof. We let $(U(e))_{e\in E(\mathbb{Z}^d)}$ be iid $U[0,1]$ random variables. For each $p \in [0,1]$, we define

$$\eta_p(e) = \begin{cases} 1 & U(e) \leq p \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathbb{P}(\eta_p(e) = 1) = \mathbb{P}(U(e) < p) = p$. Since the $U(e)$ are independent, so are η_p. Thus η_p has the law of bond percolation with probability p.

Moreover, if $p \leq q$, then $\eta_p(e) \leq \eta_q(e)$. So the result follows. $\square$

Proposition. $p_c(1) = 1$.

Theorem. For all $d \geq 2$, we have $p_c(d) \in (0,1)$.

Lemma. For $d \geq 2$, $p_c(d) > 0$.

Proof. Write Σ_n for the number of open self-avoiding paths of length n starting at 0. We then note that

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) = \mathbb{P}_p(\forall n \geq 1 : \Sigma_n \geq 1) = \lim_{n\to\infty} \mathbb{P}_p(\Sigma_n \geq 1) \leq \lim_{n\to\infty} \mathbb{E}_p[\Sigma_n].$$

We can now compute $\mathbb{E}_p[\Sigma_n]$. The point is that expectation is linear, which makes this much easier to compute. We let σ_n be the number of self-avoiding paths of length n from 0. Then we simply have

$$\mathbb{E}_p[\Sigma_n] = \sigma_n p^n.$$

We can bound σ_n by $2d \cdot (2d-1)^{n-1}$, since we have $2d$ choices of the first step, and at most $2d-1$ choices in each subsequent step. So we have

$$\mathbb{E}_p[\Sigma_n] \leq 2d(2d-1)^{n-1}p^n = \frac{2d}{2d-1}(p(2d-1))^n.$$

So if $p(2d-1) < 1$, then $\theta(p) = 0$. So we know that

$$p_c(d) \geq \frac{1}{2d-1}.$$

$\square$

Lemma. We have $\sigma_{n+m} \leq \sigma_n\sigma_m$.

Proof. A self-avoiding path of length $n+m$ can be written as a concatenation of self-avoiding paths of length n starting from 0 and another one of length m. $\square$

Lemma (Fekete's lemma)**.** If (a_n) is a subadditive sequence of real numbers, then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_k}{k} : k \geq 1\right\} \in [-\infty, \infty).$$

In particular, the limit exists.

Theorem (Duminil-Copin, Smirnov, 2010)**.** The hexagonal lattice has

$$\kappa_{\text{hex}} = \sqrt{2+\sqrt{2}}.$$

Theorem (Hara and Slade, 1991)**.** For $d \geq 5$, there exists a constant A such that

$$\sigma_n = A\kappa^n(1 + O(n^{-\varepsilon}))$$

for any $\varepsilon < \frac{1}{2}$.

Theorem (Hammersley and Welsh, 1962)**.** For all $d \geq 2$, we have

$$\sigma_n \leq C\kappa^n \exp(c'\sqrt{n})$$

for some constants C and c'.

Theorem (Hutchcroft, 2017)**.** For $d \geq 2$, we have

$$\sigma_n \leq C\kappa^n \exp(o(\sqrt{n})).$$

Lemma. $p_c(d) < 1$ for all $d \geq 2$.

Proof. It suffices to show this for $d = 2$. Suppose we perform percolation on $\mathbb{Z}^2$. Then this induces a percolation on the dual lattice by declaring an edge of the dual is open if it crosses an open edge of $\mathbb{Z}^2$, and closed otherwise.

Suppose $|\mathcal{C}(0)| < \infty$ in the primal lattice. Then there is a closed circuit in the dual lattice, given by the "boundary" of $\mathcal{C}(0)$. Let D_n be the number of closed dual circuits of length n that surround 0. Then the union bound plus Markov's inequality tells us

$$\mathbb{P}_p(|\mathcal{C}(0)| < \infty) = \mathbb{P}_p(\exists n \geq D_n \geq 1) \leq \sum_{n=4}^{\infty} \mathbb{E}_p[D_n],$$

using the union bound and Markov's inequality.

It is a simple exercise to show that

Exercise. Show that the number of dual circuits of length n that contain 0 is at most $n \cdot 4^n$.

From this, it follows that

$$\mathbb{P}_p(|\mathcal{C}(0)| < \infty) \leq \sum_{n=4}^{\infty} n \cdot 4^n (1-p)^n.$$

Thus, if p is sufficiently near 1, then $\mathbb{P}_p(|\mathcal{C}(0)| < \infty)$ is bounded away from 1. □

Proposition. Let A_∞ be the event that there is an infinite cluster.

(i) If $\theta(p) = 0$, then $\mathbb{P}_p(A_\infty) = 0$.

(ii) If $\theta(p) > 0$, then $\mathbb{P}_p(A_\infty) = 1$.

Proof.

(i) We have

$$\mathbb{P}_p(A_\infty) = \mathbb{P}_p(\exists x : |\mathcal{C}(x)| = \infty) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(|\mathcal{C}(x)| = \infty) = \sum \theta(p) = 0.$$

(ii) We need to apply the *Kolmogorov 0-1 law*. Recall that if $X_1, X_2, \ldots$ are independent random variables, and $\mathcal{F}_n = \sigma(X_k : k \geq n)$, $\mathcal{F}_\infty = \bigcap_{n \geq 0} \mathcal{F}_n$, Then $\mathcal{F}_\infty$ is trivial, i.e. for all $A \in \mathcal{F}_\infty$, $\mathbb{P}(A) \in \{0, 1\}$.

So we order the edges of $\mathbb{Z}^d$ as $e_1, e_2, \ldots$ and denote their states

$$w(e_1), w(e_2), \ldots.$$

These are iid random variables. We certainly have $\mathbb{P}_p(A_\infty) \geq \theta(p) > 0$. So if we can show that $A_\infty \in \mathcal{F}_\infty$, then we are done. But this is clear, since changing the states of a finite number of edges does not affect the occurrence of A_∞. □

Theorem (Burton and Keane)**.** If $p > p_c$, then there exists a unique infinite cluster with probability 1.

Proof. Let N be the number of infinite clusters. Then by the lemma, we know N is constant almost surely. So there is some $k \in \mathbb{N} \cup \{\infty\}$ such that $\mathbb{P}_p(N = k) = 1$. First of all, we know that $k \neq 0$, since $\theta(p) > 0$. We shall first exclude $2 \leq k < \infty$, and then exclude $k = \infty$.

Assume that $k < \infty$. We will show that $\mathbb{P}_p(n = 1) > 0$, and hence it must be the case that $\mathbb{P}_p(n = 1) = 1$.

To bound this probability, we let $B(n) = [-n, n]^d \cap \mathbb{Z}^d$ (which we will sometimes write as B_n), and let $\partial B(n)$ be its boundary. We know that

$$\mathbb{P}_p(\text{all infinite clusters intersect } \partial B(n)) \to 1$$

as $n \to \infty$. This is since with probability 1, there are only finitely many clusters by assumption, and for each of these configurations, all infinite clusters intersect $\partial B(n)$ for sufficiently large n.

In particular, we can take n large enough such that

$$\mathbb{P}_p(\text{all infinite clusters intersect } \partial B(n)) \geq \frac{1}{2}.$$

We can then bound

$$\begin{aligned}\mathbb{P}_p(N = 1) \geq \mathbb{P}_p(&\text{all infinite clusters intersect } \partial B(n) \\ &\text{and all edges in } B(n) \text{ are open}).\end{aligned}$$

Finally, note that the two events in there are independent, since they involve different edges. But the probability that all edges in $B(n)$ are open is just $p^{E(B(n))}$. So

$$\mathbb{P}_p(N = 1) \geq \frac{1}{2} p^{E(B(n))} > 0.$$

So we are done.

We now have to show that $k \neq \infty$. This involves the notion of a *trifurcation*. The idea is that we will show that if $k = \infty$, then the probability that a vertex is a trifurcation is positive. This implies the expected number of trifurcations is $\sim n^d$. We will then show deterministically that the number of trifurcations inside $B(n)$ must be $\leq |\partial B(n)|$, and so there are $O(n^{d-1})$ trifurcations, which is a contradiction.

We say a vertex x is a *trifurcation* if the following three conditions hold:

(i) x is in an infinite open cluster $\mathcal{C}_\infty$;

(ii) There exist exactly three open edges adjacent to x;

(iii) $\mathcal{C}_\infty \setminus \{x\}$ contains exactly three infinite clusters and no finite ones.

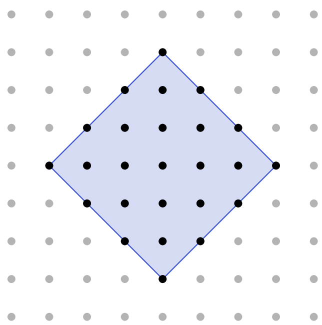
This is clearly a translation invariant notion. So

$$\mathbb{P}_p(0 \text{ is a trifurcation}) = \mathbb{P}_p(x \text{ is a trifurcation})$$

for all $x \in \mathbb{Z}^d$.

Claim. $\mathbb{P}_p(0 \text{ is a trifurcation}) > 0$.

We need to use something slightly different from $B(n)$. We define $S(n) = \{x \in \mathbb{Z}^d : \|x\|_1 \leq n\}$.



The crucial property of this is that for any $x_1, x_2, x_3 \in \partial S(n)$, there exist three disjoint self-avoiding paths joining x_i to 0 (exercise!). For each triple x_1, x_2, x_3, we arbitrarily pick a set of three such paths, and define the event

$$J(x_1, x_2, x_3) = \{\text{all edges on these 3 paths are open} \\ \text{and everything else inside } S(n) \text{ is closed}\}.$$

Next, for every possible infinite cluster in $\mathbb{Z}^d \setminus S(n)$ that intersects $\partial S(n)$ at at least one point, we pick a designated point of intersection arbitrarily.

Then we can bound

$$\mathbb{P}_p(0 \text{ is a trifurcation}) \geq \mathbb{P}_p(\exists \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \\ \text{infinite clusters which intersect } \partial S(n) \text{ at } x_1, x_2, x_3, \text{ and } J(x_1, x_2, x_3)).$$

Rewrite the right-hand probability as

$$\mathbb{P}_p(J(x_1, x_2, x_3) \mid \exists \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z} \text{ intersecting } \partial S(n)) \\ \times \mathbb{P}_p(\exists \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus \partial S(n))$$

We can bound the first term by

$$\min(p, 1-p)^{E(S(n))}.$$

To bound the second probability, we have already assumed that $\mathbb{P}_p(N = \infty) = 1$. So $\mathbb{P}_p(\exists \mathcal{C}_\infty^1, \mathcal{C}_\infty^2, \mathcal{C}_\infty^3 \subseteq \mathbb{Z}^d \setminus S(n) \text{ intersecting } \partial S(n)) \to 1$ as $n \to \infty$. We can then take n large enough such that the probability is $\geq \frac{1}{2}$. So we have shown that $c \equiv \mathbb{P}_p(0 \text{ is a trifurcation}) > 0$.

Using the linearity of expectation, it follows that

$$\mathbb{E}_p[\text{number of trifurcations inside } B(n))] \geq c|B(n)| \sim n^d.$$

On the other hand, we can bound the number of trifurcations in $B(n)$ by $|\partial B(n)|$. To see this, suppose x_1 is a trifurcation in $B(n)$. By definition, there exists 3 open paths to $\partial B(n)$. Fix three such paths. Let x_2 be another trifurcation. It also has 3 open paths to the $\partial B(n)$, and its paths to the boundary could intersect those of x_1. However, they cannot create a cycle, by definition of a trifurcation. For simplicity, we add the rule that when we produce the paths for x_2, once we intersect the path of x_1, we continue following the path of x_1.

Exploring all trifurcations this way, we obtain a forest inside $B(n)$, and the boundary points will be the leaves of the forest. Now the trifurcations have degree 3 in this forest. The rest is just combinatorics.

Exercise. For any tree, the number of degree 3 vertices is always less than the number of leaves. □

1.2 Correlation inequalities

Theorem. If N is an increasing random variable and $p_1 \leq p_2$, then

$$\mathbb{E}_{p_1}[N] \leq \mathbb{E}_{p_2}[N],$$

and if an event A is increasing, then

$$\mathbb{P}_{p_1}(A) \leq \mathbb{P}_{p_2}(A).$$

Proof. Immediate from coupling. □

Theorem (Fortuin–Kasteleyn–Ginibre (FKG) inequality)**.** Let X and Y be increasing random variables with $\mathbb{E}_p[X^2], \mathbb{E}_p[Y^2] < \infty$. Then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y].$$

In particular, if A and B are increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Equivalently,

$$\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A).$$

Proof. The plan is to first prove this in the case where X and Y depend on a finite number of edges, and we do this by induction. Afterwards, the desired result can be obtained by an application of the martingale convergence theorem. In fact, the "real work" happens when we prove it for X and Y depending on a single edge. Everything else follows from messing around with conditional probabilities.

If X and Y depend only on a single edge e_1, then for any $\omega_1, \omega_2 \in \{0, 1\}$, we claim that

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0.$$

Indeed, X and Y are both increasing. So if $\omega_1 > \omega_2$, then both terms are positive; if $\omega < \omega_2$, then both terms are negative, and there is nothing to do if they are equal.

In particular, we have

$$\sum_{\omega_1,\omega_2\in\{0,1\}} (X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2))\mathbb{P}_p(\omega(e_1) = \omega_1)\mathbb{P}_p(\omega(e_1) = \omega_2) \geq 0.$$

Expanding this, we find that the LHS is $2(\mathbb{E}_p[XY] - \mathbb{E}_p[X]\mathbb{E}_p[Y])$, and so we are done.

Now suppose the claim holds for X and Y that depend on $n < k$ edges. We shall prove the result when they depend on k edges $e_1, \ldots, e_k$. We have

$$\mathbb{E}_p[XY] = \mathbb{E}_p[\mathbb{E}_p[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})]].$$

Now after conditioning on $\omega(e_1), \ldots, \omega(e_{k-1})$, the random variables X and Y become increasing random variables of $\omega(e_k)$. Applying the first step, we get

$$\begin{aligned}\mathbb{E}_p&[XY \mid \omega(e_1), \ldots, \omega(e_{k-1})]\\ &\geq \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]. \quad (*)\end{aligned}$$

But $\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]$ is a random variable depending on the edges $e_1, \ldots, e_{k-1}$, and moreover it is increasing. So the induction hypothesis tells us

$$\begin{aligned}\mathbb{E}_p&\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right]\\ &\geq \mathbb{E}_p\left[\mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right]\mathbb{E}_p\left[\mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_{k-1})]\right]\end{aligned}$$

Combining this with (the expectation of) $(*)$ then gives the desired result.

Finally, suppose X and Y depend on the states of a countable set of edges $e_1, e_2, \ldots$. Let's define

$$\begin{aligned}X_n &= \mathbb{E}_p[X \mid \omega(e_1), \ldots, \omega(e_n)]\\ Y_n &= \mathbb{E}_p[Y \mid \omega(e_1), \ldots, \omega(e_n)]\end{aligned}$$

Then X_n and Y_n are martingales, and depend only on the states of only finitely many edges. So we know that

$$\mathbb{E}_p[X_nY_n] \geq \mathbb{E}_p[X_n]\mathbb{E}_p[Y_n] = \mathbb{E}_p[X]\mathbb{E}_p[Y].$$

By the $\mathcal{L}^2$-martingale convergence theorem, $X_n \to X$, $Y_n \to Y$ in L^2 and almost surely. So taking the limit $n \to \infty$, we get

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y]. \qquad \square$$

Theorem (BK inequality)**.** Let F be a finite set and $\Omega = \{0,1\}^F$. Let A and B be increasing events. Then

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof (Bollobás and Leader). We prove by induction on the size n of the set F. For $n = 0$, it is trivial.

Suppose it holds for $n-1$. We want to show it holds for n. For $D \subseteq \{0,1\}^F$ and $i = 0, 1$, set

$$D_i = \{(\omega_1, \ldots, \omega_{n-1}) : (\omega_1, \ldots, \omega_{n-1}, i) \in D\}.$$

Let $A, B \subseteq \{0,1\}^F$, and $C = A \circ B$. We check that

$$C_0 = A_0 \circ B_0, \quad C_1 = (A_0 \circ B_1) \cup (A_1 \circ B_0).$$

Since A and B are increasing, $A_0 \subseteq A_1$ and $B_0 \subseteq B_1$, and A_i and B_i are also increasing events. So

$$\begin{aligned} C_0 &\subseteq (A_0 \circ B_1) \cap (A_1 \circ B_0) \\ C_1 &\subseteq A_1 \circ B_1. \end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned} \mathbb{P}_p(C_0) &= \mathbb{P}_p(A_0 \circ B_0) \leq \mathbb{P}_p(A_0)\mathbb{P}_p(B_0) \\ \mathbb{P}_p(C_1) &\leq \mathbb{P}_p(A_1 \circ B_1) \leq \mathbb{P}_p(A_1)\mathbb{P}_p(B_1) \\ \mathbb{P}_p(C_0) + \mathbb{P}_p(C_1) &\leq \mathbb{P}_p((A_0 \circ B_1) \cap (A_1 \circ B_0)) + \mathbb{P}_p((A \circ B_1) \cup (A_1 \circ B_0)) \\ &= \mathbb{P}_p(A_0 \circ B_1) + \mathbb{P}_p(A_1 \circ B_0) \\ &\leq \mathbb{P}_p(A_0)\mathbb{P}_p(B_1) + \mathbb{P}_p(A_1)\mathbb{P}_p(B_0). \end{aligned}$$

Now note that for any D, we have

$$\mathbb{P}_p(D) = p\mathbb{P}_p(D_1) + (1-p)\mathbb{P}_p(D_0).$$

By some black magic, we multipy the first inequality by $(1-p)^2$, the second by p^2 and the third by $p(1-p)$. This gives

$$p\mathbb{P}_p(C_1) + (1-p)\mathbb{P}_p(C_0) \leq (p\mathbb{P}_p(A_1) + (1-p)\mathbb{P}_p(A_0))(p\mathbb{P}_p(B_1) + (1-p)\mathbb{P}_p(B_0)).$$

Expand and we are done. □

Theorem (Reimer)**.** For all events A, B depending on a finite set, we have $\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$.

Theorem. If $\chi(p) < \infty$, then there exists a positive constant c such that for all $n \geq 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq e^{-cn}.$$

Proof. Let

$$X_n = \sum_{x \in \partial B(n)} 1(0 \leftrightarrow x).$$

Now consider

$$\sum_{n=0}^{\infty} \mathbb{E}[X_n] = \sum_n \sum_{x \in \partial B(n)} \mathbb{P}_p(0 \leftrightarrow x) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x) = \chi(p).$$

Since $\chi(p)$ is finite, we in particular have $\mathbb{E}_p[X_n] \to 0$ as $n \to \infty$. Take m large enough such that $\mathbb{E}_p[X_m] < \delta < 1$.

Now we have

$$\begin{aligned}
\mathbb{P}_p(0 \leftrightarrow \partial B(m+k)) &= \mathbb{P}_p(\exists x \in \partial B(m) : 0 \leftrightarrow x \text{ and } x \leftrightarrow \partial B(m+k) \text{ disjointly}) \\
&\leq \sum_{x \in \partial B(m)} \mathbb{P}_p(0 \leftrightarrow x)\mathbb{P}_p(x \leftrightarrow \partial B(m+k)) \qquad \text{(BK)} \\
&\leq \sum_{x \in \partial B(m)} \mathbb{P}_p(0 \leftrightarrow x)\mathbb{P}_p(0 \leftrightarrow \partial B(k)) \qquad \text{(trans. inv.)} \\
&\leq \mathbb{P}_p(0 \leftrightarrow B(k))\mathbb{E}_p[X_m].
\end{aligned}$$

So for any $n > m$, write $n = qm + r$, where $r \in [0, m-1]$. Then iterating the above result, we have

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq \mathbb{P}_p(0 \leftrightarrow B(mq)) \leq \delta^q \leq \delta^{-1+\frac{n}{m}} \leq e^{-cn}. \qquad \square$$

Theorem (Russo's formula)**.** Let A be an increasing event that depends on the states of a finite number of edges. Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)],$$

where $N(A)$ is the number of pivotal edges for A.

Proof. Assume that A depends the states of m edges $e_1, \ldots, e_m$. The idea is to let each e_i be open with probability p_i, whree the $\{p_i\}$ may be distinct. We then vary the p_i one by one and see what happens.

Writing $\bar{p} = (p_1, \ldots, p_m)$, we define

$$f(p_1, \ldots, p_m) = \mathbb{P}_{\bar{p}}(A),$$

Now f is the sum of the probability of all configurations of $\{e_1, \ldots, e-m\}$ for which A happens, and is hence a finite sum of polynomials. So in particular, it is differentaible.

We now couple all percolation process. Let $(X(e) : e \in \mathbb{L}^d)$ be iid $U[0,1]$ random variables. For a vector $\bar{p} = (p(e) : e \in \mathbb{L}^d)$, we write

$$\eta_{\bar{p}}(e) = 1(X(e) \leq p(e)).$$

Then we have $\mathbb{P}_{\bar{p}}(A) = \mathbb{P}(\eta_{\bar{p}} \in A)$.

Fix an edge f and let $\bar{p}' = (p'(e))$ be such that $p'(e) = p(e)$ for all $e \neq f$, and $p'(f) = p(f) + \delta$ for some $\delta > 0$. Then

$$\begin{aligned}
\mathbb{P}_{\bar{p}'}(A) - \mathbb{P}_{\bar{p}}(A) &= \mathbb{P}(\eta_{\bar{p}'} \in A) - \mathbb{P}(\eta_{\bar{p}} \in A) \\
&= \mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \in A) + \mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \in A) - \mathbb{P}(\eta_{\bar{p}} \in A).
\end{aligned}$$

But we know A is increasing, so $\mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \in A) = \mathbb{P}(\eta_{\bar{p}} \in A)$. So the first and last terms cancel, and we have

$$\mathbb{P}_{\bar{p}'}(A) - \mathbb{P}_{\bar{p}}(A) = \mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \not\in A).$$

But we observe that we simply have

$$\mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \not\in A) = \delta \cdot \mathbb{P}_{\bar{p}}(f \text{ is pivotal for } A).$$

Indeed, we by definition of pivotal edges, we have

$$\mathbb{P}(\eta_{\bar{p}'} \in A, \eta_{\bar{p}} \notin A) = \mathbb{P}_{\bar{p}}(f \text{ is pivotal for } A,\, p(f) < X(f) \leq p(f) + \delta).$$

Since the event $\{f \text{ is pivotal for A}\}$ is independent of the state of the edge f, we obtain

$$\mathbb{P}_{\bar{p}}(f \text{ is pivotal}, p(f) < X(f) \leq p(f) + \delta) = \mathbb{P}_{\bar{p}}(f \text{ is pivotal}) \cdot \delta.$$

Therefore we have

$$\frac{\partial}{\partial p(f)} \mathbb{P}_{\bar{p}}(A) = \lim_{\delta \to 0} \frac{\mathbb{P}_{\bar{p}'}(A) - \mathbb{P}_{\bar{p}}(A)}{\delta} = \mathbb{P}_{\bar{p}}(f \text{ is pivotal for } A).$$

The desired result then follows from the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p} \mathbb{P}_p(A) &= \sum_{i=1}^m \frac{\partial}{\partial p(e_i)} \mathbb{P}_{\bar{p}}(A) \Big|_{\bar{p}=(p,\ldots,p)} \\
&= \sum_{i=1}^m \mathbb{P}_p(e_i \text{ is pivotal for } A) \\
&= \mathbb{E}_p[N(A)].
\end{aligned}$$

□

Corollary. Let A be an increasing event that depends on m edges. Let $p \leq q \in [0,1]$. Then $\mathbb{P}_q(A) \leq \mathbb{P}_p(A) \left(\frac{q}{p}\right)^m$.

Proof. We know that $\{f \text{ is pivotal for A}\}$ is independent of the state of f, and so

$$\mathbb{P}_p(\omega(f) = 1, f \text{ is pivotal for } A) = p\mathbb{P}_p(f \text{ is pivotal for } A).$$

But since A is increasing, if $\omega(f) = 1$ and f is pivotal for A, then A occurs. Conversely, if f is pivotal and A occurs, then $\omega(f) = 1$.

Thus, by Russo's formula, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p} \mathbb{P}_p(A) &= \mathbb{E}_p[N(A)] \\
&= \sum_e \mathbb{P}_p(e \text{ is pivotal for } A) \\
&= \sum_e \frac{1}{p} \mathbb{P}_p(\omega(e) = 1, e \text{ is pivotal for } A) \\
&= \sum_e \frac{1}{p} \mathbb{P}_p(e \text{ is pivotal} \mid A) \mathbb{P}_p(A) \\
&= \mathbb{P}_p(A) \frac{1}{p} \mathbb{E}_p[N(A) \mid A].
\end{aligned}$$

So we have

$$\frac{\frac{\mathrm{d}}{\mathrm{d}p} \mathbb{P}_p(A)}{\mathbb{P}_p(A)} = \frac{1}{p} \mathbb{E}_p[N(A) \mid A].$$

Integrating, we find that

$$\log \frac{\mathbb{P}_q(A)}{\mathbb{P}_p(A)} = \int_p^q \frac{1}{u} \mathbb{E}_u[N(A) \mid A] \, \mathrm{d}u.$$

Bounding $\mathbb{E}_u[N(A) \mid A] \leq m$, we obtain the desired bound. □

Theorem. Let $d \geq 2$ and $B_n = [-n, n]^d \cap \mathbb{Z}^d$.

(i) If $p < p_c$, then there exists a positive constant c for all $n \geq 1$, $\mathbb{P}_p(0 \leftrightarrow \partial B_n) \leq e^{-cn}$.

(ii) If $p > p_c$, then
$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \geq \frac{p - p_c}{p(1 - p_c)}.$$

Proof (Duminil-Copin and Tassion). If $S \subseteq V$ is finite, we write
$$\partial S = \{(x, y) \in E : x \in S, y \notin S\}.$$

We write $x \overset{S}{\leftrightarrow} y$ if there exists an open path of edges from x to y all of whose end points lie in S.

Now suppose that $0 \in S$. We define
$$\varphi_p(S) = p \sum_{(x,y) \in \partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x).$$

Define
$$\tilde{p}_c = \sup\{p \in [0, 1] : \text{exists a finite set } S \text{ with } 0 \in S \text{ and } \varphi_p(S) < 1\}.$$

Claim. It suffices to prove (i) and (ii) with p_c replaced by $\tilde{p}_c$.

Indeed, from (i), if $p < \tilde{p}_c$, then $\mathbb{P}_p(0 \leftrightarrow \partial B_n) \leq e^{-cn}$. So taking the limit $n \to \infty$, we see $\theta(p) = 0$. So $\tilde{p}_c \leq p_c$. From (ii), if $p > \tilde{p}_c$, then $\theta(p) > 0$. So $p_c \leq \tilde{p}_c$. So $p_c = \tilde{p}_c$.

We now prove (i) and (ii):

(i) Let $p < \tilde{p}_c$. Then there exists a finite set S containing 0 with $\varphi_p(S) < 1$. Since S is finite, we can pick L large enough so that $S \subseteq B_{L-1}$. We will prove that $\mathbb{P}_p(0 \leftrightarrow \partial B_{kL}) \leq (\varphi_p(S))^{k-1}$ for $k \geq 1$.

Define $\mathcal{C} = \{x \in S : 0 \overset{S}{\leftrightarrow} x\}$. Since $S \subseteq B_{L-1}$, we know $S \cap \partial B_{kL} = \emptyset$.

Now if we have an open path from 0 to ∂B_{kL}, we let x be the last element on the path that lies in $\mathcal{C}$. We can then replace the path up to x by a path that lies entirely in S, by assumption. This is then a path that lies in $\mathcal{C}$ up to x, then takes an edge on ∂S, and then lies entirely outside of $\mathcal{C}^c$. Thus,
$$\mathbb{P}_p(0 \leftrightarrow \partial B_{kL}) \leq \sum_{\substack{A \subseteq S \\ 0 \in A}} \sum_{(x,y) \in \partial A} \mathbb{P}_p(0 \overset{A}{\leftrightarrow} x,\ (x, y) \text{ open},\ \mathcal{C} = A,\ y \overset{A^C}{\leftrightarrow} \partial B_{kL}).$$

Now observe that the events $\{\mathcal{C} = A, 0 \overset{S}{\leftrightarrow} x\}$, $\{(x, y) \text{ is open}\}$ and $\{y \overset{A^c}{\leftrightarrow} \partial B_{kL}\}$ are independent. So we obtain
$$\mathbb{P}_p(0 \leftrightarrow \partial B_{kL}) \leq \sum_{A \subseteq S, 0 \in A} \sum_{(x,y) \in \partial S} p\ \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x, \mathcal{C} = A)\ \mathbb{P}_p(y \overset{A^c}{\leftrightarrow} \partial B_{kL}).$$

Since we know that $y \in B_L$, we can bound
$$\mathbb{P}_p(y \overset{A^c}{\leftrightarrow} \partial B_{kL}) \leq \mathbb{P}_p(0 \leftrightarrow \partial B_{(k-1)L}).$$

So we have

$$\begin{aligned}\mathbb{P}_p(0 \leftrightarrow \partial B_{kL}) &\le p\ \mathbb{P}_p(0 \leftrightarrow \partial B_{(k-1)L}) \sum_{A \subseteq S, 0 \in A} \sum_{(x,y) \in \partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x, \mathcal{C} = A)\\ &= \mathbb{P}_p(0 \leftrightarrow \partial B_{(k-1)L})\ p \sum_{(x,y)\in \partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x)\\ &= \mathbb{P}_p(0 \leftrightarrow \partial B_{(k-1)L}) \varphi_p(S).\end{aligned}$$

Iterating, we obtain the deseired result.

(ii) We want to use Russo's formula. We claim that it suffices to prove that

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(0 \leftrightarrow \partial B_n) \ge \frac{1}{p(1-p)} \inf_{S \subseteq B_n, 0 \in S} \varphi_p(S)(1 - \mathbb{P}_p(0 \leftrightarrow \partial B_n)).$$

Indeed, if $p > \tilde{p}_c$, we integrate from $\tilde{p}_c$ to p, use in this range $\varphi_p(S) \ge 1$, and then take the limit as $n \to \infty$.

The event $\{0 \leftrightarrow \partial B_n\}$ is increasing and only dependson a finite number of edges. So we can apply Russo's formula

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(0 \leftrightarrow \partial B_n) = \sum_{e \in B_n} \mathbb{P}_p(e \text{ is pivotal for } \{0 \leftrightarrow \partial B_n\})$$

Since being pivotal and being open/closed are independent, we can write this as

$$\begin{aligned}&= \sum_{e \in B_n} \frac{1}{1-p}\mathbb{P}_p(e \text{ is pivotal for } \{0 \leftrightarrow \partial B_n\},\ e \text{ is closed})\\ &= \sum_{e \in B_n} \frac{1}{1-p}\mathbb{P}_p(e \text{ is pivotal for } \{0 \leftrightarrow \partial B_n\},\ 0 \not\leftrightarrow \partial B_n)\end{aligned}$$

Define $\mathcal{S} = \{x \in B_n : x \not\leftrightarrow \partial B_n\}$. Then $\{0 \not\leftrightarrow \partial B_n\}$ implies $0 \in S$. So

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(0 \leftrightarrow \partial B_n) = \frac{1}{1-p} \sum_{e \in B_n} \sum_{A \subseteq B_n, 0 \in A} \mathbb{P}_p(e \text{ is pivotal},\ \mathcal{S} = A)$$

Given that $\mathcal{S} = A$, an edge $e = (x,y)$ is pivotal iff $e \in \partial A$ and $0 \overset{A}{\leftrightarrow} x$. So we know

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(0 \leftrightarrow \partial B_n) = \frac{1}{1-p} \sum_{A \subseteq B_n, 0 \in A} \sum_{(x,y) \in \partial A} \mathbb{P}_p(0 \overset{A}{\leftrightarrow} x,\ \mathcal{S} = A).$$

Observe that $\{0 \overset{A}{\leftrightarrow} x\}$ and $\{\mathcal{S} = A\}$ are independent, since to determine if $\mathcal{S} = A$, we only look at the edges on the boundary of A. So the above is equal to

$$\begin{aligned}&\frac{1}{1-p} \sum_{A \subseteq B_n, 0 \in A} \sum_{(x,y) \in \partial A} \mathbb{P}_p(0 \overset{A}{\leftrightarrow} x)\mathbb{P}_p(\mathcal{S} = A)\\ &= \frac{1}{p(1-p)} \sum_{A \subseteq B_n, 0 \in A} \varphi_p(A)\mathbb{P}_p(\mathcal{S} = A)\\ &\ge \frac{1}{p(1-p)} \inf_{S \subseteq B_n, 0 \in S} \varphi_P(S)\mathbb{P}_p(0 \not\leftrightarrow \partial B_n),\end{aligned}$$

as desired. □

1.3 Two dimensions

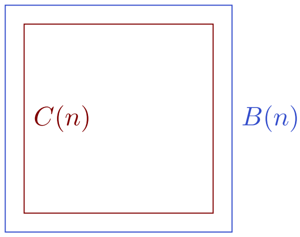
Theorem. In $\mathbb{Z}^2$, we have $\theta\left(\frac{1}{2}\right) = 0$ and $p_c = \frac{1}{2}$.

Proof. First we prove that $\theta\left(\frac{1}{2}\right) = 0$. This will imply that $p_c \geq \frac{1}{2}$.

Suppose not, and $\theta\left(\frac{1}{2}\right) > 0$. Recall that $B(n) = [-n, n]^2$, and we define

$$C(n) = [-(n-1), (n-1)]^2 + \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

in the dual lattice. The appearance of the -1 is just a minor technical inconvenience. For the same n, our $B(n)$ and $C(n)$ look like



We claim that for large n, there is a positive probability that there are open paths from the left and right edges of $B(n)$ to ∞, and also there are *closed* paths from the top and bottom edges of $C(n)$ to ∞. But we know that with probability 1, there is a *unique* infinite cluster in both the primal lattice and the dual lattice. To connect up the two infinite open paths starting from the left and right edges of $B(n)$, there must be an open left-right crossing of $B(n)$. To connect up the two infinite closed paths starting from the top and bottom of $C(n)$, there must be a closed top-bottom crossing. But these cannot both happen, since this would require an open primal edge crossing a closed dual edge, which is impossible.

To make this an actual proof, we need to show that these events do happen with positive probability. We shall always take $p = \frac{1}{2}$, and will not keep repeating it.

First note that since there is, in particular, an infinite cluster with probability 1, we have

$$\mathbb{P}(\partial B(n) \leftrightarrow \infty) \to 1.$$

So we can pick n large enough such that

$$\mathbb{P}(\partial B(n) \leftrightarrow \infty),\ \mathbb{P}(\partial C(n) \leftrightarrow \infty) \geq 1 - \frac{1}{8^4}.$$

Let $A_\ell / A_r / A_t / A_b$ be the events that the left/right/top/bottom side of $B(n)$ is connected to ∞ via an open path of edges. Similarly, let D_ℓ be the event that the left of $C(n)$ is connected to ∞ via a closed path, and same for D_r, D_r, D_b.

Of course, by symmetry, for $i, j \in \{\ell, r, t, b\}$, we have $\mathbb{P}(A_i) = \mathbb{P}(A_j)$. Using FKG, we can bound

$$\mathbb{P}(\partial S_n \not\leftrightarrow \infty) = \mathbb{P}(A_\ell^c \cap A_r^c \cap A_t^c \cap A_b^c) \geq (\mathbb{P}(A_\ell^c))^4 = (1 - \mathbb{P}(A_\ell))^4.$$

Thus, by assumption on n, we have

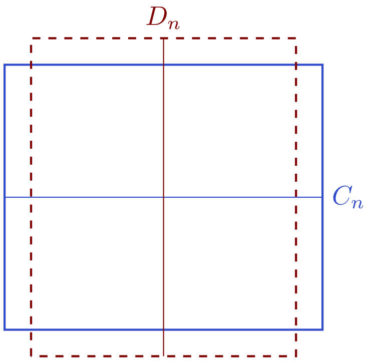
$$(1 - \mathbb{P}(A_\ell))^4 \leq \frac{1}{8^4},$$

hence
$$\mathbb{P}(A_\ell) \geq \frac{7}{8}.$$
Of course, the same is true for other A_i and D_j.

Then if $G = A_\ell \cap A_r \cap D_t \cap D_b$, which is the desired event, then we have
$$\mathbb{P}(G^c) \leq \mathbb{P}(A_\ell^c) + \mathbb{P}(A_r^c) + \mathbb{P}(D_t^c) + \mathbb{P}(D_b^c) \leq \frac{1}{2}.$$
So it follows that
$$\mathbb{P}(G) \geq \frac{1}{2},$$
which, as argued, leads to a contradiction.

So we have $p_c \geq \frac{1}{2}$. It remains to prove that $p_c \leq \frac{1}{2}$. Suppose for contradiction that $p_c > \frac{1}{2}$. Then $p = \frac{1}{2}$ is in the subcritical regime, and we expect exponential decay. Thus, (again with $p = \frac{1}{2}$ fixed) there exists a $c > 0$ such that for all $n \geq 1$,
$$\mathbb{P}(0 \leftrightarrow \partial B(n)) \leq e^{-cn}.$$
Consider $C_n = [0, n+1] \times [0, n]$, and define A_n to be the event that there exists a left-right crossing of C_n by open edges.

Again consider the dual box $D_n = [0, n] \times [-1, n] + \left(\frac{1}{2}, \frac{1}{2}\right)$.



Define B_n to be the event that there is a top-bottom crossing of D_n by closed edges of the dual.

As before, it cannot be the case that A_n and B_n both occur. In fact, A_n and B_n partition the whole space, since if A_n does not hold, then every path from left to right of C_n is blocked by a closed path of the dual. Thus, we know
$$\mathbb{P}(A_n) + \mathbb{P}(B_n) = 1.$$
But also by symmetry, we have $\mathbb{P}(A_n) = \mathbb{P}(B_n)$. So
$$\mathbb{P}(A_n) = \frac{1}{2}.$$
On the other hand, for any point on the left edge, the probability of it reaching the right edge decays exponentially with n. Thus,
$$\mathbb{P}(A_n) \leq n(n+1)\mathbb{P}(0 \leftrightarrow \partial B_n) \leq (n+1)e^{-cn}$$
which is a contradiction. So we are done. $\square$

Proposition. $\mathbb{P}_{\frac{1}{2}}(LR(\ell)) \geq \frac{1}{2}$ for all ℓ.

Proof. We have already seen that the probability of there being a left-right crossing of $[0, n+1] \times [0, n]$ is at least $\frac{1}{2}$. But if there is a left-right crossing of $[0, n+1] \times [0, n]$, then there is also a left-right crossing of $[0, n] \times [0, n]$! □

Theorem (Russo–Symour–Welsh (RSW) theorem)**.** If $\mathbb{P}_p(LR(\ell)) = \alpha$, then

$$\mathbb{P}_p(O(\ell)) \geq \left(\alpha\left(1 - \sqrt{1-\alpha}\right)^4\right)^{12}.$$

Lemma. If $\mathbb{P}_p(LR(\ell)) = \alpha$, then

$$\mathbb{P}_p\left(LR\left(\tfrac{3}{2}\ell, \ell\right)\right) \geq (1 - \sqrt{1-\alpha})^3.$$

Lemma (nth root trick)**.** If $A_1, \ldots, A_n$ are increasing events all having the same probability, then

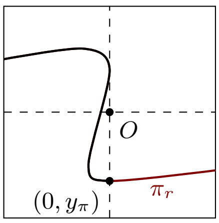
$$\mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^n A_i\right)\right)^{1/n}.$$

Proof sketch. Observe that the proof of FKG works for *decreasing* events as well, and then apply FKG to A_i^c. □

Proof sketch. Let $\mathcal{A}$ be the set of left-right crossings of $B(\ell) = [-\ell, \ell]^2$. Define a partial order on $\mathcal{A}$ by $\pi_1 \leq \pi_2$ if π_1 is contained in the closed bounded region of $B(\ell)$ below π_2.

Note that given any configuration, if the set of open left-right crossings is non-empty, then there exists a lowest one. Indeed, since $\mathcal{A}$ must be finite, it suffices to show that meets exist in this partial order, which is clear.

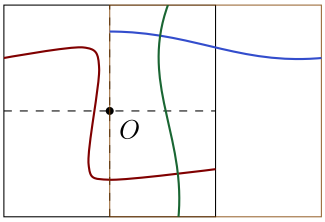
For a left-right crossing π, let $(0, y_\pi)$ be the last vertex on the vertical axis where π intersects, and let π_r be the path of the path that connects $(0, y_\pi)$ to the right.



Let

$$\begin{aligned}\mathcal{A}_- &= \{\pi \in \mathcal{A} : y_\pi \leq 0\}\\ \mathcal{A}_+ &= \{\pi \in \mathcal{A} : y_\pi \geq 0\}\end{aligned}$$

Letting $B(\ell)' = [0, 2\ell] \times [-\ell, \ell]$, our goal is to find a left-right crossing of the form



More precisely, we want the following paths:

(i) Some $\pi \in \mathcal{A}_-$

(ii) Some top-bottom crossing of $B(\ell')$ that crosses π_r.

(iii) Some left-right crossing of $B(\ell')$ that starts at the positive (i.e. non-negative) y axis.

To understand the probabilities of these events happening, we consider the "mirror" events and then apply the square root trick.

Let π_r' be the reflection of π_r on $\{(\ell, k) : k \in \mathbb{Z}\}$. For any $\pi \in \mathcal{A}$, we define

$$\begin{aligned}
V_\pi &= \{\text{all edges of } \pi \text{ are open}\}\\
M_\pi &= \{\text{exists open crossing from top of } B(\ell)' \text{ to } \pi_r \cup \pi_r'\}\\
M_\pi^- &= \{\text{exists open crossing from top of } B(\ell)' \text{ to } \pi_r\}\\
M_\pi^+ &= \{\text{exists open crossing from top of } B(\ell)' \text{ to } \pi_r'\}\\
L^+ &= \{\text{exists open path in } \mathcal{A}_+\}\\
L^- &= \{\text{exists open path in } \mathcal{A}_-\}\\
L_\pi &= \{\pi \text{ is the lowest open LR crossing of } B(\ell)\}\\
N &= \{\text{exists open LR crossing of } B(\ell)'\}\\
N^+ &= \{\text{exists open LR crossing in } B(\ell)' \text{ starting from positive vertical axis}\}\\
N^- &= \{\text{exists open LR crossing in } B(\ell)' \text{ starting from negative vertical axis}\}
\end{aligned}$$

In this notation, our previous observation was that

$$\underbrace{\bigcup_{\pi \in \mathcal{A}_-} (V_\pi \cap M_\pi^-)}_{G} \cap N^+ \subseteq LR\left(\frac{3}{2}\ell, \ell\right)$$

So we know that

$$\mathbb{P}_p\left(LR\left(\frac{3}{2}\ell, \ell\right)\right) \geq \mathbb{P}_p(G \cap N') \geq \mathbb{P}_p(G)\mathbb{P}_p(N^+),$$

using FKG.

Now by the "square root trick", we know

$$\mathbb{P}_p(N^+) \geq 1 - \sqrt{1 - \mathbb{P}_p(N^+ \cup N^-)}.$$

Of course, we have $\mathbb{P}_p(N^+ \cup N^-) = \mathbb{P}_p(LR(\ell)) = \alpha$. So we know that

$$\mathbb{P}_p(N^+) \geq 1 - \sqrt{1 - \alpha}.$$

We now have to understand G. To bound its probability, we try to bound it by the union of some disjoint events. We have

$$\begin{aligned}\mathbb{P}_p(G) &= \mathbb{P}_p\left(\bigcup_{\pi\in\mathcal{A}_-}(V_\pi \cap M_\pi^-)\right)\\ &\geq \mathbb{P}_p\left(\bigcup_{\pi\in A_-}(M_\pi^- \cap L_\pi)\right)\\ &= \sum_{\pi\in\mathcal{A}_-}\mathbb{P}_p(M_\pi^- \mid L_\pi)\mathbb{P}_p(L_\pi).\end{aligned}$$

Claim.

$$\mathbb{P}_p(M_\pi^- \mid L_\pi) \geq 1 - \sqrt{1-\alpha}.$$

Note that if π intersects the vertical axis in one point, then $\mathbb{P}_p(M_\pi^- \mid L_\pi) = \mathbb{P}_p(M_\pi^- \mid V_\pi)$, since L_π^- tells us what happens below π, and this does not affect the occurrence of M_π^-.

Since M_π^- and V_π are increasing events, by FKG, we have

$$\mathbb{P}_p(M_\pi^- \mid V_\pi) \geq \mathbb{P}_p(M_\pi^-) \geq 1 - \sqrt{1-\mathbb{P}_p(M_\pi^- \cup M_\pi^+)} = 1 - \sqrt{1-\mathbb{P}_p(M_\pi)}.$$

Since $\mathbb{P}_p(M_\pi) \geq \mathbb{P}_p(LR(\ell)) = \alpha$, the claim follows.

In the case where π is more complicated, we will need an extra argument, which we will not provide.

Finally, we have

$$\mathbb{P}_p(G) \geq \sum_{\pi\in\mathcal{A}_-}\mathbb{P}_p(L_\pi)(1-\sqrt{1-\alpha}) = (1-\sqrt{1-\alpha})\mathbb{P}_p(L^-).$$

But again by the square root trick,

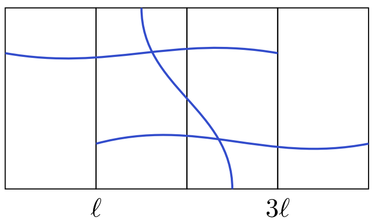
$$\mathbb{P}_p(L^-) \geq 1 - \sqrt{1-\mathbb{P}_p(L^+ \cup L^-)} = 1 - \sqrt{1-\alpha},$$

and we are done. □

Lemma.

$$\begin{aligned}\mathbb{P}_p(LR(2\ell,\ell)) &\geq \mathbb{P}_p(LR(\ell))\left(\mathbb{P}_p\left(LR\left(\frac{3}{2}\ell,\ell\right)\right)\right)^2\\ \mathbb{P}_p(LR(3\ell,\ell)) &\geq \mathbb{P}_p(LR(\ell))\left(\mathbb{P}_p\left(LR\left(2\ell,\ell\right)\right)\right)^2\\ \mathbb{P}_p(O(\ell)) &\geq \mathbb{P}_p(LR(3\ell,\ell))^4\end{aligned}$$

Proof. To prove the first inequality, consider the box $[0,4\ell]\times[-\ell,\ell]$.



We let

$$\begin{aligned} LR_1 &= \{\text{exists left-right crossing of } [0, 3\ell] \times [-\ell, \ell]\} \\ LR_2 &= \{\text{exists left-right crossing of } [\ell, 4\ell] \times [-\ell, \ell]\} \\ TB_1 &= \{\text{exists top-bottom crossing of } [\ell, 3\ell] \times [-\ell, \ell]\}. \end{aligned}$$

Then by FKG, we find that

$$\mathbb{P}_p(LR(2\ell, \ell)) \geq \mathbb{P}_p(LR_1 \cap LR_2 \cap TB_1) \geq \mathbb{P}_p(LR_1)\mathbb{P}_p(LR_2)\mathbb{P}_p(TB_1).$$

The others are similar. □

Theorem. There exists positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, A_1, A_2, A_4$ such that

$$\begin{aligned} \mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B(n)) &\leq A_1 n^{-\alpha_1} \\ \mathbb{P}_{\frac{1}{2}}(|\mathcal{C}(0)| \geq n) &\leq A_2 n^{-\alpha_2} \\ \mathbb{E}(|\mathcal{C}(0)|^{\alpha_3}) &\leq \infty \end{aligned}$$

Moreover, for $p > p_c = \frac{1}{2}$, we have

$$\theta(p) \leq A_4 \left(p - \frac{1}{2}\right)^{\alpha_4}.$$

Proof.

(i) We first prove the first inequality. Define dual boxes

$$B(k)_d = B(k) + \left(\frac{1}{2}, \frac{1}{2}\right).$$

The dual annuli $A(\ell)_d$ are defined by

$$A(\ell)_d = B(3\ell)_d \setminus B(\ell)_d.$$

We let $O(\ell)_d$ be the event that there is a closed dual circuit in $A(\ell)_d$ containing $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then RSW tells us there is some $\zeta \in (0, 1)$ such that

$$\mathbb{P}_{\frac{1}{2}}(O(\ell)_d) \geq \zeta,$$

independent of ℓ. Then observe that

$$\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B(3^k + 1)) \leq \mathbb{P}_p(O(3^r)_d \text{ does not occur for all } r < k).$$

Since the annuli $(A(3^r)_d)$ are disjoint, the events above are independent. So

$$\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B(3^k + 1)) \leq (1 - \zeta)^k,$$

and this proves the first inequality.

(ii) The second inequality follows from the first inequality plus the fact that $|\mathcal{C}(0)| \geq n$ implies $0 \leftrightarrow \partial B(g(n))$, for some function $g(n) \sim \sqrt{n}$.

(iii) To show that $\mathbb{E}_{\frac{1}{2}}[|\mathcal{C}(0)|^{\alpha_3}] < \infty$ for some α_3, we observe that this expectation is just
$$\sum_n \mathbb{P}_{\frac{1}{2}}(|\mathcal{C}(0)|^{\alpha_3} \geq n).$$

(iv) To prove the last part, note that
$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty) \leq \mathbb{P}_p(0 \leftrightarrow \partial B_n)$$
for all n. By the corollary of Russo's formula, and since $\{0 \leftrightarrow \partial B_n\}$ only depends on the edges in B_n, which are $\leq 18n^2$, we get that
$$\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B_n) \geq \left(\frac{1}{2p}\right)^{18n^2} \mathbb{P}_p(0 \leftrightarrow \partial B_n).$$
So
$$\theta(p) \leq (2p)^{18n^2} \mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B_n) \leq A_1 (2p)^{18n^2} n^{-\alpha_1}.$$
Now take $n = \lfloor (\log 2p)^{-1/2} \rfloor$. Then as $p \searrow \frac{1}{2}$, we have
$$n \sim \frac{1}{(2p-1)^{\frac{1}{2}}}.$$
Substituting this in, we get
$$\theta(p) \leq C\left(p - \frac{1}{2}\right)^{\alpha_1/2}. \qquad \square$$

Theorem. When $d = 2$ and $p > p_c$, there exists a positive constant c such that
$$\mathbb{P}_p(0 \leftrightarrow \partial B(n), |\mathcal{C}(0)| < \infty) \leq e^{-cn}.$$

Theorem (Grimmett–Marstrand)**.** Let F be an infinite-connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$. Then for all $\eta > 0$, there exists $k \in \mathbb{N}$ such that
$$p_c(2kF + B_k) \leq p_c + \eta.$$
In particular, for all $d \geq 3$, $p_c^{slab} = p_c$.

Theorem. If $d \geq 3$ and $p > p_c$, then there exists $c > 0$ such that
$$\mathbb{P}_p(0 \leftrightarrow \partial B(n), |\mathcal{C}(0)| < \infty) \leq e^{-cn}.$$

1.4 Conformal invariance and SLE in $d = 2$

Theorem (Smirnov, 2001)**.** Suppose (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$ are conformally equivalent. Then
$$\mathbb{P}(ac \leftrightarrow bd \text{ in } \Omega) = \mathbb{P}(a'c' \leftrightarrow b'd' \text{ in } \Omega').$$

2 Random walks

2.1 Random walks in finite graphs

Proposition. Let P be an irreducible matrix on Ω and $B \subseteq \Omega$, $f : B \to \mathbb{R}$ a function. Then

$$h(x) = \mathbb{E}_x[f(X_{\tau_B})]$$

is the unique extension of f which is harmonic on $\Omega \setminus B$.

Proof. It is obvious that $h(x) = f(x)$ for $x \in B$. Let $x \notin B$. Then

$$\begin{aligned} h(x) &= \mathbb{E}_x[f(X_{\tau_B})] \\ &= \sum_y \mathbb{P}(x,y)\mathbb{E}_x[f(X_{\tau_B}) \mid X_1 = y] \\ &= \sum_y \mathbb{P}(x,y)\mathbb{E}_y[f(X_{\tau_B})] \\ &= \sum_Y \mathbb{P}(x,y)h(y) \end{aligned}$$

So h is harmonic.

To show uniqueness, suppose h' be another harmonic extension. Take $g = h - h'$. Then $g = 0$ on B. Set

$$A = \left\{ x : g(x) = \max_{y \in \Omega \setminus B} g(y) \right\},$$

and let $x \in A \setminus B$. Now since $g(x)$ is the weighted average of its neighbours, and $g(y) \leq g(x)$ for all neighbours y of x, it must be the case that $g(x) = g(y)$ for all neighbours of y.

Since we assumed G is connected, we can construct a path from x to the boundary, where g vanishes. So $g(x) = 0$. In other words, $\max g = 0$. Similarly, $\min g = 0$. So $g = 0$. □

Proposition. Let θ be a flow from a to z satisfying the cycle law for any cycle. Let I the current flow associated to a voltage W. If $\|\theta\| = \|I\|$, then $\theta = I$.

Proof. Take $f = \theta - I$. Then f is a flow which satisfies Kirchhoff's node law at all vertices and the cycle law for any cycle. We want to show that $f = 0$. Suppose not. The we can find some e_1 such that $f(e_i) > 0$. But

$$\sum_{y \sim x} f(x,y) = 0$$

for all x, so there must exist an edge e_2 that e_1 leads to such that $f(e_2) > 0$. Continuing this way, we will get a cycle of ordered edges where $f > 0$, which violates the cycle law. □

Proposition. Take a weighted random walk on G. Then

$$\mathbb{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a)R_{\mathrm{eff}}(a,z)},$$

where $\tau_a^+ = \min\{t \geq 1 : X_t = a\}$.

Proof. Let
$$f(x) = \mathbb{P}_x(\tau_z < \tau_a).$$
Then $f(a) = 0$ and $f(z) = 1$. Moreover, f is harmonic on $\Omega \setminus \{a, z\}$. Let W be a voltage. By uniqueness, we know
$$f(x) = \frac{W(a) - W(x)}{W(a) - W(z)}.$$
So we know
$$\begin{aligned}
\mathbb{P}_a(\tau_z < \tau_a^+) &= \sum_{x \sim a} \mathbb{P}(a, x) f(x) \\
&= \sum_{x \sim a} \frac{c(a, x)}{c(a)} \frac{W(a) - W(x)}{W(a) - W(z)} \\
&= \frac{1}{c(a)(W(a) - W(z))} \sum_{x \sim a} I(a, x) \\
&= \frac{\|I\|}{c(a)(W(a) - W(z))} \\
&= \frac{1}{c(a) R_{\mathrm{eff}}(a, z)}.
\end{aligned}$$
□

Corollary. For any reversible chain and all a, z, we have
$$G_{\tau_z}(a, a) = c(a) R_{\mathrm{eff}}(a, z).$$

Proof. By the Markov property, the number of visits to a starting from a until τ_z is the geometric distribution $Geo(\mathbb{P}_a(\tau_z < \tau_a^+))$. So
$$G_{\tau_z}(a, a) = \frac{1}{\mathbb{P}_a(\tau_z < \tau_a^+)} = c(a) R_{\mathrm{eff}}(a, z).$$
□

Theorem (Thomson's principle)**.** Let G be a finite connected graph with conductances $(c(e))$. Then for any a, z, we have
$$R_{\mathrm{eff}}(a, z) = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } a \text{ to } z\}.$$
Moreover, the unit current flow from a to z is the unique minimizer.

Proof. Let i be the unit current flow associated to potential φ. Let j be another unit flow $a \to z$. We need to show that $\mathcal{E}(j) \geq \mathcal{E}(i)$ with equality iff $j = i$.

Let $k = j - i$. Then k is a flow of 0 strength. We have
$$\mathcal{E}(j) = \mathcal{E}(i + k) = \sum_e (i(e) + k(e))^2 r(e) = \mathcal{E}(i) + \mathcal{E}(k) + 2 \sum_e i(e) k(e) r(e).$$
It suffices to show that the last sum is zero. By definition, we have
$$i(x, y) = \frac{\varphi(x) - \varphi(y)}{r(x, y)}.$$

So we know

$$\begin{aligned}\sum_e i(e)k(e)r(e) &= \frac{1}{2}\sum_x \sum_{y\sim x} (\varphi(x) - \varphi(y))k(x,y)\\ &= \frac{1}{2}\sum_x \sum_{y\sim x} \varphi(x)k(x,y) + \frac{1}{2}\sum_x \sum_{y\sim x} \varphi(x)k(x,y).\end{aligned}$$

Now note that $\sum_{y\sim x} k(x,y) = 0$ for any x since k has zero strength. So this vanishes.

Thus, we get

$$\mathcal{E}(j) = \mathcal{E}(i) + \mathcal{E}(k),$$

and so $\mathcal{E}(j) \geq \mathcal{E}(i)$ with equality iff $\mathcal{E}(k) = 0$, i.e. $k(e) = 0$ for all e. □

Theorem (Rayleigh's monotonicity principle)**.** Let G be a finite connected graph and $(r(e))_e$ and $(r'(e))_e$ two sets of resistances on the edges such that $r(e) \leq r'(e)$ for all e. Then

$$R_{\text{eff}}(a, z; r) \leq R_{\text{eff}}(a, z; r').$$

for all $a, z \in G$.

Proof. By definition of energy, for any current i, we have

$$\mathcal{E}(i; r) \leq \mathcal{E}(i; r').$$

Then take the infimum and conclude by Thompson. □

Corollary. Suppose we add an edge to G which is not adjacent to a. This increases the escape probability

$$\mathbb{P}_a(\tau_z < \tau_a^+).$$

Proof. We have

$$\mathbb{P}_a(\tau_z < \tau_z^+) = \frac{1}{c(a)R_{\text{eff}}(a,z)}.$$

We can think of the new edge as an edge having infinite resistance in the old graph. By decreasing it to a finite number, we decrease R_{eff}. □

Theorem (Nash–Williams inequality)**.** Let (Π_k) be disjoint edge-cutsets separating a from z. Then

$$R_{\text{eff}}(a,z) \geq \sum_k \left(\sum_{e\in\Pi_k} c(e)\right)^{-1}.$$

Proof. By Thompson, it suffices to prove that for any unit flow θ from a to z, we have

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \left(\sum_{e\in\Pi_k} c(e)\right)^{-1}.$$

Certainly, we have

$$\sum_e (\theta(e))^2 r(e) \geq \sum_k \sum_{e\in\Pi_k} (\theta(e))^2 r(e).$$

We now use Cauchy–Schwarz to say

$$\left(\sum_{e\in\Pi_k}(\theta(e))^2 r(e)\right)\left(\sum_{e\in\Pi_k} c(e)\right) \geq \left(\sum_{e\in\Pi_k}|\theta(e)|\sqrt{r(e)c(e)}\right)^2$$
$$= \left(\sum_{e\in\Pi_k}|\theta(e)|\right)^2.$$

The result follows if we show that the final sum is ≥ 1.

Let F be the component of $G\setminus\Pi_k$ containing a. Let F' be the set of all edges that start in F and land outside F. Then we have

$$1 = \sum_{x\in F}\operatorname{div}\theta(x) \leq \sum_{e\in F'}|\theta(e)| \leq \sum_{e\in\Pi_k}|\theta(e)|. \qquad \square$$

Corollary. Consider $B_n = [1,n]^2\cap\mathbb{Z}^2$. Then

$$R_{\mathrm{eff}}(a,z) \geq \frac{1}{2}\log(n-1).$$

Proof. Take

$$\Pi_k = \{(v,u)\in B_n : \|v\|_\infty = k-1, \|u\|_\infty = k\}.$$

Then

$$|\Pi_k| = 2(k-1),$$

and so

$$R_{eff}(a,z) \geq \sum_{k=1}^{-1}\frac{1}{|\Pi_k|} \geq \frac{1}{2}\log(n-1). \qquad \square$$

Proposition. Let X be an irreducible Markov chain on a finite state space. Let τ be a stopping time such that $\mathbb{P}_a(X_\tau = a) = 1$ and $\mathbb{E}_a[\tau] < \infty$ for some a in the state space. Then

$$G_\tau(a,x) = \pi(x)\mathbb{E}_a[\tau].$$

Theorem (Commute time identity)**.** Let X be a reversible Markov chain on a finite state space. Then for all a,b, we have

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c(G)R_{\mathrm{eff}}(a,b),$$

where

$$c(G) = 2\sum_e c(e).$$

Proof. Let $\tau_{a,b}$ be the first time we hit a after we hit b. Then $\mathbb{P}_a(X_{\tau_{a,b}} = a) = 1$. By the proposition, we have

$$G_{\tau_{a,b}}(a,a) = \pi(a)\mathbb{E}_a[\tau_{a,b}].$$

Since starting from a, the number of times we come back to A by $\tau_{a,b}$ is the same as up to time τ_b, we get

$$G_{\tau_{a,b}}(a,a) = G_{\tau_b}(a,a) = c(a)R_{\mathrm{eff}}(a,b).$$

Combining the two, we get

$$\pi(a)\mathbb{E}_a[\tau_{a,b}] = c(a)R_{\mathrm{eff}}(a,b).$$

Thus, the result follows. $\square$

2.2 Infinite graphs

Theorem. Let G be an infinite connected graph with conductances $(c(e))_e$. Then

(i) Random walk on G is recurrent iff $R_{\mathrm{eff}}(0, \infty) = \infty$.

(ii) The random walk is transient iff there exists a unit flow i from 0 to ∞ of finite energy
$$\mathcal{E}(i) = \sum_e (i(e))^2 r(e).$$

Proof. The first part is immediate since
$$\mathbb{P}_0(\tau_0^+ = \infty) = \frac{1}{c(0) R_{\mathrm{eff}}(0, \infty)}.$$

To prove the second part, let θ be a unit flow from 0 to ∞ of finite energy. We want to show that the effective resistance is finite, and it suffices to extend Thompson's principle to infinite graphs.

Let i_n be the unit current flow from 0 to z_n in G_n^*. Let $v_n(x)$ be the associated potential. Then by Thompson,
$$R_{\mathrm{eff}}(0, z_n; G_n^*) = \mathcal{E}(i_n).$$

Let θ_n be the restriction of θ to G_n^*. Then this is a unit flow on G_n^* from 0 to z_n. By Thompson, we have
$$\mathcal{E}(i_n) \leq \mathcal{E}(\theta_n) \leq \mathcal{E}(\theta) < \infty.$$

So
$$R_{\mathrm{eff}}(0, \infty) = \lim_{n \to \infty} \mathcal{E}(i_n).$$

So by the first part, the flow is transient.

Conversely, if the random walk is transient, we want to construct a unit flow from 0 to ∞ of finite energy. The idea is to define it to be the limit of $(i_n)(x, y)$.

By the first part, we know $R_{\mathrm{eff}}(0, \infty) = \lim \mathcal{E}(i_n) < \infty$. So there exists $M > 0$ such that $\mathcal{E}(i_n) \leq M$ for all n. Starting from 0, let $Y_n(x)$ be the number of visits to x up to time τ_{z_n}. Let $Y(x)$ be the total number of visits to x. Then $Y_n(x) \nearrow Y(x)$ as $n \to \infty$ almost surely.

By monotone convergence, we get that
$$\mathbb{E}_0[Y_n(x)] \nearrow \mathbb{E}_0[Y(x)] < \infty,$$

since the walk is transient. On the other hand, we know
$$E_0[Y_n(x)] = G_{\tau_{z_n}}(0, x).$$

It is then easy to check that $\frac{G_{\tau_{z_n}}(0,x)}{c(x)}$ is a harmonic function outside of 0 and z_n with value 0 at z_n. So it has to be equal to the voltage $v_n(x)$. So
$$v_n(x) = \frac{G_{\tau_{z_n}}(0, x)}{c(x)}.$$

Therefore there exists a function v such that

$$\lim_{n\to\infty} c(x)v_n(x) = c(x)v(x).$$

We define

$$i(x,y) = c(x,y)(v(x) - v(y)) = \lim_{n\to\infty} c(x,y)(v_n(x) - v_n(y)) = \lim_{n\to\infty} i_n(x,y).$$

Then by dominated convergence, we know $\mathcal{E}(i) \leq M$, and also i is a flow from 0 to ∞. □

Corollary. Let $G' \subseteq G$ be connected graphs.

(i) If a random walk on G is recurrent, then so is random walk on G'.

(ii) If random walk on G' is transient, so is random walk on G.

Theorem (Polya's theorem)**.** Random walk on $\mathbb{Z}^2$ is recurrent and transient on $\mathbb{Z}^d$ for $d \geq 3$.

Proof sketch. For $d = 2$, if we glue all vertices at distance n from 0, then

$$R_{\text{eff}}(0,\infty) \geq \sum_{i=1}^{n-1} \frac{1}{8i-4} \geq c \cdot \log n$$

using the parallel and series laws. So $R_{\text{eff}}(0,\infty) = \infty$. So we are recurrent.

For $d = 3$, we can construct a flow as follows — let S be the sphere of radius $\frac{1}{4}$ centered at the origin. Given any edge $\mathbf{e}$, take the unit square centered at the midpoint $\mathbf{m}_e$ of $\mathbf{e}$ and has normal $\mathbf{e}$. We define $|\theta(\mathbf{e})|$ to be the area of the radial projection on the sphere, with positive sign iff $\langle \mathbf{m}_e, \mathbf{e}\rangle > 0$.

One checks that θ satisfies Kirchoff's node law outside of 0. Then we find that

$$\mathcal{E}(\theta) \leq C \sum_n n^2 \cdot \left(\frac{1}{n^2}\right)^2 < \infty,$$

since there are $\sim n^2$ edges at distance n, and the flow has magnitude $\sim \frac{1}{n^2}$. □

Another proof sketch. We consider a binary tree T_ρ with edges joining generation to $n+1$ having resistance ρ^n for ρ to be determined. Then

$$R_{\text{eff}}(T_\rho) = \sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n,$$

and this is finite when $\rho < 2$.

Now we want to embed this in $\mathbb{Z}^3$ in such a way that neighbours in T_ρ of generation n and $n+1$ are separated by a path of length of order ρ^n. In generation n of T_ρ, there are 2^n vertices. On the other hand, the number of vertices at distance n in $\mathbb{Z}^3$ will be of order $(\rho^n)^2$. So we need $(\rho^n)^2 \geq 2^n$. So we need $\rho > \sqrt{2}$. We can then check that

$$R_{\text{eff}}(0,\infty;\mathbb{Z}^3) \leq c \cdot R_{\text{eff}}(T_\rho) < \infty.$$ □

3 Uniform spanning trees

3.1 Finite uniform spanning trees

Theorem (Foster's theorem)**.** Let $G = (V, E)$ be a finite weighted graph on n vertices. Then

$$\sum_{e \in E} R_{\mathrm{eff}}(e) = n - 1.$$

Theorem. Let $e \neq f \in E$. Then

$$\mathbb{P}(e \in T \mid f \in T) \leq \mathbb{P}(e \in T).$$

Theorem (Kirchoff)**.** Let T be a uniform spanning tree, e an edge. Then

$$\mathbb{P}(e \in T) = R_{\mathrm{eff}}(e)$$

Theorem. Define, for every edge $e = (a, b)$,

$$i(a, b) = \frac{N(s, a, b, t) - N(s, b, a, t)}{N}.$$

Then i is a unit flow from s to t satisfying Kirchoff's node law and the cycle law.

Proof. We first show that i is a flow from s to t. It is clear that i is anti-symmetric. To check Kirchoff's node law, pick $a \notin \{s, t\}$. We need to show that

$$\sum_{x \sim a} i(a, x) = 0.$$

To show this, we count how much each spanning tree contributes to the sum. In each spanning tree, the unique path from s to t may or may not contain a. If it does not contain a, it contributes 0. If it contains A, then there is one edge entering a and one edge leaving a, and it also contributes 0. So every spanning tree contributes exactly zero to the sum.

So we have to prove that i satisfies the cycle law. Let $C = (v_1, v_2, \ldots, v_{n+1} = v_1)$ be a cycle. We need to show that

$$\sum_{i=1}^{n} i(v_i, v_{i+1}) = 0.$$

To prove this, it is easier to work with "bushes" instead of trees. An s/t *bush* is a forest that consists of exactly 2 trees, T_s and T_t, such that $s \in T_s$ and $t \in T_t$. Let $e = (a, b)$ be an edge. Define $\mathcal{B}(s, a, b, t)$ to be the set of s/t bushes such that $a \in T_s$ and $b \in T_t$.

We claim that $|\mathcal{B}(s, a, b, t)| = N(s, a, b, t)$. Indeed, given a bush in $\mathcal{B}(s, a, b, t)$, we can add the edge $e = (a, b)$ to get a spanning tree whose unique path from s to t passes through e, and vice versa.

Instead of considering the contribution of each tree to the sum, we count the contribution of each bush to the set. Then this is easy. Let

$$\begin{aligned} F_+ &= |\{(v_j, v_{j+1}) : B \in \mathcal{B}(s, v_j, v_{j+1}, t)\} \\ F_- &= |\{(v_j, v_{j+1}) : B \in \mathcal{B}(s, v_{j+1}, v_j, t)\} \end{aligned}$$

Then the contribution of B is
$$\frac{F_+ - F_-}{N}.$$
By staring it at long enough, since we have a cycle, we realize that we must have $F_+ = F_-$. So we are done.

Finally, we need to show that i is a unit flow. In other words,
$$\sum_{x \sim s} i(s, x) = 1.$$
But this is clear, since each spanning tree contributes $\frac{1}{N}$ to $i(s, x)$, and there are N spanning trees. $\square$

Theorem. Let $e \neq f \in E$. Then
$$\mathbb{P}(e \in T \mid f \in T) \leq \mathbb{P}(e \in T).$$

Proof. Define the new graph $G.f$ to be the graph obtained by gluing both endpoints of f to a single vertex (keeping multiple edges if necessary). This gives a correspondence between spanning trees of G containing f and spanning trees of $G.f$. But
$$\mathbb{P}(e \in T \mid f \in T) = \frac{\text{number of spanning trees of } G.f \text{ containing } e}{\text{number of spanning trees of } G.f}.$$
But this is just $\mathbb{P}(e \in \text{UST of } G.f)$, and this is just $R_{\mathrm{eff}}(e; G.f)$. So it suffices to show that
$$R_{\mathrm{eff}}(e; G.f) \leq R_{\mathrm{eff}}(e; G).$$
But this is clear by Rayleigh's monotone principle, since contracting f is the same as setting the resistance of the edge to 0. $\square$

Theorem (Wilson)**.** The resulting tree is a uniform spanning tree.

Lemma. The order in which cycles are popped is irrelevant, in the sense that either the popping will never stop, or the same set of cycles will be popped, thus leaving the same spanning tree lying underneath.

Proof. Given an edge (x, S_x^i), where S_x^i is the ith instruction under x, colour (x, S_x^i) with colour i. A colour is now coloured, but not necessarily with the same colour.

Suppose C is a cycle that can be popped in the order $C_1, C_2, \ldots, C_n$, with $C_n = C$. Let C' be any cycle in the original directed graph. We claim that either C' does not intersect $C_1, \ldots, C_n$, or $C' = C_k$ for some k, and C' does not intersect $C_1, \ldots C_{k-1}$. Indeed, if they intersect, let $x \in C' \cap C_k$, where k is the smallest index where they intersect. Then the edge coming out of x will have colour 1. Then S_x^1 is also in the intersection of C' and C_k, and so the same is true for the edge coming out of S_x^1. Continuing this, we see that we must have $C_k = C'$. So popping $C_k, C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_n$ gives the same result as popping $C_1, \ldots, C_n$.

Thus, by induction, if C is a cycle that is popped, then after performing a finite number of pops, C is still a cycle that can be popped. So either there is an infinite number of cycles that can be popped, so popping can never stop, or every cycle that can be popped will be popped, thus in this case giving the same spanning tree. $\square$

Corollary (Cayley's formula)**.** The number of labeled unrooted trees on n-vertices is equal to n^{n-2}.

Proof. Exercise. □

3.2 Infinite uniform spanning trees and forests

Proposition. Let G be a transient graph. The wired uniform spanning forest is the same as the spanning forest generated using Wilson's method rooted at infinity.

Proof. Let $e_1, \ldots, e_M$ be a finite set of edges. Let $T(n)$ be the uniform spanning tree on G_n^W. Let $\mathcal{F}$ be the limiting law of $T(n)$. Look at G_n^W and generate $T(n)$ using Wilson's method rooted at z_n. Start the random walks from $u_1, u_2, \ldots, u_L$ in this order, where (u_i) are all the end points of the $e_1, \ldots, e_M$ (L could be less than $2M$ if the edges share end points).

Start the first walk from u_1 and wait until it hits z_n; Then start from the next vertex and wait until it hits the previous path. We use the same infinite path to generate all the walks, i.e. for all i, let $(X_k(u_i))_{k \geq 0}$ be a simple random walk on G started from u_i. When considering G_n^W, we stop these walks when they exit G_n, .e. if they hit z_n. In this way, we couple all walks together and all the spanning trees.

Let τ_i^n be the first time the ith walk hits the tree the previous $i-1$ walks have generated in G_n^W. Now

$$\mathbb{P}(e_1, \ldots, e_M \in T(n)) = \mathbb{P}\left(e_1, \ldots, e_M \in \bigcup_{j=1}^{L} LE(X_k(u_j)) : k \leq \tau_{j^n}\right).$$

Let τ_j be the stopping times corresponding to Wilson's method rooted at infinity. By induction on j, we have $\tau_j^n \to \tau_j$ as $n \to \infty$, and by transience, we have $LE(X_k(u_j) : k \leq \tau_j^k) \to LE(X_k(u_j) : k \leq \tau_j)$. So we are done. □

Theorem (Pemantle, 1991)**.** The uniform spanning forest on $\mathbb{Z}^d$ is a single tree almost surely if and only if $d \leq 4$.

Proposition (Pemantle)**.** The uniform spanning forest is a single tree iff starting from every vertex, a simple random walk intersects an independent loop erased random walk infinitely many times with probability 1. Moreover, the probability that x and y are in the same tree of the uniform spanning forest is equal to the probability that simple random walk started from x intersects an independent loop-erased random walk started from y.

Theorem (Lyons, Peres, Schramm)**.** Two independent simple random walks intersect infinitely often with probability 1 if one walks intersects the loop erasure of the other one infinitely often with probability 1.

Theorem. The uniform spanning forest is not a tree for $d \geq 5$ with probability 1.

Proof of Pemantle's theorem. Let X, Y be two independent simple random walks in $\mathbb{Z}^d$. Write

$$I = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \mathbf{1}(X_t = Y_s).$$

Then we have

$$\mathbb{E}_{x,y}[I] = \sum_t \sum_s \mathbb{P}_{x-y}(X_{t+s} = 0) \approx \sum_{t=\|x-y\|} t\mathbb{P}_{x-y}(X_t = 0).$$

It is an elementary exercise to show that

$$\mathbb{P}_x(X_t = 0) \leq \frac{c}{t^{d/2}},$$

so

$$\mathbb{P}_x(X_t = 0) \leq \sum_{t=\|x-y\|} \frac{1}{t^{d/2-1}}.$$

For $d \geq 5$, for all $\varepsilon > 0$, we take x, y such that

$$\mathbb{E}_{x,y}[I] < \varepsilon.$$

Then

$$\mathbb{P}(\text{USF is connected}) \leq \mathbb{P}_{x,y}(I > 0) \leq \mathbb{E}_{x,y}[I] < \varepsilon.$$

Since this is true for every ε, it follows that $\mathbb{P}(\text{USF is connected}) = 0$. $\square$