Part III — Percolation and Random Walks on Graphs

Theorems

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A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

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0 Introduction

1 Percolation

1.1 The critical probability

Lemma. θ is an increasing function of p.

Proposition. $p_c(1) = 1$.

Theorem. For all $d \geq 2$, we have $p_c(d) \in (0, 1)$.

Lemma. For $d \geq 2$, $p_c(d) > 0$.

Lemma. We have $\sigma_{n+m} \leq \sigma_n \sigma_m$.

Lemma (Fekete's lemma)**.** If (a_n) is a subadditive sequence of real numbers, then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_k}{k} : k \geq 1\right\} \in [-\infty, \infty).$$

In particular, the limit exists.

Theorem (Duminil-Copin, Smirnov, 2010)**.** The hexagonal lattice has

$$\kappa_{\mathrm{hex}} = \sqrt{2 + \sqrt{2}}.$$

Theorem (Hara and Slade, 1991)**.** For $d \geq 5$, there exists a constant A such that

$$\sigma_n = A\kappa^n(1 + O(n^{-\varepsilon}))$$

for any $\varepsilon < \frac{1}{2}$.

Theorem (Hammersley and Welsh, 1962)**.** For all $d \geq 2$, we have

$$\sigma_n \leq C\kappa^n \exp(c'\sqrt{n})$$

for some constants C and c'.

Theorem (Hutchcroft, 2017)**.** For $d \geq 2$, we have

$$\sigma_n \leq C\kappa^n \exp(o(\sqrt{n})).$$

Lemma. $p_c(d) < 1$ for all $d \geq 2$.

Proposition. Let A_∞ be the event that there is an infinite cluster.

(i) If $\theta(p) = 0$, then $\mathbb{P}_p(A_\infty) = 0$.

(ii) If $\theta(p) > 0$, then $\mathbb{P}_p(A_\infty) = 1$.

Theorem (Burton and Keane)**.** If $p > p_c$, then there exists a unique infinite cluster with probability 1.

1.2 Correlation inequalities

Theorem. If N is an increasing random variable and $p_1 \leq p_2$, then

$$\mathbb{E}_{p_1}[N] \leq \mathbb{E}_{p_2}[N],$$

and if an event A is increasing, then

$$\mathbb{P}_{p_1}(A) \leq \mathbb{P}_{p_2}(A).$$

Theorem (Fortuin–Kasteleyn–Ginibre (FKG) inequality)**.** Let X and Y be increasing random variables with $\mathbb{E}_p[X^2], \mathbb{E}_p[Y^2] < \infty$. Then

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y].$$

In particular, if A and B are increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Equivalently,

$$\mathbb{P}_p(A \mid B) \geq \mathbb{P}_p(A).$$

Theorem (BK inequality)**.** Let F be a finite set and $\Omega = \{0, 1\}^F$. Let A and B be increasing events. Then

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Theorem (Reimer)**.** For all events A, B depending on a finite set, we have $\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$.

Theorem. If $\chi(p) < \infty$, then there exists a positive constant c such that for all $n \geq 1$,

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq e^{-cn}.$$

Theorem (Russo's formula)**.** Let A be an increasing event that depends on the states of a finite number of edges. Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)],$$

where $N(A)$ is the number of pivotal edges for A.

Corollary. Let A be an increasing event that depends on m edges. Let $p \leq q \in [0, 1]$. Then $\mathbb{P}_q(A) \leq \mathbb{P}_p(A)\left(\frac{q}{p}\right)^m$.

Theorem. Let $d \geq 2$ and $B_n = [-n, n]^d \cap \mathbb{Z}^d$.

(i) If $p < p_c$, then there exists a positive constant c for all $n \geq 1$, $\mathbb{P}_p(0 \leftrightarrow \partial B_n) \leq e^{-cn}$.

(ii) If $p > p_c$, then
$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \geq \frac{p - p_c}{p(1 - p_c)}.$$

1.3 Two dimensions

Theorem. In $\mathbb{Z}^2$, we have $\theta\left(\frac{1}{2}\right) = 0$ and $p_c = \frac{1}{2}$.

Proposition. $\mathbb{P}_{\frac{1}{2}}(LR(\ell)) \geq \frac{1}{2}$ for all ℓ.

Theorem (Russo–Symour–Welsh (RSW) theorem)**.** If $\mathbb{P}_p(LR(\ell)) = \alpha$, then

$$\mathbb{P}_p(O(\ell)) \geq \left(\alpha\left(1 - \sqrt{1-\alpha}\right)^4\right)^{12}.$$

Lemma. If $\mathbb{P}_p(LR(\ell)) = \alpha$, then

$$\mathbb{P}_p\left(LR\left(\tfrac{3}{2}\ell, \ell\right)\right) \geq (1 - \sqrt{1-\alpha})^3.$$

Lemma (nth root trick)**.** If $A_1, \ldots, A_n$ are increasing events all having the same probability, then

$$\mathbb{P}_p(A_1) \geq 1 - \left(1 - \mathbb{P}_p\left(\bigcup_{i=1}^n A_i\right)\right)^{1/n}.$$

Lemma.

$$\begin{aligned}
\mathbb{P}_p(LR(2\ell, \ell)) &\geq \mathbb{P}_p(LR(\ell))\left(\mathbb{P}_p\left(LR\left(\frac{3}{2}\ell, \ell\right)\right)\right)^2\\
\mathbb{P}_p(LR(3\ell, \ell)) &\geq \mathbb{P}_p(LR(\ell))\left(\mathbb{P}_p\left(LR\left(2\ell, \ell\right)\right)\right)^2\\
\mathbb{P}_p(O(\ell)) &\geq \mathbb{P}_p(LR(3\ell, \ell))^4
\end{aligned}$$

Theorem. There exists positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, A_1, A_2, A_4$ such that

$$\begin{aligned}
\mathbb{P}_{\frac{1}{2}}(0 \leftrightarrow \partial B(n)) &\leq A_1 n^{-\alpha_1}\\
\mathbb{P}_{\frac{1}{2}}(|\mathcal{C}(0)| \geq n) &\leq A_2 n^{-\alpha_2}\\
\mathbb{E}(|\mathcal{C}(0)|^{\alpha_3}) &\leq \infty
\end{aligned}$$

Moreover, for $p > p_c = \frac{1}{2}$, we have

$$\theta(p) \leq A_4\left(p - \frac{1}{2}\right)^{\alpha_4}.$$

Theorem. When $d = 2$ and $p > p_c$, there exists a positive constant c such that

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n), |\mathcal{C}(0)| < \infty) \leq e^{-cn}.$$

Theorem (Grimmett–Marstrand)**.** Let F be an infinite-connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$. Then for all $\eta > 0$, there exists $k \in \mathbb{N}$ such that

$$p_c(2kF + B_k) \leq p_c + \eta.$$

In particular, for all $d \geq 3$, $p_c^{slab} = p_c$.

Theorem. If $d \geq 3$ and $p > p_c$, then there exists $c > 0$ such that

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n), |\mathcal{C}(0)| < \infty) \leq e^{-cn}.$$

1.4 Conformal invariance and SLE in $d = 2$

Theorem (Smirnov, 2001)**.** Suppose (Ω, a, b, c, d) and $(\Omega', a', b', c', d')$ are conformally equivalent. Then

$$\mathbb{P}(ac \leftrightarrow bd \text{ in } \Omega) = \mathbb{P}(a'c' \leftrightarrow b'd' \text{ in } \Omega').$$

2 Random walks

2.1 Random walks in finite graphs

Proposition. Let P be an irreducible matrix on Ω and $B \subseteq \Omega$, $f : B \to \mathbb{R}$ a function. Then

$$h(x) = \mathbb{E}_x[f(X_{\tau_B})]$$

is the unique extension of f which is harmonic on $\Omega \setminus B$.

Proposition. Let θ be a flow from a to z satisfying the cycle law for any cycle. Let I the current flow associated to a voltage W. If $\|\theta\| = \|I\|$, then $\theta = I$.

Proposition. Take a weighted random walk on G. Then

$$\mathbb{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a) R_{\mathrm{eff}}(a, z)},$$

where $\tau_a^+ = \min\{t \geq 1 : X_t = a\}$.

Corollary. For any reversible chain and all a, z, we have

$$G_{\tau_z}(a, a) = c(a) R_{\mathrm{eff}}(a, z).$$

Theorem (Thomson's principle)**.** Let G be a finite connected graph with conductances $(c(e))$. Then for any a, z, we have

$$R_{\mathrm{eff}}(a, z) = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } a \text{ to } z\}.$$

Moreover, the unit current flow from a to z is the unique minimizer.

Theorem (Rayleigh's monotonicity principle)**.** Let G be a finite connected graph and $(r(e))_e$ and $(r'(e))_e$ two sets of resistances on the edges such that $r(e) \leq r'(e)$ for all e. Then

$$R_{\mathrm{eff}}(a, z; r) \leq R_{\mathrm{eff}}(a, z; r').$$

for all $a, z \in G$.

Corollary. Suppose we add an edge to G which is not adjacent to a. This increases the escape probability

$$\mathbb{P}_a(\tau_z < \tau_a^+).$$

Theorem (Nash–Williams inequality)**.** Let (Π_k) be disjoint edge-cutsets separating a from z. Then

$$R_{\mathrm{eff}}(a, z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e)\right)^{-1}.$$

Corollary. Consider $B_n = [1, n]^2 \cap \mathbb{Z}^2$. Then

$$R_{\mathrm{eff}}(a, z) \geq \frac{1}{2} \log(n - 1).$$

Proposition. Let X be an irreducible Markov chain on a finite state space. Let τ be a stopping time such that $\mathbb{P}_a(X_\tau = a) = 1$ and $\mathbb{E}_a[\tau] < \infty$ for some a in the state space. Then

$$G_\tau(a, x) = \pi(x)\mathbb{E}_a[\tau].$$

Theorem (Commute time identity)**.** Let X be a reversible Markov chain on a finite state space. Then for all a, b, we have

$$\mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c(G)R_{\mathrm{eff}}(a, b),$$

where

$$c(G) = 2\sum_e c(e).$$

2.2 Infinite graphs

Theorem. Let G be an infinite connected graph with conductances $(c(e))_e$. Then

(i) Random walk on G is recurrent iff $R_{\mathrm{eff}}(0, \infty) = \infty$.

(ii) The random walk is transient iff there exists a unit flow i from 0 to ∞ of finite energy

$$\mathcal{E}(i) = \sum_e (i(e))^2 r(e).$$

Corollary. Let $G' \subseteq G$ be connected graphs.

(i) If a random walk on G is recurrent, then so is random walk on G'.

(ii) If random walk on G' is transient, so is random walk on G.

Theorem (Polya's theorem)**.** Random walk on $\mathbb{Z}^2$ is recurrent and transient on $\mathbb{Z}^d$ for $d \geq 3$.

3 Uniform spanning trees

3.1 Finite uniform spanning trees

Theorem (Foster's theorem)**.** Let $G = (V, E)$ be a finite weighted graph on n vertices. Then
$$\sum_{e \in E} R_{\mathrm{eff}}(e) = n - 1.$$

Theorem. Let $e \neq f \in E$. Then
$$\mathbb{P}(e \in T \mid f \in T) \leq \mathbb{P}(e \in T).$$

Theorem (Kirchoff)**.** Let T be a uniform spanning tree, e an edge. Then
$$\mathbb{P}(e \in T) = R_{\mathrm{eff}}(e)$$

Theorem. Define, for every edge $e = (a, b)$,
$$i(a, b) = \frac{N(s, a, b, t) - N(s, b, a, t)}{N}.$$
Then i is a unit flow from s to t satisfying Kirchoff's node law and the cycle law.

Theorem. Let $e \neq f \in E$. Then
$$\mathbb{P}(e \in T \mid f \in T) \leq \mathbb{P}(e \in T).$$

Theorem (Wilson)**.** The resulting tree is a uniform spanning tree.

Lemma. The order in which cycles are popped is irrelevant, in the sense that either the popping will never stop, or the same set of cycles will be popped, thus leaving the same spanning tree lying underneath.

Corollary (Cayley's formula)**.** The number of labeled unrooted trees on n-vertices is equal to n^{n-2}.

3.2 Infinite uniform spanning trees and forests

Proposition. Let G be a transient graph. The wired uniform spanning forest is the same as the spanning forest generated using Wilson's method rooted at infinity.

Theorem (Pemantle, 1991)**.** The uniform spanning forest on $\mathbb{Z}^d$ is a single tree almost surely if and only if $d \leq 4$.

Proposition (Pemantle)**.** The uniform spanning forest is a single tree iff starting from every vertex, a simple random walk intersects an independent loop erased random walk infinitely many times with probability 1. Moreover, the probability that x and y are in the same tree of the uniform spanning forest is equal to the probability that simple random walk started from x intersects an independent loop-erased random walk started from y.

Theorem (Lyons, Peres, Schramm)**.** Two independent simple random walks intersect infinitely often with probability 1 if one walks intersects the loop erasure of the other one infinitely often with probability 1.

Theorem. The uniform spanning forest is not a tree for $d \geq 5$ with probability 1.