Part III — Percolation and Random Walks on Graphs

Definitions

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A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

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0 Introduction

1 Percolation

1.1 The critical probability

Notation. We write $x \leftrightarrow y$ if there is an open path of edges from x to y.

Notation. We write $\mathcal{C}(x) = \{y \in V : y \leftrightarrow x\}$, the *cluster* of x.

Notation. We write $x \leftrightarrow \infty$ if $|\mathcal{C}(x)| = \infty$.

Definition ($\theta(p)$)**.** We define $\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$.

Definition (Coupling)**.** Let μ and ν be two probability measures on (potentially) different probability spaces. A *coupling* is a pair of random variables (X, Y) defined on the same probability space such that the marginal distribution of X is μ and the marginal distribution of Y is ν.

Definition (Critical probability)**.** We define $p_c(d) = \sup\{p \in [0, 1] : \theta(p) = 0\}$.

Definition (σ_n)**.** We write σ_n for the number of self-avoiding paths of length n starting from 0.

Definition (λ and κ)**.** We define

$$\lambda = \lim_{n\to\infty} \frac{\log \sigma_n}{n}, \quad \kappa = e^\lambda.$$

κ is known as the *connective constant*.

Definition (Planar graph)**.** A graph G is called planar if it can be embedded on the plane in such a way that no two edges cross.

Definition (Dual graph)**.** Let G be a planar graph (which we call the *primal graph*). We define the *dual graph* by placing a vertex in each face of G, and connecting 2 vertices if their faces share a boundary edge.

1.2 Correlation inequalities

Notation ($\leq$)**.** Given $\omega, \omega' \in \Omega$, we write $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all $e \in E$.

Definition (Increasing random variable)**.** A random variable X is increasing if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$, and is decreasing if $-X$ is increasing.

Definition (Increasing event)**.** An event A is increasing (resp. decreasing) if the indicator $1(A)$ is increasing (resp. decreasing)

Definition (Disjoint occurrence)**.** Let F be a set and $\Omega = \{0, 1\}^F$. If A and B are events, then the event that A and B occurs *disjointly* is

$$A \circ B = \{\omega \in \Omega : \exists S \subseteq F \text{ s.t. } [\omega]_S \subseteq A \text{ and } [\omega]_{F\setminus S} \subseteq B\}.$$

Definition (Pivotal edge)**.** Let A be an event and ω a percolation configuration. The edge e is *pivotal* for (A, ω) if

$$1(\omega \in A) \neq 1(\omega' \in A),$$

where ω' is defined by

$$\omega'(f) = \begin{cases} \omega(f) & f \neq e \\ 1 - \omega(f) & f = e \end{cases}.$$

The event that e is pivotal for A is defined to be the set of all ω such that e is pivotal for (A, ω).

1.3 Two dimensions

1.4 Conformal invariance and SLE in $d = 2$

2 Random walks

2.1 Random walks in finite graphs

Definition (Flow)**.** A *flow* θ on G is a function defined on oriented edges which is anti-symmetric, i.e. $\theta(x, y) = -\theta(y, x)$.

Definition (Divergence)**.** The *divergence* of a flow θ is

$$\operatorname{div}\theta(x) = \sum_{y\sim x} \theta(x, y).$$

Definition (Flow from a to z)**.** A flow θ from a to z is a flow such that

(i) $\operatorname{div}\theta(x) = 0$ for all $x \notin \{a, z\}$. (*Kirchhoff's node law*)

(ii) $\operatorname{div}\theta(a) \geq 0$.

The *strength* of the flow from a to z is $\|\theta\| = \operatorname{div}\theta(a)$.We say this is a *unit flow* if $\|\theta\| = 1$.

Definition (Harmonic function)**.** Let P be a transition matrix on Ω. We call h *harmonic* for P at the vertex x if

$$h(x) = \sum_y \mathbb{P}(x, y)h(y).$$

Definition (Voltage)**.** A *voltage* W is a function on Ω that is harmonic on $\Omega \setminus \{a, z\}$.

Definition (Current flow)**.** The *current flow* associated to the voltage W is

$$I(x, y) = \frac{W(x) - W(y)}{r(x, y)} = c(x, y)(W(x) - W(y)).$$

Definition (Effective resistance)**.** The *effective resistance* $R_{\text{eff}}(a, z)$ of an electric network is defined to be the ratio

$$R_{\text{eff}}(a, z) = \frac{W(a) - W(z)}{\|I\|}$$

for any voltage W with associated current I. The *effective conductance* is $C_{\text{eff}}(a, z) = R_{\text{eff}}(a, z)^{-1}$.

Definition (Green kernel)**.** Let τ be a stopping time. We define the *Green kernel* to be

$$G_\tau(a, x) = \mathbb{E}_a\left[\sum_{t=0}^{\infty} \mathbf{1}(X_t = x, t < \tau)\right].$$

Definition (Energy)**.** Let θ be a flow on G with conductances $(c(e))$. Then the *energy* is

$$\mathcal{E}(\theta) = \sum_e (\theta(e))^2 r(e).$$

Here we sum over unoriented edges.

Definition (Edge cutset)**.** A set of edges Π is an edge-cutset separating a from z if every path from a to z uses an edge of Π.

2.2 Infinite graphs

Definition (Flow)**.** Let G be an infinite graph. A *flow* θ from 0 to ∞ is an anti-symmetric function on the edges such that $\operatorname{div}\theta(x) = 0$ for all $x \neq 0$.

3 Uniform spanning trees

3.1 Finite uniform spanning trees

Definition (Spanning tree)**.** Let $G = (V, E)$ be a finite connected graph. A *spanning tree* T of G is a connected subgraph of G which is a tree (i.e. there are no cycles) and contains all the vertices in G.

Notation. Fix two vertices s, t of G. For all every edge $e = (a, b)$, define $\mathcal{N}(s, a, b, t)$ to be the set of spanning tress of G whose unique path from s to t passes along the edge (a, b) in the direction from a to b. Write

$$N(s, a, b, t) = |\mathcal{N}(s, a, b, t)|,$$

and N the total number of spanning trees.

Definition (Loop erasure)**.** Let $x = \langle x_1, \ldots x_n\rangle$ be a finite path in the graph G. We define the *loop erasure* as follows: for any pair $i < j$ such that $x_i = x_j$, remove $x_{i+1}, x_{i+2}, \ldots, x_j$, and keep repeating until no such pairs exist.

3.2 Infinite uniform spanning trees and forests