Part III — Local Fields

Theorems with proof

Based on lectures by H. C. Johansson

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The p-adic numbers $\mathbb{Q}_p$ (where p is any prime) were invented by Hensel in the late 19th century, with a view to introduce function-theoretic methods into number theory. They are formed by completing $\mathbb{Q}$ with respect to the p-adic absolute value $|-|_p$, defined for non-zero $x \in \mathbb{Q}$ by $|x|_p = p^{-n}$, where $x = p^n a/b$ with $a, b, n \in \mathbb{Z}$ and a and b are coprime to p. The p-adic absolute value allows one to study congruences modulo all powers of p simultaneously, using analytic methods. The concept of a local field is an abstraction of the field $\mathbb{Q}_p$, and the theory involves an interesting blend of algebra and analysis. Local fields provide a natural tool to attack many number-theoretic problems, and they are ubiquitous in modern algebraic number theory and arithmetic geometry.

Topics likely to be covered include:

The p-adic numbers. Local fields and their structure.

Finite extensions, Galois theory and basic ramification theory.

Polynomial equations; Hensel's Lemma, Newton polygons.

Continuous functions on the p-adic integers, Mahler's Theorem.

Local class field theory (time permitting).

Pre-requisites

Basic algebra, including Galois theory, and basic concepts from point set topology and metric spaces. Some prior exposure to number fields might be useful, but is not essential.

Contents

0 Introduction

1 Basic theory

1.1 Fields

Proposition. $||x| - |y|| \leq |x - y|$. Here the outer absolute value on the left hand side is the usual absolute value of $\mathbb{R}$, while the others are the absolute values of the relevant field.

Proposition. Let K be a field, and $| \cdot |, | \cdot |'$ be absolute values on K. Then the following are equivalent.

(i) $| \cdot |$ and $| \cdot |'$ are equivalent

(ii) $|x| < 1$ implies $|x|' < 1$ for all $x \in K$

(iii) There is some $s \in \mathbb{R}_{>0}$ such that $|x|^s = |x|'$ for all $x \in K$.

Proof. (i) ⇒ (ii) and (iii) ⇒ (i) are easy exercises. Assume (ii), and we shall prove (iii). First observe that since $|x^{-1}| = |x|^{-1}$, we know $|x| > 1$ implies $|x|' > 1$, and hence $|x| = 1$ implies $|x|' = 1$. To show (iii), we have to show that the ratio $\frac{\log |x|}{\log |x'|}$ is independent of x.

Suppose not. We may assume

$$\frac{\log |x|}{\log |x|'} < \frac{\log |y|}{\log |y|'},$$

and moreover the logarithms are positive. Then there are $m, n \in \mathbb{Z}_{>0}$ such that

$$\frac{\log |x|}{\log |y|} < \frac{m}{n} < \frac{\log |x|'}{\log |y|'}.$$

Then rearranging implies

$$\left|\frac{x^n}{y^m}\right| < 1 < \left|\frac{x^n}{y^m}\right|',$$

a contradiction. □

Proposition. Let $(K, | \cdot |)$ be a non-archimedean valued field, and let $x \in K$ and $r \in \mathbb{R}_{>0}$. Let $z \in B(x, r)$. Then

$$B(x, r) = B(z, r).$$

Proof. Let $y \in B(z, r)$. Then

$$|x - y| = |(x - z) + (z - y)| \leq \max(|x - z|, |z - y|) \leq r.$$

So $y \in B(x, r)$. By symmetry, $y \in B(x, r)$ implies $y \in B(z, r)$. □

Corollary. Closed balls are open.

Proof. To show that $B(x, r)$ is open, we let $z \in B(x, r)$. Then we have

$$\{y : |y - z| < r\} \subseteq B(z, r) = B(x, r).$$

So we know the open ball of radius r around z is contained in $B(x, r)$. So $B(x, r)$ is open. □

Proposition. Let K be a non-archimedean valued field, and $x, y \in K$. If $|x| > |y|$, then $|x + y| = |x|$.

More generally, if $x = \sum_{c=0}^{\infty} x_i$ and the non-zero $|x_i|$ are distinct, then $|x| = \max |x_i|$.

Proof. On the one hand, we have $|x + y| \leq \max\{|x|, |y|\}$. On the other hand, we have

$$|x| = |(x + y) - y| \leq \max(|x + y|, |y|) = |x + y|,$$

since we know that we cannot have $|x| \leq |y|$. So we must have $|x| = |x + y|$. □

Proposition. Let K be a valued field.

(i) Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$, then x_n is Cauchy.

If we assume further that K is complete, then

(ii) Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$, then a sequence (x_n) in K converges.

(iii) Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges.

Proof.

(i) Pick $\varepsilon > 0$ and N such that $|x_n - x_{n+1}| < \varepsilon$ for all $n \geq N$. Then given $m \geq n \geq N$, we have

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots - x_n| \\ &\leq \max(|x_m - x_{m-1}|, \cdots, |x_{n+1} - x_n|) \\ &< \varepsilon. \end{aligned}$$

So the sequence is Cauchy.

(ii) Follows from (1) and the definition of completeness.

(iii) Follows from the definition of convergence of a series and (2). □

Proposition. Let K be a valued field. Then

$$\mathcal{O}_K = \{x : |x| \leq 1\}$$

is an open subring of K. Moreover, for each $r \in (0, 1]$, the subsets $\{x : |x| < r\}$ and $\{x : |x| \leq r\}$ are open ideals of $\mathcal{O}_K$. Moreover, $\mathcal{O}_K^\times = \{x : |x| = 1\}$.

Proof. We know that these sets are open since all balls are open.

To see $\mathcal{O}_K$ is a subring, we have $|1| = |-1| = 1$. So $1, -1 \in \mathcal{O}_K$. If $x, y \in \mathcal{O}_K$, then $|x + y| \leq \max(|x|, |y|) \leq 1$. So $x + y \in \mathcal{O}_K$. Also, $|xy| = |x||y| \leq 1 \cdot 1 = 1$. So $xy \in \mathcal{O}_K$.

That the other sets are ideals of $\mathcal{O}_K$ is checked in the same way.

To check the units, we have $x \in \mathcal{O}_K^\times \Leftrightarrow |x|, |x^{-1}| \leq 1 \Leftrightarrow |x| = |x|^{-1} = 1$. □

1.2 Rings

Theorem. Let $R \subseteq S$ be rings. Then $s_1, \cdots, s_n \in S$ are all integral iff $R[s_1, \cdots, s_n] \subseteq S$ is a finitely-generated R-module.

Proposition. For any A, we have $A^*A = AA^* = \det(A)I$, where I is the identity matrix.

Proof of theorem. Note that we can construct $R[s_1, \cdots, s_n]$ by a sequence

$$R \subseteq R[s_1] \subseteq R[s_1, s_2] \subseteq \cdots \subseteq R[s_1, \cdots, s_n] \subseteq S,$$

and each s_i is integral over $R[s_1, \cdots, s_{n-1}]$. Since the finite extension of a finite extension is still finite, it suffices to prove it for the case $n = 1$, and we write s for s_1.

Suppose $f(x) \in R[x]$ is monic such that $f(s) = 0$. If $g(x) \in R[x]$, then there is some $q, r \in R[x]$ such that $g(x) = f(x)q(x) + r(x)$ with $\deg r < \deg f$. Then $g(s) = r(s)$. So any polynomial expression in s can be written as a polynomial expression with degree less than $\deg f$. So $R[s]$ is generated by $1, s, \cdots, s^{\deg f - 1}$.

In the other direction, let $t_1, \cdots, t_d$ be R-module generators of $R[s_1, \cdots, s_n]$. We show that in fact any element of $R[s_1, \cdots, s_n]$ is integral over R. Consider any element $b \in R[s_1, \cdots, s_n]$. Then there is some $a_{ij} \in R$ such that

$$bt_i = \sum_{j=1}^{d} a_{ij} t_j.$$

In matrix form, this says

$$(bI - A)t = 0.$$

We now multiply by $(bI - A)^*$ to obtain

$$\det(bI - A)t_j = 0$$

for all j. Now we know $1 \in R$. So $1 = \sum c_j t_j$ for some $c_j \in R$. Then we have

$$\det(bI - A) = \det(bI - A) \sum c_j t_j = \sum c_j (\det(bI - A)t_j) = 0.$$

Since $\det(bI - A)$ is a monic polynomial in b, it follows that b is integral. □

Corollary. Let $R \subseteq S$ be rings. If $s_1, s_2 \in S$ are integral over R, then $s_1 + s_2$ and $s_1 s_2$ are integral over R. In particular, the set $\tilde{R} \subseteq S$ of all elements in S integral over R is a ring, known as the integral closure of R in S.

Proof. If s_1, s_2 are integral, then $R[s_1, s_2]$ is a finite extension over R. Since $s_1 + s_2$ and $s_1 s_2$ are elements of $R[s_1, s_2]$, they are also integral over R. □

1.3 Topological rings

Proposition. The set of all I-adically open sets form a topology on R, called the *I-adic topology*.

Proof. By definition, we have $\emptyset$ and R are open, and arbitrary unions are clearly open. If U, V are I-adically open, and $x \in U \cap V$, then there are n, m such that $x + I^n \subseteq U$ and $x + I^m \subseteq V$. Then $x + I^{\max(m,n)} \subseteq U \cap V$. □

Proposition. The inverse limit topology is a ring topology.

Proof sketch. We can fit the addition and multiplication maps into diagrams

$$\begin{tikzcd}
\varprojlim R_n \times \varprojlim R_n \arrow[r] & \varprojlim R_n \\
\prod R_n \times \prod R_n \arrow[r] \arrow[u, hook] & \prod R_n \arrow[u, hook]
\end{tikzcd}$$

By the definition of the subspace topology, it suffices to show that the corresponding maps on $\prod R_n$ are continuous. By the universal property of the product, it suffices to show that the projects $\prod R_n \times \prod R_n \to R_m$ is continuous for all m. But this map can alternatively be obtained by first projecting to R_m, then doing multiplication in R_m, and projection is continuous. So the result follows. □

Proposition. Giving a continuous ring homomorphism $g : S \to \varprojlim R_n$ is the same as giving a continuous ring homomorphism $g_n : S \to R_n$ for each n, such that each of the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "g_n"] \arrow[dr, "g_{n-1}"'] & R_n \arrow[d, "f_{n-1}"] \\
& R_{n-1}
\end{tikzcd}$$

1.4 The p-adic numbers

Proposition. The p-adic absolute value is an absolute value.

Proof. It is clear that $|x|_p = 0$ iff $x = 0$.

Suppose we have

$$x = p^n \frac{a}{b}, \quad y = p^m \frac{c}{d}.$$

We wlog $m \geq n$. Then we have

$$|xy|_p = \left|p^{n+m}\frac{ac}{bd}\right| = p^{-m-n} = |x|_p|y|_p.$$

So this is multiplicative. Finally, we have

$$|x+y|_p = \left|p^n \frac{ab + p^{m-n}cb}{bd}\right| \leq p^{-n} = \max(|x|_p, |y|_p).$$

Note that we must have bd coprime to p, but $ab + p^{m-n}cb$ need not be. However, any extra powers of p could only decrease the absolute value, hence the above result. □

Proposition. $\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$.

Proof. If $x \in \mathbb{Z}$ is non-zero, then $x = p^n a$ with $n \geq 0$. So $|x|_p \leq 1$. So $\mathbb{Z} \subseteq \mathbb{Z}_p$.

We now want to show that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. We know the set

$$\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} : |x|_p \leq 1\}$$

is dense inside $\mathbb{Z}_p$, essentially by definition. So it suffices to show that $\mathbb{Z}$ is dense in $\mathbb{Z}_{(p)}$. We let $x \in \mathbb{Z}_{(p)} \setminus \{0\}$, say

$$x = p^n \frac{a}{b}, \quad n \geq 0.$$

It suffices to find $x_i \in \mathbb{Z}$ such that $x_i \to \frac{1}{b}$. Then we have $p^n a x_i \to x$.

Since $(b, p) = 1$, we can find $x_i, y_i \in \mathbb{Z}$ such that $bx_i + p^i y_i = 1$ for all $i \geq 1$. So

$$\left|x_i - \frac{1}{b}\right|_p = \left|\frac{1}{b}\right|_p |bx_i - 1|_p = |p^i y_i|_p \leq p^{-i} \to 0.$$

So done. □

Proposition. The non-zero ideals of $\mathbb{Z}_p$ are $p^n\mathbb{Z}_p$ for $n \geq 0$. Moreover,

$$\frac{\mathbb{Z}}{p^n\mathbb{Z}} \cong \frac{\mathbb{Z}_p}{p^n\mathbb{Z}_p}.$$

Proof. Let $0 \neq I \subseteq \mathbb{Z}_p$ be an ideal, and pick $x \in I$ such that $|x|_p$ is maximal. This supremum exists and is attained because the possible values of the absolute values are discrete and bounded above. If $y \in I$, then by maximality, we have $|y|_p \leq |x|_p$. So we have $|yx^{-1}|_p \leq 1$. So $yx^{-1} \in \mathbb{Z}_p$, and this implies that $y = (yx^{-1})x \in x\mathbb{Z}_p$. So $I \subseteq x\mathbb{Z}_p$, and we obviously have $x\mathbb{Z}_p \subseteq I$. So we have $I = x\mathbb{Z}_p$.

Now if $x = p^n \frac{a}{b}$, then since $\frac{a}{b}$ is invertible in $\mathbb{Z}_p$, we have $x\mathbb{Z}_p = p^n\mathbb{Z}_p$. So $I = p^n\mathbb{Z}_p$.

To show the second part, consider the map

$$f_n : \mathbb{Z} \to \frac{\mathbb{Z}_p}{p^n\mathbb{Z}_p}$$

given by the inclusion map followed by quotienting. Now $p^n\mathbb{Z}_p = \{x : |x|_p \leq p^{-n}$. So we have

$$\ker f_n = \{x \in \mathbb{Z} : |x|_p \leq p^{-n}\} = p^n\mathbb{Z}.$$

Now since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we know the image of f_n is dense in $\mathbb{Z}_p/p^n\mathbb{Z}_p$. But $\mathbb{Z}_p/p^n\mathbb{Z}_p$ has the discrete topology. So f_n is surjective. So f_n induces an isomorphism $\mathbb{Z}/p^n Z \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$. □

Corollary. $\mathbb{Z}_p$ is a PID with a unique prime element p (up to units).

Proposition. The topology on $\mathbb{Z}$ induced by $|\cdot|_p$ is the p-adic topology (i.e. the $p\mathbb{Z}$-adic topology).

Proof. Let $U \subseteq \mathbb{Z}$. By definition, U is open wrt $|\cdot|_p$ iff for all $x \in U$, there is an $n \in \mathbb{N}$ such that

$$\{y \in \mathbb{Z} : |y - x|_p \leq p^{-n}\} \subseteq U.$$

On the other hand, U is open in the p-adic topology iff for all $x \in U$, there is some $n \geq 0$ such that $x + p^n\mathbb{Z} \subseteq U$. But we have

$$\{y \in \mathbb{Z} : |y - x|_p \leq p^{-n}\} = x + p^n\mathbb{Z}.$$

So done. □

Proposition. $\mathbb{Z}_p$ is p-adically complete and is (isomorphic to) the p-adic completion of $\mathbb{Z}$.

Proof. The second part follows from the first as follows: we have the maps

$$\mathbb{Z}_p \xrightarrow{\nu} \varprojlim \mathbb{Z}_p/(p^n\mathbb{Z}_p) \xleftarrow{(f_n)_n} \lim \mathbb{Z}/(p^n\mathbb{Z})$$

We know the map induced by $(f_n)_n$ is an isomorphism. So we just have to show that ν is an isomorphism

To prove the first part, we have $x \in \ker \nu$ iff $x \in p^n\mathbb{Z}_p$ for all n iff $|x|_p \leq p^{-n}$ for all n iff $|x|_p = 0$ iff $x = 0$. So the map is injective.

To show surjectivity, we let

$$z_n \in \varprojlim \mathbb{Z}_p/p^n\mathbb{Z}_p.$$

We define $a_i \in \{0, 1, \cdots, p-1\}$ recursively such that

$$x_n = \sum_{i=0}^{n-1} a_i p^i$$

is the unique representative of z_n in the set of integers $\{0, 1, \cdots, p^n - 1\}$. Then

$$x = \sum_{i=0}^{\infty} a_i p^i$$

exists in $\mathbb{Z}_p$ and maps to $x \equiv x_n \equiv z_n \pmod{p^n}$ for all $n \geq 0$. So $\nu(x) = (z_n)$. So the map is surjective. So ν is bijective. □

Corollary. Every $a \in \mathbb{Z}_p$ has a unique expansion

$$a = \sum_{i=0}^{\infty} a_i p^i.$$

with $a_i \in \{0, \cdots, p-1\}$.

More generally, for any $a \in \mathbb{Q}^\times$, there is a unique expansion

$$a = \sum_{i=n}^{\infty} a_i p^i$$

for $a_i \in \{0, \cdots, p-1\}$, $a_n \neq 0$ and

$$n = -\log_p |a|_p \in \mathbb{Z}.$$

Proof. The second part follows from the first part by multiplying a by p^{-n}. □

2 Valued fields

2.1 Hensel's lemma

Theorem (Hensel's lemma)**.** Let K be a complete valued field, and let $f \in K[x]$ be primitive. Put $\bar{f} = f \bmod \mathfrak{m} \in k[x]$. If there is a factorization

$$\bar{f}(x) = \bar{g}(x)\bar{h}(x)$$

with $(\bar{g}, \bar{h}) = 1$, then there is a factorization

$$f(x) = g(x)h(x)$$

in $\mathcal{O}[x]$ with

$$\bar{g} = g, \quad \bar{h} = h \mod \mathfrak{m},$$

with $\deg g = \deg \bar{g}$.

Proof. Let g_0, h_0 be arbitrary lifts of $\bar{g}$ and $\bar{h}$ to $\mathcal{O}[x]$ with $\deg \bar{g} = g_0$ and $\deg \bar{h} = h_0$. Then we have

$$f = g_0 h_0 \mod \mathfrak{m}.$$

The idea is to construct a "Taylor expansion" of the desired g and h term by term, starting from g_0 and h_0, and using completeness to guarantee convergence. To proceed, we use our assumption that $\bar{g}, \bar{h}$ are coprime to find some $a, b \in \mathcal{O}[x]$ such that

$$ag_0 + bh_0 \equiv 1 \mod \mathfrak{m}. \tag{$\dagger$}$$

It is easier to work modulo some element π instead of modulo the ideal $\mathfrak{m}$, since we are used to doing Taylor expansion that way. Fortunately, since the equations above involve only finitely many coefficients, we can pick an $\pi \in \mathfrak{m}$ with absolute value large enough (i.e. close enough to 1) such that the above equations hold with $\mathfrak{m}$ replaced with π. Thus, we can write

$$f = g_0 h_0 + \pi r_0, \quad r_0 \in \mathcal{O}[x].$$

Plugging in ($\dagger$), we get

$$f = g_0 h_0 + \pi r_0 (a g_0 + b h_0) + \pi^2(\text{something}).$$

If we are lucky enough that $\deg r_0 b < \deg g_0$, then we group as we learnt in secondary school to get

$$f = (g_0 + \pi r_0 b)(h_0 + \pi r_0 a) + \pi^2(\text{something}).$$

We can then set

$$\begin{aligned} g_1 &= g_0 + \pi r_0 b \\ h_1 &= h_0 + \pi r_0 a, \end{aligned}$$

and then we can write

$$f = g_1 h_1 + \pi^2 r_1, \quad r_1 \in \mathcal{O}[x], \quad \deg g_1 = \deg \bar{g}. \tag{$*$}$$

If it is not true that $\deg r_0 b \leq \deg g_0$, we use the division algorithm to write

$$r_0 b = q g_0 + p.$$

Then we have

$$f = g_0 h_0 + \pi((r_0 a + q) g_0 + p h_0),$$

and then proceed as above.

Given the factorization $(*)$, we replace r_1 by $r_1(a g_0 + b h_0)$, and then repeat the procedure to get a factorization

$$f \equiv g_2 h_2 \mod \pi^3, \quad \deg g_2 = \deg \bar{g}.$$

Inductively, we constrict g_k, h_k such that

$$\begin{aligned} f &\equiv g_k h_k \mod \pi^{k+1} \\ g_k &\equiv g_{k-1} \mod \pi^k \\ h_k &\equiv h_{k-1} \mod \pi^k \\ \deg g_k &= \deg \bar{g} \end{aligned}$$

Note that we may drop the terms of h_k whose coefficient are in $\pi^{k+1}\mathcal{O}$, and the above equations still hold. Moreover, we can then bound $\deg h_k \leq \deg f - \deg g_k$. It now remains to set

$$g = \lim_{k \to \infty} g_k, \quad h = \lim_{k \to \infty} h_k. \qquad \square$$

Corollary. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ where K is complete and $a_0, a_n \neq 0$. If f is irreducible, then

$$|a_\ell| \leq \max(|a_0|, |a_n|)$$

for all ℓ.

Proof. By scaling, we can wlog f is primitive. We then have to prove that $\max(|a_0|, |a_n|) = 1$. If not, let r be minimal such that $|a_r| = 1$. Then $0 < r < n$. Moreover, we can write

$$f(x) \equiv x^r(a_r + a_{r+1}x + \cdots + a_n x^{n-r}) \mod \mathfrak{m}.$$

But then Hensel's lemma says this lifts to a factorization of f, a contradiction. $\square$

Corollary (of Hensel's lemma)**.** Let $f \in \mathcal{O}[x]$ be monic, and K complete. If f mod $\mathfrak{m}$ has a simple root $\bar{\alpha} \in k$, then f has a (unique) simple root $\alpha \in \mathcal{O}$ lifting $\bar{\alpha}$.

2.2 Extension of norms

Theorem. Let K be a complete valued field, and let L/K be a finite extension. Then the absolute value on K has a unique extension to an absolute value on L, given by

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|},$$

where $n = [L : K]$ and $N_{L/K}$ is the field norm. Moreover, L is complete with respect to this absolute value.

Corollary. Let K be complete and M/K be an algebraic extension of K. Then $|\cdot|$ extends uniquely to an absolute value on M.

Corollary. Let K be a complete valued field and L/K a finite extension. If $\sigma \in \mathrm{Aut}(L/K)$, then $|\sigma(\alpha)|_L = |\alpha|_L$.

Proof. We check that $\alpha \mapsto |\sigma(\alpha)|_L$ is also an absolute value on L extending the absolute value on K. So the result follows from uniqueness. □

Proposition. Let K be a complete valued field, and V a finite-dimensional K-vector space. Then V is complete under the max norm.

Proof. Given a Cauchy sequence in V under the max norm, take the limit of each coordinate to get the limit of the sequence, using the fact that K is complete. □

Proposition. Let K be a complete valued field, and V a finite-dimensional K-vector space. Then any norm $\|\cdot\|$ on V is equivalent to $\|\cdot\|_{\max}$.

Corollary. V is complete with respect to any norm.

Proof. Let $\|\cdot\|$ be a norm. We need to find $C, D > 0$ such that

$$C\left\|x\right\|_{\max} \leq \|x\| \leq D\left\|x\right\|_{\max}.$$

We set $D = \max_i(\|x_i\|)$. Then we have

$$\|x\| = \left\|\sum a_i x_i\right\| \leq \max\left(|a_i|\left\|x_i\right\|\right) \leq (\max|a_i|)D = \left\|x\right\|_{\max} D.$$

We find C by induction on n. If $n = 1$, then $\|x\| = \|a_1 x_1\| = |a_1|\left\|x\right\| = \left\|x\right\|_{\max}\left\|x_1\right\|$. So $C = \|x_1\|$ works.

For $n \geq 2$, we let

$$\begin{aligned} V_i &= Kx_1 \oplus \cdots \oplus Kx_{i-1} \oplus Kx_{i+1} \oplus \cdots \oplus Kx_n \\ &= \mathrm{span}\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}. \end{aligned}$$

By the induction hypothesis, each V_i is complete with respect to (the restriction of) $\|\cdot\|$. So in particular V_i is closed in V. So we know that the union

$$\bigcup_{i=1}^n x_i + V_i$$

is also closed. By construction, this does not contain 0. So there is some $C > 0$ such that if $x \in \bigcup_{i=1}^n x_i + V_i$, then $\|x\| \geq C$. We claim that

$$C\left\|x\right\|_{\max} \leq \|x\|.$$

Indeed, take $x = \sum a_i x_i \in V$. Let r be such that

$$|a_r| = \max_i(|a_i|) = \left\|x\right\|_{\max}.$$

Then

$$\begin{aligned} \left\|x\right\|_{\max}^{-1}\left\|x\right\| &= \left\|a_r^{-1} x\right\| \\ &= \left\|\frac{a_1}{a_r}x_1 + \cdots + \frac{a_{r-1}}{a_r}x_{r-1} + x_r + \frac{a_{r+1}}{a_r}x_{r+1} + \cdots + \frac{a_n}{a_r}x_n\right\| \\ &\geq C, \end{aligned}$$

since the last vector is an element of $x_r + V_r$. □

Lemma. Let K be a valued field. Then the valuation ring $\mathcal{O}_K$ is integrally closed in K.

Proof. Let $x \in K$ and $|x| > 1$. Suppose we have $a_{n-1}, \cdots, a_0 \in \mathcal{O}_K$. Then we have

$$|x^n| > |a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}|.$$

So we know

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has non-zero norm, and in particular is non-zero. So x is not integral over $\mathcal{O}_K$. So $\mathcal{O}_K$ is integrally closed. □

Lemma. Let L be a field and $|\cdot|$ a function that satisfies all axioms of an absolute value but the strong triangle inequality. Then $|\cdot|$ is an absolute value iff $|\alpha| \leq 1$ implies $|\alpha + 1| \leq 1$.

Proof. It is clear that if $|\cdot|$ is an absolute value, then $|\alpha| \leq 1$ implies $|\alpha+1| \leq 1$.

Conversely, if this holds, and $|x| \leq |y|$, then $|x/y| \leq 1$. So $|x/y + 1| \leq 1$. So $|x + y| \leq |y|$. So $|x + y| \leq \max\{|x|, |y|\}$. □

Theorem. Let K be a complete valued field, and let L/K be a finite extension. Then the absolute value on K has a unique extension to an absolute value on L, given by

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|},$$

where $n = [L : K]$ and $N_{L/K}$ is the field norm. Moreover, L is complete with respect to this absolute value.

Proof. For uniqueness and completeness, if $|\cdot|_L$ is an absolute value on L, then it is in particular a K-norm on L as a finite-dimensional vector space. So we know L is complete with respect to $|\cdot|_L$.

If $|\cdot|'_L$ is another absolute value extending $|\cdot|$, then we know $|\cdot|_L$ and $|\cdot|'_L$ are equivalent in the sense of inducing the same topology. But then from one of the early exercises, when *field* norms are equivalent, then we can find some $s > 0$ such that $|\cdot|_L^s = |\cdot|'_L$. But the two norms agree on K, and they are non-trivial. So we must have $s = 1$. So the norms are equal.

To show existence, we have to prove that

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|}$$

is a norm.

(i) If $|\alpha|_L = 0$, then $N_{L/K}(\alpha) = 0$. This is true iff $\alpha = 0$.

(ii) The multiplicativity of $|\alpha|$ and follows from the multiplicativity of $N_{L/K}$, $|\cdot|$ and $\sqrt[n]{\cdot}$.

To show the strong triangle inequality, it suffices to show that $|\alpha|_L \leq 1$ implies $|\alpha + 1|_L \leq 1$.

Recall that

$$\mathcal{O}_L = \{\alpha \in L : |\alpha|_L \leq 1\} = \{\alpha \in L : N_{L/K}(\alpha) \in \mathcal{O}_K\}.$$

We claim that $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L. This implies what we want, since the integral closure is closed under addition (and 1 is in the integral closure).

Let $\alpha \in \mathcal{O}_L$. We may assume $\alpha \neq 0$, since that case is trivial. Let the minimal polynomial of α over K be

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n \in K[x].$$

We need to show that $a_i \in \mathcal{O}_K$ for all i. In other words, $|a_i| \leq 1$ for all i. This is easy for a_0, since

$$N_{L/K}(\alpha) = \pm a_0^m,$$

and hence $|a_0| \leq 1$.

By the corollary of Hensel's lemma, for each i, we have

$$|a_i| \leq \max(|a_0|, 1)$$

By general properties of the field norm, there is some $m \in \mathbb{Z}_{\geq 1}$ such that $N_{L/K}(\alpha) = \pm a_0^m$. So we have

$$|a_i| \leq \max\left(\left|N_{L/K}(\alpha)^{1/m}\right|, 1\right) = 1.$$

So $f \in \mathcal{O}_K[x]$. So α is integral over $\mathcal{O}_K$.

On the other hand, suppose α is integral over $\mathcal{O}_K$. Let $\bar{K}/K$ be an algebraic closure of K. Note that

$$N_{L/K}(\alpha) = \left(\prod_{\sigma: L \hookrightarrow \bar{K}} \sigma(\alpha)\right)^d,$$

for some $d \in \mathbb{Z}_{\geq 1}$, and each $\sigma(\alpha)$ is integral over $\mathcal{O}_K$, since α is (apply σ to the minimal polynomial). This implies that $N_{L/K}(\alpha)$ is integral over $\mathcal{O}_K$ (and lies in K). So $N_{L/K}(\alpha) \in \mathcal{O}_K$ since $\mathcal{O}_K$ is integrally closed in K. □

Corollary (of the proof)**.** Let K be a complete valued field, and L/K a finite extension. We equip L with $|\cdot|_L$ extending $|\cdot|$ on K. Then $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.

2.3 Newton polygons

Theorem. Let K be complete valued field, and v the valuation on K. We let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x].$$

Let L be the splitting field of f over K, equipped with the unique extension w of v.

If $(r, v(a_r)) \to (s, v(a_s))$ is a line segment of the Newton polygon of f with slope $-m \in \mathbb{R}$, then f has precisely $s - r$ roots of valuation m.

Proof. Dividing by a_n only shifts the polygon vertically, so we may wlog $a_n = 1$. We number the roots of f such that

$$\begin{aligned} w(\alpha_1) = \cdots = w(\alpha_{s_1}) &= m_1 \\ w(\alpha_{s_1+1}) = \cdots = w(\alpha_{s_2}) &= m_2 \\ &\vdots \\ w(\alpha_{s_t}) = \cdots = w(\alpha_n) &= m_{t+1}, \end{aligned}$$

where we have

$$m_1 < m_2 < \cdots < m_{t+1}.$$

Then we know

$$\begin{aligned} v(a_n) &= v(1) = 0 \\ v(a_{n-1}) &= w\left(\sum \alpha_i\right) \geq \min_i w(\alpha_i) = m_1 \\ v(a_{n-2}) &= w\left(\sum \alpha_i\alpha_j\right) \geq \min_{i\neq j} w(\alpha_i\alpha_j) = 2m_1 \\ &\vdots \\ v(a_{n-s_1}) &= w\left(\sum_{i_1\neq\ldots\neq i_{s_1}} \alpha_{i_1\ldots\alpha_{i_{s_1}}}\right) = \min w(\alpha_{i_1}\cdots\alpha_{i_{s_1}}) = s_1m_1. \end{aligned}$$

It is important that in the last one, we have equality, not an inequality, because there is one term in the sum whose valuation is less than all the others.

We can then continue to get

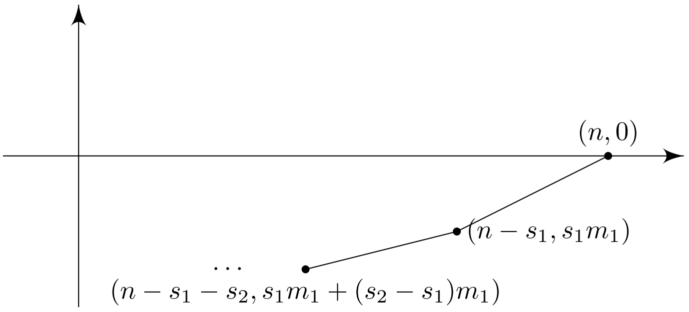
$$v(\alpha_{n-s_1-1}) \geq \min w(\alpha_{i_1}\cdots\alpha_{i_{s_1+1}}) = s_1m_1 + m_2,$$

until we reach

$$v(\alpha_{n-s_1-s_2}) = s_1m_1 + (s_2 - s_1)m_2.$$

We keep going on.

We draw the Newton polygon.



We don't know where exactly the other points are, but the inequalities imply that the $(i, v(a_i))$ are above the lines drawn. So this is the Newton polygon.

Counting from the right, the first line segment has length $n - (n - s_1) = s_1$ and slope

$$\frac{0 - s_1m_1}{n - (n - s_1)} = -m_1.$$

In general, the kth segment has length $(n - s_{k-1}) - (n - s_k) = s_k - s_{k-1}$, and slope

$$\frac{\left(s_1m_1 + \sum_{i=1}^{k-2}(s_{i+1} - s_i)m_{i+1}\right) - \left(s_1m_1 + \sum_{i=1}^{k-1}(s_{i+1} - s_i)m_{i+1}\right)}{s_k - s_{k-1}}$$
$$= \frac{-(s_k - s_{k-1})m_k}{s_k - s_{k-1}} = -m_k.$$

and the others follow similarly. □

Corollary. If f is irreducible, then the Newton polygon has a single line segment.

Proof. We need to show that all roots have the same valuation. Let α, β be in the splitting field L. Then there is some $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma(\alpha) = \beta$. Then $w(\alpha) = w(\sigma(\alpha)) = \beta$. So done. □

3 Discretely valued fields

Proposition. Let K be a discretely valued field with uniformizer π. Let $S \subseteq \mathcal{O}_K$ be a set of coset representatives of $\mathcal{O}_k/\mathfrak{m}_k = k_K$ containing 0. Then

(i) The non-zero ideals of $\mathcal{O}_K$ are $\pi^n\mathcal{O}_K$ for $n \geq 0$.

(ii) The ring $\mathcal{O}_K$ is a PID with unique prime π (up to units), and $\mathfrak{m}_K = \pi\mathcal{O}_K$.

(iii) The topology on $\mathcal{O}_K$ induced by the absolute value is the π-adic topology.

(iv) If K is complete, then $\mathcal{O}_K$ is π-adically complete.

(v) If K is complete, then any $x \in K$ can be written uniquely as

$$x = \sum_{n \gg -\infty}^{\infty} a_n \pi^n,$$

where $a_n \in S$, and

$$|x| = |\pi|^{-\inf\{n : a_n \neq 0\}}.$$

(vi) The completion $\hat{K}$ is also discretely valued and π is a uniformizer, and moreover the natural map

$$\frac{\mathcal{O}_k}{\pi^n\mathcal{O}_k} \xrightarrow{\sim} \frac{\mathcal{O}_{\hat{K}}}{\pi^n\mathcal{O}_{\hat{K}}}$$

is an isomorphism.

Proof. The same as for $\mathbb{Q}_p$ and $\mathbb{Z}_p$, with π instead of p. □

Proposition. Let K be a discretely valued field. Then K is a local field iff $\mathcal{O}_K$ is compact.

Proof. If $\mathcal{O}_K$ is compact, then $\pi^{-n}\mathcal{O}_K$ is compact for all $n \geq 0$ (where π is the uniformizer), and in particular complete. So

$$K = \bigcup_{n \geq 0}^{\infty} \pi^{-n}\mathcal{O}_K$$

is complete, as this is an increasing union, and Cauchy sequences are bounded. Also, we know the quotient map $\mathcal{O}_K \to k_K$ is continuous when k_K is given the discrete topology, by definition of the π-adic topology. So k_K is compact and discrete, hence finite.

In the other direction, if K is local, then we know $\mathcal{O}_K/\pi^n\mathcal{O}_K$ is finite for all $n \geq 0$ (by induction and finiteness of k_K). We let (x_i) be a sequence in $\mathcal{O}_K$. Then by finiteness of $\mathcal{O}_K/\pi\mathcal{O}_K$, there is a subsequence $(x_{1,i})$ which is constant modulo π. We keep going, choosing a subsequence $(x_{n+1,i})$ of (x_{n_i}) such that $(x_{n+1,i})$ is constant modulo π^{n+1}. Then $(x_{i,i})_{i=1}^{\infty}$ converges, since it is Cauchy as

$$|x_{ii} - x_{jj}| \leq |\pi|^j$$

for $j \leq i$. So $\mathcal{O}_K$ is sequentially compact, hence compact. □

Proposition. R is a DVR iff $R \cong \mathcal{O}_K$ for some DVF K.

Proof. We have already seen that valuation rings of discrete valuation fields are DVRs. In the other direction, let R be a DVR, and π a prime. Let $x \in \mathbb{R} \setminus \{0\}$. Then we can find a unique unit $u \in R^\times$ and $n \in \mathbb{Z}_{\geq 0}$ such that $x = \pi^n u$ (say, by unique factorization of PIDs). We define

$$v(x) = \begin{cases} n & x \neq 0 \\ \infty & x = 0 \end{cases}$$

This is then a discrete valuation of R. This extends uniquely to the field of fractions K. It remains to show that $R = \mathcal{O}_K$. First note that

$$K = R\left[\frac{1}{\pi}\right].$$

This is since any non-zero element in $R\left[\frac{1}{\pi}\right]$ looks like $\pi^n u, u \in R^\times, n \in \mathbb{Z}$, and is already invertible. So it must be the field of fractions. Then we have

$$v(\pi^n u) = n \in \mathbb{Z}_{\geq 0} \iff \pi^n u \in R.$$

So we have $R = \mathcal{O}_K$. □

3.1 Teichmüller lifts

Theorem. Let R be a ring, and let $x \in R$. Assume that R is x-adically complete and that R/xR is perfect of characteristic p. Then there is a unique map $[-] : R/xR \to R$ such that

$$[a] \equiv a \mod x$$

and

$$[ab] = [a][b].$$

for all $a, b \in R/xR$. Moreover, if R has characteristic p, then $[-]$ is a ring homomorphism.

Lemma. Let R be a ring with $x \in R$ such that R/xR has characteristic p. Let $\alpha, \beta \in R$ be such that

$$\alpha = \beta \mod x^k \tag{†}$$

Then we have

$$\alpha^p = \beta^p \mod x^{k+1}.$$

Proof. It is left as an exercise to modify the proof to work for $p = 2$ (it is actually easier). So suppose p is odd. We take the pth power of (†) to obtain

$$\alpha^p - \beta^p + \sum_{i=1}^{p-1} \binom{p}{i} \alpha^{p-i}\beta^i \in x^{p(k+1)}R.$$

We can now write

$$\begin{aligned} \sum_{i=1}^{p-1} (-1)^i \binom{p}{i} \alpha^{p-i}\beta^i &= \sum_{i=1}^{\frac{p-1}{2}} (-1)^i \binom{p}{i} (\alpha\beta)^i \left(\alpha^{p-2i} - \beta^{p-2i}\right) \\ &= p(\alpha - \beta)(\text{something}). \end{aligned}$$

Now since R/xR has characteristic p, we know $p \in xR$. By assumption, we know $\alpha - \beta \in x^{k+1}R$. So this whole mess is in $x^{k+2}R$, and we are done. □

Proof of theorem. Let $a \in R/xR$. For each n, there is a unique $a^{p^{-n}} \in R/xR$. We lift this arbitrarily to some $\alpha_n \in R$ such that

$$\alpha_n \equiv a^{p^{-n}} \mod x.$$

We define

$$\beta_n = \alpha_n^{p^n}.$$

The claim is that

$$[a] = \lim_{n\to\infty} \beta_n$$

exists and is independent of the choices.

Note that if the limit exists no matter how we choose the α_n, then it must be independent of the choices. Indeed, if we had choices β_n and β'_n, then $\beta_1, \beta'_2, \beta_3, \beta'_4, \beta_5, \beta'_6, \cdots$ is also a respectable choice of lifts, and thus must converge. So β_n and β'_n must have the same limit.

Since the ring is x-adically complete and is discretely valued, to show the limit exists, it suffices to show that $\beta_{n+1} - \beta_n \to 0$ x-adically. Indeed, we have

$$\beta_{n+1} - \beta_n = (\alpha_{n+1}^p)^{p^n} - \alpha_n^{p^n}.$$

We now notice that

$$\alpha_{n+1}^p \equiv (a^{p^{-n-1}})^p = a^{p^{-n}} \equiv \alpha_n \mod x.$$

So by applying the previous the lemma many times, we obtain

$$(\alpha_{n+1}^p)^{p^n} \equiv \alpha_n^{p^n} \mod x^{n+1}.$$

So $\beta_{n+1} - \beta_n \in x^{n+1}R$. So $\lim \beta_n$ exists.

To see $[a] = a \mod x$, we just have to note that

$$\lim_{n\to\infty} \alpha_n^{p^n} \equiv \lim_{n\to\infty} (a^{p^{-n}})^{p^n} = \lim a = a \mod x.$$

(here we are using the fact that the map $R \to R/xR$ is continuous when R is given the x-adic topology and R/xR is given the discrete topology)

The remaining properties then follow trivially from the uniqueness of the above limit.

For multiplicativity, if we have another element $b \in R/xR$, with $\gamma_n \in R$ lifting $b^{p^{-n}}$ for all n, then $\alpha_n\gamma_n$ lifts $(ab)^{p^{-n}}$. So

$$[ab] = \lim \alpha_n^{p^n} \gamma_n^{p^n} = \lim \alpha_n^{p^n} \lim \gamma_n^{p^n} = [a][b].$$

If R has characteristic p, then $\alpha_n + \gamma_n$ lifts $a^{p^{-n}} + b^{p^{-n}} = (a+b)^{p^{-n}}$. So

$$[a+b] = \lim(\alpha_n + \gamma_n)^{p^n} = \lim \alpha_n^{p^n} + \lim \gamma_n^{p^n} = [a] + [b].$$

Since 1 is a lift of 1 and 0 is a lift of 0, it follows that this is a ring homomorphism.

Finally, to show uniqueness, suppose $\phi : R/xR \to R$ is a map with these properties. Then we note that $\phi(a^{p^{-n}}) \equiv a^{p^{-n}} \mod x$, and is thus a valid choice of α_n. So we have

$$[a] = \lim_{n\to\infty} \phi(a^{p^{-n}})^{p^n} = \lim \phi(a) = \phi(a).$$ □

Theorem. Let K be a complete discretely valued field of equal characteristic p, and assume that k_K is perfect. Then $K \cong k_K((T))$.

Proof. Let K be a complete DVF. Since every DVF the field of fractions of its valuation ring, it suffices to prove that $\mathcal{O}_K \cong k_K[[T]]$. We know $\mathcal{O}_K$ has characteristic p. So $[-] : k_K \to \mathcal{O}_K$ is an injective ring homomorphism. We choose a uniformizer $\pi \in \mathcal{O}_K$, and define

$$k_K[[T]] \to \mathcal{O}_K$$

by

$$\sum_{n=0}^{\infty} a_n T^n \mapsto \sum_{n=0}^{\infty} [a_n] \pi^n.$$

Then this is a ring homomorphism since $[-]$ is. The bijectivity follows from property (v) in our list of properties of complete DVF's. □

Corollary. Let K be a local field of equal characteristic p. Then $k_K \cong \mathbb{F}_q$ for some q a power of p, and $K \cong F_q((T))$.

3.2 Witt vectors*

Lemma. Let A be a strict p-ring. Then any element of A can be written uniquely as

$$a = \sum_{n=0}^{\infty} [a_n] p^n,$$

for a *unique* $a_n \in A/pA$.

Proof. We recursively construct the a_n by

$$\begin{aligned} a_0 &= a \pmod p \\ a_1 &\equiv p^{-1}(a - [a_0]) \pmod p \\ &\vdots \end{aligned}$$

□

Lemma. Let A and B be strict p-rings and let $f : A/pA \to B/pB$ be a ring homomorphism. Then there is a unique homomorphism $F : A \to B$ such that $f = F \bmod p$, given by

$$F\left(\sum [a_n] p^n\right) = \sum [f(a_n)] p^n.$$

Proof sketch. We define F by the given formula and check that it works. First of all, by the formula, F is p-adically continuous, and the key thing is to check that it is additive (which is slightly messy). Multiplicativity then follows formally from the continuity and additivity.

To show uniqueness, suppose that we have some ψ lifting f. Then $\psi(p) = p$. So ψ is p-adically continuous. So it suffices to show that $\psi([a]) = [\psi(a)]$.

We take $\alpha_n \in A$ lifting $a^{p^{-n}} \in A/pA$. Then $\psi(\alpha_n)$ lifts $f(a)^{p^{-n}}$. So

$$\psi([a]) = \lim \psi(\alpha_n^{p^{-n}}) = \lim \psi(\alpha_n)^{p^{-n}} = [f(a)].$$

So done. □

Proposition. Let A be a strict p-ring and B be a ring with an element x such that B is x-adically complete and B/xB is perfect of characteristic p. If $f : A/pA \to B/xB$ is a ring homomorphism. Then there exists a unique ring homomorphism $F : A \to B$ with $f = F \mod x$, i.e. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d] & B \arrow[d] \\ A/pA \arrow[r, "f"] & B/xB \end{tikzcd} .$$

Theorem. Let R be a perfect ring. Then there is a unique (up to isomorphism) strict p-ring $W(B)$ called the *Witt vectors* of R such that $W(R)/pW(R) \cong R$.

Moreover, for any other perfect ring R, the reduction mod p map gives a bijection

$$\mathrm{Hom}_{\mathrm{Ring}}(W(R), W(R')) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ring}}(R, R') .$$

Proof sketch. If $W(R)$ and $W(R')$ are such strict p-rings, then the second part follows from the previous lemma. Indeed, if C is a strict p-ring with $C/pC \cong R \cong W(R)/pW(R)$, then the isomorphism $\bar{\alpha} : W(R)/pW(R) \to C/pC$ and its inverse $\bar{\alpha}^{-1}$ have unique lifts $\gamma : W(R) \to C$ and $\gamma^{-1} : C \to W(R)$, and these are inverses by uniqueness of lifts.

To show existence, let R be a perfect ring. We form

$$\begin{aligned} \mathbb{F}_p[x_r^{p^{-\infty}} \mid r \in R] &\to R \\ x_r &\mapsto r \end{aligned}$$

Then we know that the p-adic completion of $\mathbb{Z}[x_r^{p^{-\infty}} \mid r \in R]$, written A, is a strict p-ring with

$$A/pA \cong \mathbb{F}_p[x_r^{p^{-\infty}} \mid r \in R].$$

We write

$$I = \ker(\mathbb{F}_p[x_r^{p^{-\infty}} \mid r \in R] \to R).$$

Then define

$$J = \left\{ \sum_{n=0}^{\infty} [a_k] p^n \in A : a_n \in I \text{ for all } n \right\}.$$

This turns out to be an ideal.

$$\begin{tikzcd} & J \arrow[r, dashed] \arrow[d] & A \arrow[r] \arrow[d] & R \arrow[d, equal] & \\ 0 \arrow[r] & I \arrow[r] & A/pA \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

We put $W(R) = A/J$. We can then painfully check that this has all the required properties. For example, if

$$x = \sum_{n=0}^{\infty} [a_n] p^n \in A,$$

and

$$px = \sum_{n=0}^{\infty} [a_n] p^{n+1} \in J,$$

then by definition of J, we know $[a_n] \in I$. So $x \in J$. So $W(R)/J$ is p-torsion free. By a similar calculation, one checks that

$$\bigcap_{n=0}^{\infty} p^n W(R) = \{0\}.$$

This implies that $W(R)$ injects to its p-adic completion. Using that A is p-adically complete, one checks the surjectivity by hand.

Also, we have

$$\frac{W(R)}{pW(R)} \cong \frac{A}{J + pA}.$$

But we know

$$J + pA = \left\{\sum_n [a_n]p^n \mid a_0 \in I\right\}.$$

So we have

$$\frac{W(R)}{pW(R)} \cong \frac{\mathbb{F}_p[x_r^{p^{-\infty}} \mid r \in R]}{I} \cong R.$$

So we know that $W(R)$ is a strict p-ring. □

Proposition. A complete DVR A of mixed characteristic with perfect residue field and such that p is a uniformizer is the same as a strict p-ring A such that A/pA is a field.

Proof. Let A be a complete DVR such that p is a uniformizer and A/pA is perfect. Then A is p-torsion free, as A is an integral domain of characteristic 0. Since it is also p-adically complete, it is a strict p-ring.

Conversely, if A is a strict p-ring, and A/pA is a field, then we have $A^\times \subseteq A \setminus pA$, and we claim that $A^\times = A \setminus pA$. Let

$$x = \sum_{n=0}^{\infty} [x_n]p^n$$

with $x_0 \neq 0$, i.e. $x \notin pA$. We want to show that x is a unit. Since A/pA is a field, we can multiply by $[x_0^{-1}]$, so we may wlog $x_0 = 1$. Then $x = 1 - py$ for some $y \in A$. So we can invert this with a geometric series

$$x^{-1} = \sum_{n=0}^{\infty} p^n y^n.$$

So x is a unit. Now, looking at Teichmüller expansions and factoring out multiple of p, any non-zero element z can be written as $p^n u$ for a unique $n \geq \mathbb{Z}_{\geq 0}$ and $u \in A^\times$. Then we have

$$v(z) = \begin{cases} n & z \neq 0 \\ \infty & z = 0 \end{cases}$$

is a discrete valuation on A. □

Corollary. Let R be a complete DVR of mixed characteristic with absolute ramification index 1 and perfect residue field k. Then $R \cong W(k)$.

Proof. Having absolute ramification index 1 is the same as saying p is a uniformizer. So R is a strict p-ring with $R/pR \cong k$. By uniqueness of the Witt vector, we know $R \cong W(k)$. □

Theorem. Let R be a complete DVR of mixed characteristic p with a perfect residue field k and uniformizer π. Then R is finite over $W(k)$.

Proof. We need to first exhibit $W(k)$ as a subring of R. We know that $\mathrm{id} : k \to k$ lifts to a homomorphism $W(k) \to R$. The kernel is a prime ideal because R is an integral domain. So it is either 0 or $pW(k)$. But R has characteristic 0. So it can't be $pW(k)$. So this must be an injection.

Let e be the absolute ramification index of R. We want to prove that

$$R = \bigoplus_{i=0}^{e-1} \pi^i W(k).$$

Looking at valuations, one sees that $1, \pi, \pi, \cdots, \pi^{e-1}$ are linearly independent over $W(k)$. So we can form

$$M = \bigoplus_{i=0}^{e-1} \pi^i W(k) \subseteq R.$$

We consider R/pR. Looking at Teichmüller expansions

$$\sum_{n=0}^{\infty} [x_n]\pi^n \equiv \sum_{n=0}^{e-1} [x_n]\pi^n \mod pR,$$

we see that $1, \pi, \cdots, \pi^{e-1}$ generate R/pR as $W(k)$-modules (all the Teichmüller lifts live in $W(k)$). Therefore $R = M + pR$. We iterate to get

$$R = M + p(M + pR) = M + p^2 r = \cdots = M + p^m R$$

for all $m \geq 1$. So M is dense in R. But M is also p-adically complete, hence closed in R. So $M = R$. □

Corollary. Let K be a mixed characteristic local field. Then K is a finite extension of $\mathbb{Q}_p$.

Proof. Let $\mathbb{F}_q$ be the residue field of K. Then $\mathcal{O}_K$ is finite over $W(\mathbb{F}_q)$ by the previous theorem. So it suffices to show that $W(\mathbb{F}_q)$ is finite over $W(\mathbb{F}_p) = \mathbb{Z}_p$. Again the inclusion $\mathbb{F}_p \subseteq \mathbb{F}_q$ gives an injection $W(\mathbb{F}_p) \hookrightarrow W(\mathbb{F}_q)$. Write $q = p^d$, and let $x_1, \cdots, x_d \in W(F_q)$ be lifts of an $\mathbb{F}_p$-bases of $\mathbb{F}_q$.. Then we have

$$W(\mathbb{F}_q) = \bigoplus_{i=1}^{d} x_d \mathbb{Z}_p + pW(\mathbb{F}_q),$$

and then argue as in the end of the previous theorem to get

$$W(\mathbb{F}_q) = \bigoplus_{i=1}^{d} x_d \mathbb{Z}_p.$$ □

4 Some p-adic analysis

Proposition. Let K be a complete valued field with an absolute value $|\cdot|$ and assume that $K \supseteq \mathbb{Q}_p$ and $|\cdot|$ restricts to the usual p-adic norm on $\mathbb{Q}_p$. Then $\exp(x)$ converges for $|x| < p^{-1/(p-1)}$ and $\log(1+x)$ converges for $|x| < 1$, and then define continuous maps

$$\begin{aligned} \exp &: \{x \in K : |x| < p^{-1/(p-1)}\} \to \mathcal{O}_K \\ \log &: \{1 + x \in K : |x| < 1\} \to K. \end{aligned}$$

Proof. We let $v = -\log_p |\cdot|$ be a valuation extending v_p. Then we have the dumb estimate

$$v(n) \leq \log_p n.$$

Then we have

$$v\left(\frac{x^n}{n}\right) \geq n \cdot v(x) - \log_p n \to \infty$$

if $v(x) > 0$. So log converges.

For exp, we have

$$v(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the sum of the p-adic digits of n. Then we have

$$v\left(\frac{x^n}{n!}\right) \geq n \cdot v(x) - \frac{n}{p-1} = n \cdot \left(v(x) - \frac{1}{p-1}\right) \to \infty$$

if $v(x) > 1/(p-1)$. Since $v\left(\frac{x^n}{n!}\right) \geq 0$, this lands in $\mathcal{O}_K$.

For the continuity, we just use uniform convergence as in the real case. □

Theorem (Mahler's theorem)**.** Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be any continuous function. Then there is a unique sequence $(a_n)_{n \geq 0}$ with $a_n \in \mathbb{Q}_p$ and $a_n \to 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

and moreover

$$\sup_{x \in \mathbb{Z}_p} |f(x)| = \max_{k \in \mathbb{N}} |a_k|.$$

Proposition. The norm $\|\cdot\|$ defined above is in fact a (non-archimedean) norm, and that $C(\mathbb{Z}_p, \mathbb{Q}_p)$ is complete under this norm.

Lemma. Let $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. Then there exists some $k \geq 1$ such that

$$\|\Delta^{p^k} f\| \leq \frac{1}{p}\|f\|.$$

Proof. If $f = 0$, there is nothing to prove. So we will wlog $\|f\| = 1$ by scaling (this is possible since the norm is attained at some x_0, so we can just divide by $f(x_0)$). We want to find some k such that

$$\Delta^{p^k} f(x) \equiv 0 \mod p$$

for all x. To do so, we use the explicit formula

$$\Delta^{p^k} f(x) = \sum_{i=0}^{p^k} (-1)^i \binom{p^k}{i} f(x + p^k - i) \equiv f(x + p^k) - f(x) \pmod p$$

because the binomial coefficients $\binom{p^k}{i}$ are divisible by p for $i \neq 0, p^k$. Note that we do have a negative sign in front of $f(x)$ because $(-1)^{p^k}$ is -1 as long as p is odd, and $1 = -1$ if $p = 2$.

Now $\mathbb{Z}_p$ is compact. So f is uniformly continuous. So there is some k such that $|x - y|_p \leq p^{-k}$ implies $|f(x) - f(y)|_p \leq p^{-1}$ for all $x, y \in \mathbb{Z}_p$. So take this k, and we're done. □

Proposition. The map $f \mapsto (a_n(f))_{n=0}^\infty$ defines an injective norm-decreasing linear map $C(\mathbb{Z}_p, \mathbb{Q}_p) \to c_0$.

Proof. First we prove that $a_n(f) \to 0$. We know that

$$\|a_n(f)\|_p \leq \|\Delta^n f\|.$$

So it suffices to show that $\|\Delta^n f\| \to 0$. Since $\|\Delta\| \leq 1$, we know $\|\Delta^n f\|$ is monotonically decreasing. So it suffices to find a subsequence that tends to 0. To do so, we simply apply the lemma repeatedly to get $k_1, k_2, \cdots$ such that

$$\left\| \Delta^{p^k_1 + \cdots + k_n} \right\| \leq \frac{1}{p^n} \|f\|.$$

This gives the desired sequence.

Note that

$$|a_n(f)|_p \leq \|\Delta^n\| \leq \|f\|.$$

So we know

$$\|(a_n(f))_n\| = \max |a_n(f)|_p \leq \|f\|.$$

So the map is norm-decreasing. Linearity follows from linearity of Δ. To finish, we have to prove injectivity.

Suppose $a_n(f) = 0$ for all $n \geq 0$. Then

$$a_0(f) = f(0) = 0,$$

and by induction,we have that

$$f(n) = \Delta^k f(0) = a_n(f) = 0.$$

for all $n \geq 0$. So f is constantly zero on $\mathbb{Z}_{\geq 0}$. By continuity, it must be zero everywhere on $\mathbb{Z}_p$. □

Lemma. We have

$$\binom{x}{n} + \binom{x}{n-1} = \binom{x+1}{n}$$

for all $n \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{Z}_p$.

Proof. It is well known that this is true when $x \in \mathbb{Z}_{\geq n}$. Since the expressions are polynomials in x, them agreeing on infinitely many values implies that they are indeed the same. □

Proposition. Let $a = (a_n)_{n=0}^\infty \in c_0$. We define $f_a : \mathbb{Z}_p \to \mathbb{Q}_p$ by

$$f_a(x) = \sum_{n=0}^\infty a_n \binom{x}{n}.$$

This defines a norm-decreasing linear map $c_0 \to C(\mathbb{Z}_p, \mathbb{Q}_p)$. Moreover $a_n(f_a) = a_n$ for all $n \geq 0$.

Proof. Linearity is clear. Norm-decreasing follows from

$$|f_a(x)| = \left|\sum a_n \binom{x}{n}\right| \leq \sup_n |a_n|_p \left|\binom{x}{n}\right|_p \leq \sup_n |a_n|_p = \|a_n\|,$$

where we used the fact that $\binom{x}{n} \in \mathbb{Z}_p$, hence $\left|\binom{x}{n}\right|_p \leq 1$.

Taking the supremum, we know that

$$\|f_a\| \leq \|a\|.$$

For the last statement, for all $k \in \mathbb{Z}_{\geq 0}$, we define

$$a^{(k)} = (a_k, a_{k+1}, a_{k+1}, \cdots).$$

Then we have

$$\begin{aligned}
\Delta f_a(x) &= f_a(x+1) - f_a(x)\\
&= \sum_{n=1}^\infty a_n \left(\binom{x+1}{n} - \binom{x}{n}\right)\\
&= \sum_{n=1}^\infty a_n \binom{x}{n-1}\\
&= \sum_{n=0}^\infty a_{n+1} \binom{x}{n}\\
&= f_{a^{(1)}}(x)
\end{aligned}$$

Iterating, we have

$$\Delta^k f_a = f_{a^{(k)}}.$$

So we have

$$a_n(f_a) = \Delta^n f_a(0) = f_{a^{(n)}}(0) = a_n.$$

□

Lemma. Suppose V, W are normed spaces, and $F : V \to W$, $G : W \to V$ are maps such that F is injective and norm-decreasing, and G is norm-decreasing and $FG = \mathrm{id}_W$. Then $GF = \mathrm{id}_V$ and F and G are norm-preserving.

Proof. Let $v \in V$. Then

$$F(v - GFv) = Fv - FGFv = (F - F)v = 0.$$

Since F is injective, we have

$$v = GFv.$$

Also, we have

$$\|v\| \geq \|Fv\| \geq \|GFv\| = \|v\|.$$

So we have equality throughout. Similarly, we have $\|v\| = \|Gv\|$. □

5 Ramification theory for local fields

5.1 Ramification index and inertia degree

Theorem. Let L/K be a finite extension. Then

$$[L : K] = e_{L/K} f_{L/K}.$$

Proposition. Let K be a local field, and L/K a finite extension of degree n. Then $\mathcal{O}_L$ is a finitely-generated and free $\mathcal{O}_K$ module of rank n, and k_L/k_K is an extension of degree $\leq n$.

Moreover, L is also a local field.

Proof. Choose a K-basis $\alpha_1, \cdots, \alpha_n$ of L. Let $\|\cdot\|$ denote the maximum norm on L.

$$\left\|\sum_{i=1}^n x_i \alpha_i\right\| = \max_{i=1,\ldots,n} |x_i|$$

as before. Again, we know that $\|\cdot\|$ is equivalent to the extended norm $|\cdot|$ on L as K-norms. So we can find $r > s > 0$ such that

$$M = \{x \in L : \|x\| \leq s\} \subseteq \mathcal{O}_L \subseteq N = \{x \in L : \|x\| \leq r\}.$$

Increasing r and decreasing s if necessary, we wlog $r = |a|$ and $s = |b|$ for some $a, b \in K$.

Then we can write

$$M = \bigoplus_{i=1}^n \mathcal{O}_k b \alpha_i \subseteq \mathcal{O}_L \subseteq N = \bigoplus_{i=1}^n \mathcal{O}_K a \alpha_i.$$

We know that N is finitely generated and free of rank n over $\mathcal{O}_K$, and so is M. So $\mathcal{O}_L$ must be finitely generated and free of rank n over $\mathcal{O}_K$.

Since $\mathfrak{m}_k = \mathfrak{m}_k \cap \mathcal{O}_K$, we have a natural injection

$$\frac{\mathcal{O}_K}{\mathfrak{m}_k} \hookrightarrow \frac{\mathcal{O}_L}{\mathfrak{m}_L} = k_L.$$

Since $\mathcal{O}_L$ is generated over $\mathcal{O}_K$ by n elements, we know that k_K is generated by n elements over k_K, so it has rank at most n.

To see that L is a local field, we know that k_L/k_K is finite and k_K is finite, so k_L is finite. It is complete under the norm because it is a finite-dimensional vector space over a complete field.

Finally, to see that the valuation is discrete, suppose we have a normalized valuation on K, and w the unique extension of v_K to L. Then we have

$$w(\alpha) = \frac{1}{n} v_K(N_{L/K}(\alpha)).$$

So we have

$$w(L^\times) \subseteq \frac{1}{n} v(K^\times) = \frac{1}{n}\mathbb{Z}.$$

So it is discrete. □

Theorem. Let L/K be a finite extension. Then

$$[L:K] = e_{L/K} f_{L/K},$$

and there is some $\alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

Proof. We will be lazy and write $e = e_{L/K}$ and $f = f_{L/K}$. We first note that k_L/k_K is separable, so there is some $\bar{\alpha} \in k_L$ such that $k_L = k_K(\bar{\alpha})$ by the primitive element theorem. Let

$$\bar{f}(x) \in k_K[x]$$

be the minimal polynomial of $\bar{\alpha}$ over k_K and let $f \in \mathcal{O}_L[x]$ be a monic lift of $\bar{f}$ with $\deg f = \deg \bar{f}$.

We first claim that there is some $\alpha \in \mathcal{O}_L$ lifting $\bar{\alpha}$ such that $v_L(f(\alpha)) = 1$ (note that it is always ≥ 1). To see this, we just take any lift β. If $v_L(f(\beta)) = 1$, then we are happy and set $\alpha = \beta$. If it doesn't work, we set $\alpha = \beta + \pi_L$, where π_L is the uniformizer of L.

Then we have

$$f(\alpha) = f(\beta + \pi_L) = f(\beta) + f'(\beta)\pi_L + b\pi_L^2$$

for some $b \in \mathcal{O}_L$, by Taylor expansion around β. Since $v_L(f(\beta)) \geq 2$ and $v_L(f'(\beta)) = 0$ (since $\bar{f}$ is separable, we know $f'(\beta)$ does not vanish when we reduce mod $\mathfrak{m}$), we know $v_L(f(\alpha)) = 1$. So $f(\alpha)$ is a uniformizer of L.

We now claim that the elements $\alpha^i \pi^j$ for $i = 0, \cdots, f-1$ and $j = 0, \cdots, e-1$ are an $\mathcal{O}_K$-basis of $\mathcal{O}_L$. Suppose we have

$$\sum_{i,j} a_{ij} \alpha^i \pi^j = 0$$

for some $a_{ij} \in K$ not all 0. We put

$$s_j = \sum_{i=0}^{f-1} a_{ij} \alpha^i.$$

We know that $1, \alpha, \cdots, \alpha^{f-1}$ are linearly independent over K since their reductions are linearly independent over k_K. So there must be some j such that $s_j \neq 0$.

The next claim is that if $s_j \neq 0$, then $e \mid v_L(s_j)$. We let k be an index for which $|a_{kj}|$ is maximal. Then we have

$$a_{kj}^{-1} s_j = \sum_{i=0}^{f-1} a_{kj}^{-1} a_{ij} \alpha^i.$$

Now note that by assumption, the coefficients on the right have absolute value ≤ 1, and is 1 when $i = k$. So we know that

$$a_{kj}^{-1} s_j \not\equiv 0 \mod \pi_L,$$

because $1, \bar{\alpha}, \cdots, \bar{\alpha}^{f-1}$ are linearly independent. So we have

$$v_L(a_{kj}^{-1} s_j) = 0.$$

So we must have

$$v_L(s_j) = v_L(a_{kj}) + v_L(a_{kj}^{-1}s_j) \in v_L(K^\times) = ev_L(L^\times) = e\mathbb{Z}.$$

Now we write

$$\sum a_{ij}\alpha^i\pi^j = \sum_{j=0}^{e-1} s_j\pi^j = 0.$$

If $s_j \neq 0$, then we have $v_L(s_j\pi^j) = v_L(s_j) + j \in j + e\mathbb{Z}$. So no two non-zero terms in $\sum_{j=0}^{e-1} s_j\pi^j$ have the same valuation. This implies that $\sum_{j=0}^{e-1} s_j\pi^j \neq 0$, which is a contradiction.

We now want to prove that

$$\mathcal{O}_L = \bigoplus_{i,j} \mathcal{O}_K\alpha^i\pi^j.$$

We let

$$M = \bigoplus_{i,j} \mathcal{O}_K\alpha^i\pi^j,$$

and put

$$N = \bigoplus_{i=0}^{f-1} \mathcal{O}_L\alpha^i.$$

Then we have

$$M = N + \pi N + \pi^2 N + \cdots + \pi^{e-1}N.$$

We are now going to use the fact that $1, \bar{\alpha}, \cdots, \bar{\alpha}^{f-1}$ span k_L over k_K. So we must have that $\mathcal{O}_L = N + \pi\mathcal{O}_L$. We iterate this to obtain

$$\begin{aligned}\mathcal{O}_L &= N + \pi(N + \mathcal{O}_L)\\ &= N + \pi N + \pi^2\mathcal{O}_L\\ &= \cdots\\ &= N + \pi N + \pi^2 N + \cdots + \pi^{e-1}N + \pi^n\mathcal{O}_L\\ &= M + \pi_K\mathcal{O}_L,\end{aligned}$$

using the fact that π_K and π^e have the same valuation, and thus they differ by a unit in $\mathcal{O}_L$. Iterating this again, we have

$$\mathcal{O}_L = M + \pi_k^n\mathcal{O}_L$$

for all $n \geq 1$. So M is dense in $\mathcal{O}_L$. But M is the closed unit ball in the subspace

$$\bigoplus_{i,j} K\alpha^i\pi^j \subseteq l$$

with respect to the maximum norm with respect to the given basis. So it must be complete, and thus $M = \mathcal{O}_L$.

Finally, since $\alpha^i\pi^j = \alpha^i f(\alpha)^j$ is a polynomial in α, we know that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. □

Corollary. If $M/L/K$ is a tower of finite extensions of local fields, then

$$\begin{aligned} f_{M/K} &= f_{L/K} f_{M/L} \\ e_{M/K} &= e_{L/K} e_{M/L} \end{aligned}$$

Proof. The multiplicativity of $f_{M/K}$ follows from the tower law for the residue fields, and the multiplicativity of $e_{M/K}$ follows from the tower law for the local fields and that $f_{M/K} e_{M/K} = [M : K]$. □

5.2 Unramified extensions

Theorem. Let K be a local field. For every finite extension ℓ/k_K, there is a *unique* (up to isomorphism) finite unramified extension L/K with $k_L \cong \ell$ over k_K. Moreover, L/K is Galois with

$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(\ell/k_K).$$

Proof. We start with existence. Let $\bar{\alpha}$ be a primitive element of ℓ/k_K with minimal polynomial $\bar{f} \in k_K[x]$. Take a monic lift $f \in \mathcal{O}_K[x]$ of $\bar{f}$ such that $\deg f = \deg \bar{f}$. Note that since $\bar{f}$ is irreducible, we know f is irreducible. So we can take $L = K(\alpha)$, where α is a root of f (i.e. $L = K[x]/f$). Then we have

$$[L : K] = \deg f = \deg(\bar{f}) = [\ell : k_K].$$

Moreover, k_L contains a root of $\bar{f}$, namely the reduction α. So there is an embedding $\ell \hookrightarrow k_L$, sending $\bar{\alpha}$ to the reduction of α. So we have

$$[k_L : k_K] \geq [\ell : k_L] = [L : K].$$

So L/K must be unramified and $k_L \cong \ell$ over k_K.

Uniqueness and the Galois property follow from the following lemma: □

Lemma. Let L/K be a finite unramified extension of local fields and let M/K be a finite extension. Then there is a natural bijection

$$\mathrm{Hom}_{K\text{ - Alg}}(L, M) \longleftrightarrow \mathrm{Hom}_{k_K\text{ - Alg}}(k_L, k_M)$$

given in one direction by restriction followed by reduction.

Proof. By the uniqueness of extended absolute values, any K-algebra homomorphism $\varphi : L \hookrightarrow M$ is an isometry for the extended absolute values. In particular, we have $\varphi(\mathcal{O}_L) \subseteq \mathcal{O}_M$ and $\varphi(\mathfrak{m}_L) \subseteq \mathfrak{m}_M$. So we get an induced k_K-algebra homomorphism $\bar{\varphi} : k_L \to k_M$.

So we obtain a map

$$\mathrm{Hom}_{K\text{-Alg}}(L, M) \to \mathrm{Hom}_{k_K\text{-Alg}}(k_L, k_M)$$

To see this is bijective, we take a primitive element $\bar{\alpha} \in k_L$ over k_K, and take a minimal polynomial $\bar{f} \in k_K[x]$. We take a monic lift of $\bar{f}$ to $\mathcal{O}_k[x]$, and $\alpha \in \mathcal{O}_L$ the unique root of f which lifts $\bar{\alpha}$, which exists by Hensel's lemma. Then by counting dimensions, the fact that the extension is unramified tells us that

$$k_L = k_K(\bar{\alpha}), \quad L = K(\alpha).$$

So we can construct the following diagram:

$$\begin{tikzcd}
\varphi \arrow[d, mapsto] & \mathrm{Hom}_{K\text{-Alg}}(L, M) \arrow[r, "\text{reduction}"] \arrow[d, "\cong"] & \mathrm{Hom}_{k_K\text{-Alg}}(k_L, k_M) \arrow[d, "\cong"] & \bar{\varphi} \arrow[d, mapsto] \\
\varphi(\alpha) & \{x \in M : f(x) = 0\} \arrow[r, "\text{reduction}"] & \{\bar{x} \in k_M : \bar{f}(\bar{x}) = 0\} & \bar{\varphi}(\bar{\alpha})
\end{tikzcd}$$

But the bottom map is a bijection by Hensel's lemma. So done. □

Proof of theorem (continued). To finish off the proof of the theorem, we just note that an isomorphism $\bar{\varphi} : k_L \cong k_M$ over k_K between unramified extensions. Then $\bar{\varphi}$ lifts to a K-embedding $\varphi : L \hookrightarrow M$ and $[L : K] = [M : K]$ implies that φ is an isomorphism.

To see that the extension is Galois, we just notice that

$$|\operatorname{Aut}_K(L)| = |\operatorname{Aut}_{k_K}(k_L)| = [k_L : k_K] = [L : K].$$

So L/K is Galois. Moreover, the map $\operatorname{Aut}_K(L) \to \operatorname{Aut}_{k_K}(k_L)$ is really a homomorphism, hence an isomorphism. □

Proposition. Let K be a local field, and L/K a finite unramified extension, and M/K finite. Say L, M are subfields of some fixed algebraic closure $\bar{K}$ of K. Then LM/M is unramified. Moreover, any subextension of L/K is unramified over K. If M/K is unramified as well, then LM/K is unramified.

Proof. Let $\bar{\alpha}$ be a primitive element of k_K/k_L, and $\bar{f} \in k_K[x]$ a minimal polynomial of $\bar{\alpha}$, and $f \in \mathcal{O}_k[x]$ a monic lift of $\bar{f}$, and $\alpha \in \mathcal{O}_L$ a unique lift of f lifting $\bar{\alpha}$. Then $L = K(\alpha)$. So $LM = M(\alpha)$.

Let $\bar{g}$ be the minimal polynomial of $\bar{\alpha}$ over k_M. Then $\bar{g} \mid \bar{f}$. By Hensel's lemma, we can factorize $f = gh$ in $\mathcal{O}_M[x]$, where g is monic and lifts $\bar{g}$. Then $g(\alpha) = 0$ and g is irreducible in $M[x]$. So g is the minimal polynomial of α over M. So we know that

$$[LM : M] = \deg g = \deg \bar{g} \leq [k_{LM} : k_M] \leq [LM : M].$$

So we have equality throughout and LM/M is unramified.

The second assertion follows from the multiplicativity of $f_{L/K}$, as does the third. □

Corollary. Let K be a local field, and L/K finite. Then there is a unique maximal subfield $K \subseteq T \subseteq L$ such that T/K is unramified. Moreover, $[T : K] = f_{L/K}$.

Proof. Let T/K be the unique unramified extension with residue field extension k_L/k_K. Then $\mathrm{id} : k_T = k_L \to k_L$ lifts to a K-embedding $T \hookrightarrow L$. Identifying T with its image, we know

$$[T : K] = f_{L/K}.$$

Now if T' is any other unramified extension, then $T'T$ is an unramified extension over K, so

$$[T : K] \leq [TT' : K] \leq f_{L/K} = [T : K].$$

So we have equality throughout, and $T' \subseteq T$. So this is maximal. □

5.3 Totally ramified extensions

Theorem (Eisenstein criterion)**.** Let K be a local field, and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{O}_K[x]$. Let π_K be the uniformizer of K. If $\pi_K \mid a_{n-1}, \cdots, a_0$ and $\pi_K^2 \nmid a_0$, then f is irreducible.

Proof. Left as an exercise. You've probably seen this already in a much more general context, but in this case there is a neat proof using Newton polygons. □

Proposition. Let L/K be an extension of local fields, and v_K be the normalized valuation. Let w be the unique extension of v_K to L. Then the ramification index $e_{L/K}$ is given by

$$e_{L/K}^{-1} = w(\pi_L) = \min\{w(x) : x \in \mathfrak{m}_L\},$$

Proof. We know w and v_L differ by a constant. To figure out what this is, we have

$$1 = w(\pi_K) = e_{L/K}^{-1} v_L(\pi_K).$$

So for any $x \in L$, we have

$$w(x) = e_{L/K}^{-1} v_L(x).$$

In particular, putting $x = \pi_L$, we have

$$w(\pi_L) = e_{L/K}^{-1} v_L(\pi_L) = e_{L/K}^{-1}.$$

The equality

$$w(\pi_L) = \min\{w(x) : x \in \mathfrak{m}_L\},$$

is trivially true because the minimum is attained by π_L. □

Proposition. Let L/K be a totally ramified extension of local fields. Then $L = K(\pi_L)$ and the minimal polynomial of π_L over K is Eisenstein.

Conversely, if $L = K(\alpha)$ and the minimal polynomial of α over K is Eisenstein, then L/K is totally ramified and α is a uniformizer of L.

Proof. Let $n = [L : K]$, v_K be the valuation of K, and w the unique extension to L. Then

$$[K(\pi_L) : K]^{-1} \leq e_{K(\pi_L)/K}^{-1} = \min_{x \in \mathfrak{m}_{K(\pi_L)}} w(c) \leq \frac{1}{n},$$

where the last inequality follows from the fact that $\pi_L \in \mathfrak{m}_{L(\pi_L)}$.

But we also know that

$$[K(\pi_L) : K] \leq [L : K].$$

So we know that $L = K(\pi_L)$.

Now let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{O}_K[x]$ be the minimal polynomial of π_L/K. Then we have

$$\pi_L^n = -(a_0 + a_1\pi_L + \cdots + a_{n-1}\pi_L^{n-1}).$$

So we have

$$1 = w(\pi_L^n) = w(a_0 + a_1\pi_L + \cdots + a_{n-1}\pi_L^{n-1}) = \min_{i=0,\ldots,n-1}\left(v_k(a_i) + \frac{i}{n}\right).$$

This implies that $v_K(a_i) \geq 1$ for all i, and $v_K(x_0) = 1$. So it is Eisenstein.

For the converse, if $K = K(\alpha)$ and $n = [L : K]$, take

$$g(x) = x^n + b_{n-1}x^{n-1} + .. + b_0 \in \mathcal{O}_K[x]$$

be the minimal polynomial of α. So all roots have the same valuation. So we have

$$1 = w(b_0) = n \cdot w(\alpha).$$

So we have $w(\alpha) = \frac{1}{n}$. So we have

$$e_{L/K}^{-1} = \min_{x \in \mathfrak{m}_L} w(x) \leq \frac{1}{n} = [L : K]^{-1}.$$

So $[L : K] = e_{L/K} = n$. So L/K is totally ramified and α is a uniformizer. □

6 Further ramification theory

6.1 Some filtrations

Proposition. We have

$$\begin{aligned} U_K/U_K^{(1)} &\cong (k_K^\times, \cdot\,), \\ U_K^{(s)}/U_K^{(s+1)} &\cong (k_K, +). \end{aligned}$$

for $s \geq 1$.

Proof. We have a surjective homomorphism $\mathcal{O}_K^\times \to k_K^\times$ which is just reduction mod π_K, and the kernel is just things that are 1 modulo π_K, i.e. $U_K^{(1)}$. So this gives the first part.

For the second part, we define a surjection $U_K^{(s)} \to k_K$ given by

$$1 + \pi_K^s x \mapsto x \mod \pi_k.$$

This is a group homomorphism because

$$(1 + \pi_K^s x)(1 + \pi_K^s y) = 1 + \pi^S(x + y + \pi^s xy),$$

and this gets mapped to

$$x + y + \pi^s x + y \cong x + y \mod \pi_K.$$

Then almost by definition, the kernel is $U_K^{(s+1)}$. □

Proposition. Let L/K be a finite Galois extension of local fields. Then the homomorphism

$$\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$$

given by reduction is surjective.

Proof. Let T/K be maximal unramified subextension. Then by Galois theory, the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(T/K)$ is a surjection. Moreover, we know that $k_T = k_L$. So we have a commutative diagram

$$\begin{tikzcd} \mathrm{Gal}(L/K) \arrow[r] \arrow[d, two heads] & \mathrm{Gal}(k_L/k_K) \arrow[d, equal] \\ \mathrm{Gal}(T/K) \arrow[r, "\sim"] & \mathrm{Gal}(k_T/k_K). \end{tikzcd}$$

So the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ is surjective. □

Lemma. Let L/K be a finite Galois extension of local fields, and let $\sigma \in I(L/K)$. Then $\sigma([x]) = [x]$ for all x.

More generally, let $x \in k_L$ and $\sigma \in \mathrm{Gal}(L/K)$ with image $\bar{\sigma} \in \mathrm{Gal}(k_L/k_K)$. Then we have

$$[\bar{\sigma}(x)] = \sigma([x]).$$

Proof. Consider the map $k_L \to \mathcal{O}_L$ given by

$$f : x \mapsto \sigma^{-1}([\bar{\sigma}(x)]).$$

This is multiplicative, because every term is multiplicative, and

$$\sigma^{-1}([\bar{\sigma}(x)]) \equiv x \mod \pi_L.$$

So this map f has to be the Teichmüller lift by uniqueness. □

Proposition. Let L/K be a finite Galois extension of local fields, and v_L the normalized valuation of L. Let π_L be the uniformizer of L. Then $G_{s+1}(L/K)$ is a normal subgroup of $G_s(L/K)$ for $s \in \mathbb{Z}_{\geq 0}$, and the map

$$\frac{G_s(L/K)}{G_{s+1}(L/K)} \to \frac{U_L^{(s)}}{U_L^{(s+1)}}$$

given by

$$\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L}$$

is a well-defined injective group homomorphism, independent of the choice of π_L.

Proof. We define the map

$$\begin{aligned} \phi : G_s(L/K) &\to \frac{U_L^{(s)}}{U_L^{(s+1)}} \\ \sigma &\mapsto \sigma(\pi_L)/\pi_L. \end{aligned}$$

We want to show that this has kernel $G_{s+1}(L/K)$.

First we show it is well-defined. If $\sigma \in G_s(L/K)$, we know

$$\sigma(\pi_L) = \pi_L + \pi_L^{s+1} x$$

for some $x \in \mathcal{O}_L$. So we know

$$\frac{\sigma(\pi_L)}{\pi_L} = 1 + \pi_L^s x \in U_L^{(s)}.$$

So it has the right image. To see this is independent of the choice of π_L, we let $u \in \mathcal{O}_L^\times$. Then $\sigma(u) = u + \pi_L^{s+1} y$ for some $y \in \mathcal{O}_L$.

Since any other uniformizer must be of the form $\pi_L u$, we can compute

$$\begin{aligned} \frac{\sigma(\pi_L u)}{\pi_L u} &= \frac{(\pi_L + \pi_L^{s+1})(u + \pi_L^{s+1} y)}{\pi_L u} \\ &= (1 + \pi_L^s x)(1 + \pi_L^{s+1} u^{-1} y) \\ &\equiv 1 \pi_L^s x \pmod{U_L^{s+1}}. \end{aligned}$$

So they represent the same element in in $U_L^{(s)}/U_L^{(s+1)}$.

To see this is a group homomorphism, we know

$$\phi(\sigma\tau) = \frac{\sigma(\tau(\pi_L))}{\pi_L} = \frac{\sigma(\tau(\pi_L))}{\tau(\pi_L)} \frac{\tau(\pi_L)}{\pi_L} = \phi(\sigma)\phi(t),$$

using the fact that $\tau(\pi_L)$ is also a uniformizer.

Finally, we have to show that $\ker \phi = G_{s+1}(L/K)$. We write down

$$\ker \phi = \{\sigma \in G_s(L/K) : v_L(\sigma(\pi_L) - \pi_L) \geq s + 2\}.$$

On the other hand, we have

$$G_{s+1}(L/K) = \{\sigma \in G_s(L/K) : v_L(\sigma(z) - z) \geq s + 2 \text{ for all } z \in \mathcal{O}_L\}.$$

So we trivially have $G_{s+1}(L/K) \subseteq \ker \phi$. To show the converse, let $x \in \mathcal{O}_L$ and write

$$x = \sum_{n=0}^{\infty} [x_n] \pi_L^n.$$

Take $\sigma \in \ker \phi \subseteq G_s(L/K) \subseteq I(L/K)$. Then we have

$$\sigma(\pi_L) = \pi_L + \pi_L^{s+2} y, \quad y \in \mathcal{O}_L.$$

Then by the previous lemma, we know

$$\begin{aligned} \sigma(x) - x &= \sum_{n=1}^{\infty} [x_n] \left((\sigma(\pi_L))^n - \pi_L^n\right) \\ &= \sum_{n=1}^{\infty} [x_n] \left((\pi_L + \pi_L^{s+2} y)^n - \pi_L^n\right) \\ &= \pi_L^{s+2}(\text{things}). \end{aligned}$$

So we know $v_L(\sigma(x) - x) \geq s + 2$. □

Corollary. $\operatorname{Gal}(L/K)$ is solvable.

Proof. Note that

$$\bigcap_s G_s(L/K) = \{\mathrm{id}\}.$$

So $(G_s(L/K))_{s \in \mathbb{Z}_{\geq -1}}$ is a subnormal series of $\operatorname{Gal}(L/K)$, and all quotients are abelian, because they embed into $\frac{U_L^{(s)}}{U_L^{(s+1)}} \cong (k_K, +)$ (and $s = -1$ can be checked separately). □

Proposition. $G_1(L/K)$ is always a p-group.

6.2 Multiple extensions

Proposition. Let $M/L/K$ be finite extensions of local fields, and M/K Galois. Then

$$G_s(M/K) \cap \operatorname{Gal}(M/L) = G_s(M/L).$$

Proof. We have

$$G_s(M/K) = \{\sigma \in \operatorname{Gal}(M/L) : v_M(\sigma x - x) \geq s + 1\} = G_s(M/K) \cap \operatorname{Gal}(M/L).$$

□

Theorem (Herbrand's theorem)**.** Let $M/L/K$ be finite extensions of local fields with M/K and L/K Galois. Then there is some function $\eta_{M/L}$ such that

$$G_t(L/K) \cong \frac{G_s(M/K)}{G_s(M/L)}$$

for all s, where $t = \eta_{M/L}(s)$.

Proposition. Let L/K be a finite Galois extension of local fields, and pick $\alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. Then

$$i_{L/K}(\sigma) = v_L(\sigma(\alpha) - \alpha).$$

Proof. Fix a σ. It is clear that $i_{L/K}(\sigma) \leq v_L(\sigma(\alpha) - \alpha)$. Conversely, for any $x \in \mathcal{O}_L$, we can find a polynomial $g \in \mathcal{O}_K[t]$ such that

$$x = g(\alpha) = \sum b_i \alpha^i,$$

where $b_i \in \mathcal{O}_K$. In particular, b_i is fixed by σ.

Then we have

$$\begin{aligned} v_L(\sigma(x) - x) &= v_L(\sigma g(\alpha) - g(\alpha)) \\ &= v_L\left(\sum_{i=1}^n b_i(\sigma(\alpha)^i - \alpha^i)\right) \\ &\geq v_L(\sigma(\alpha) - \alpha), \end{aligned}$$

using the fact that $\sigma(\alpha) - \alpha \mid \sigma(\alpha)^i - \alpha^i$ for all i. So done. □

Proposition. Let $M/L/K$ be a finite extension of local fields, such that M/K and L/K are Galois. Then for $\sigma \in \mathrm{Gal}(L/K)$, we have

$$i_{L/K}(\sigma) = e_{M/L}^{-1} \sum_{\substack{\tau \in \mathrm{Gal}(M/K) \\ \tau|_L = \sigma}} i_{M/K}(\tau).$$

Proof. If $\sigma = 1$, then both sides are infinite by convention, and equality holds. So we assume $\sigma \neq 1$. Let $\mathcal{O}_M = \mathcal{O}_L[\alpha]$ and $\mathcal{O}_L = \mathcal{O}_K[\beta]$, where $\alpha \in \mathcal{O}_M$ and $\beta \in \mathcal{O}_L$. Then we have

$$e_{M/L} i_{L/K}(\sigma) = e_{M/L} v_L(\sigma\beta - \beta) = v_M(\sigma\beta - \beta).$$

Now if $\tau \in \mathrm{Gal}(M/K)$, then

$$i_{M/K}(\tau) = v_M(\tau\alpha - \alpha)$$

Now fix a τ such that $\tau|_L = \sigma$. We set $H = \mathrm{Gal}(M/L)$. Then we have

$$\sum_{\tau' \in \mathrm{Gal}(M/K), \tau'|_L = \sigma} i_{M/K}(\tau') = \sum_{g \in H} v_M(\tau g(\alpha) - \alpha) = v_M\left(\prod_{g \in H} (\tau g(\alpha) - \alpha)\right).$$

We let

$$b = \sigma(\beta) - \beta = \tau(\beta) - \beta$$

and

$$a = \prod_{g \in H} (\tau g(\alpha) - \alpha).$$

We want to prove that $v_M(b) = v_M(a)$. We will prove that $a \mid b$ and $b \mid a$.

We start with a general observation about elements in $\mathcal{O}_L$. Given $z \in \mathcal{O}_L$, we can write

$$z = \sum_{i=1}^{n} z_i \beta^i, \quad z_i \in \mathcal{O}_K.$$

Then we know

$$\tau(z) - z = \sum_{i=1}^{n} z_i (\tau(\beta)^i - \beta^i)$$

is divisible by $\tau(\beta) - \beta = b$.

Now let $F(x) \in \mathcal{O}_L[x]$ be the minimal polynomial of α over L. Then explicitly, we have

$$F(x) = \prod_{g \in H} (x - g(\alpha)).$$

Then we have

$$(\tau F)(x) = \prod_{g \in H} (x - \tau g(\alpha)),$$

where τF is obtained from F by applying τ to all coefficients of F. Then all coefficients of $\tau F - F$ are of the form $\tau(z) - z$ for some $z \in \mathcal{O}_L$. So it is divisible by b. So b divides every value of this polynomial, and in particular

$$b \mid (\tau F - F)(\alpha) = \prod_{g \in H} (\alpha - g(\alpha)) = \pm a,$$

So $b \mid a$.

In other direction, we pick $f \in \mathcal{O}_K[x]$ such that $f(\alpha) = \beta$. Then $f(\alpha) - \beta = 0$. This implies that the polynomial $f(x) - \beta$ divides the minimal polynomial of α in $\mathcal{O}_L[x]$. So we have

$$f(x) - \beta = F(x)h(x)$$

for some $h \in \mathcal{O}_L[x]$.

Then noting that f has coefficients in $\mathcal{O}_K$, we have

$$(f - \tau\beta)(x) = (\tau f - \tau b)(x) = (\tau F)(x)(\tau h)(x).$$

Finally, set $x = \alpha$. Then

$$-b = \beta - \tau\beta = \pm a(\tau h)(\alpha).$$

So $a \mid b$. □

Theorem (Herbrand's theorem)**.** Let $M/L/K$ be a finite extension of local fields with M/K and L/K Galois. We set

$$H = \mathrm{Gal}(M/L), \quad t = \eta_{M/L}(s).$$

Then we have

$$\frac{G_s(M/K)H}{H} = G_t(L/K).$$

By some isomorphism theorem, and the fact that $H \cap G_s(M/K) = G_s(M/L)$, this is equivalent to saying

$$G_t(L/K) \cong \frac{G_s(M/K)}{G_s(M/L)}.$$

Proof. Let $G = \mathrm{Gal}(M/K)$. Fix a $\sigma \in \mathrm{Gal}(L/K)$. We let $\tau \in \mathrm{Gal}(M/K)$ be an extension of σ to M that maximizes $i_{M/K}$, i.e.

$$i_{M/K}(\tau) \geq i_{M/K}(\tau g)$$

for all $g \in H$. This is possible since H is finite.

We claim that

$$i_{L/K}(\sigma) - 1 = \eta_{M/L}(i_{M/K}(\tau) - 1).$$

If this were true, then we would have

$$\begin{aligned} \sigma \in \frac{G_s(M/K)H}{H} &\Leftrightarrow \tau \in G_s(M/K) \\ &\Leftrightarrow i_{M/K}(\tau) - 1 \geq s \end{aligned}$$

Since $\eta_{M/L}$ is strictly increasing, we have

$$\begin{aligned} &\Leftrightarrow \eta_{M/L}(i_{M/K}(\tau) - 1) \geq \eta_{M/L}(s) = t \\ &\Leftrightarrow i_{L/K}(\sigma) - 1 \geq t \\ &\Leftrightarrow \sigma \in G_t(L/K), \end{aligned}$$

and we are done.

To prove the claim, we now use our known expressions for $i_{L/K}(\sigma)$ and $\eta_{M/L}(i_{M/K}(\tau) - 1)$ to rewrite it as

$$e_{M/L}^{-1} \sum_{g \in H} i_{M/K}(\tau g) = e_{M/L}^{-1} \sum_{g \in H} \min(i_{M/L}(g), i_{M/K}(\tau)).$$

We then make the *stronger* claim

$$i_{M/K}(\tau g) = \min(i_{M/L}(g), i_{M/K}(\tau)).$$

We first note that

$$\begin{aligned} i_{M/K}(\tau g) &= v_M(\tau g(\alpha) - \alpha) \\ &= v_M(\tau g(\alpha) - g(\alpha) + g(\alpha) - \alpha) \\ &\geq \min(v_M(\tau g(\alpha) - g(\alpha)), v_M(g(\alpha) - \alpha)) \\ &= \min(i_{M/K}(\tau), i_{M/K}(g)) \end{aligned}$$

We cannot conclude our (stronger) claim yet, since we have a $\geq$ in the middle. We now have to split into two cases.

(i) If $i_{M/K}(g) \geq i_{M/K}(\tau)$, then the above shows that $i_{M/K}(\tau g) \geq i_{M/K}(\tau)$. But we also know that it is bounded above by m. So $i_{M/K}(\tau g) = i_{M/K}(\tau)$. So our claim holds.

(ii) If $i_{M/K}(g) < i_{M/K}(\tau)$, then the above inequality is in fact an equality as the two terms have different valuations. So our claim also holds.

So done. □

Proposition. Write $G = \mathrm{Gal}(L/K)$. Then

$$\eta_{L/K}(s) = \int_0^s \frac{\mathrm{d}x}{(G_0(L/K) : G_x(L/K))}.$$

When $-1 \leq x < 0$, our convention is that

$$\frac{1}{(G_0(L/K) : G_x(L/K))} = (G_x(L/K) : G_0(L/K)),$$

which is just equal to 1 when $-1 < x < 0$. So

$$\eta_{L/K}(s) = s \text{ if } -1 \leq s \leq 0.$$

Proof. We denote the RHS by $\theta(s)$. It is clear that both $\eta_{L/K}(s)$ and $\theta(s)$ are piecewise linear and the break points are integers (since $i_{L/K}(\sigma)$ is always an integer). So to see they are the same, we see that they agree at a point, and that they have equal derivatives. We have

$$\eta_{L/K}(0) = \frac{|\{\sigma \in G : i_{L/K}(\sigma) \geq 1\}|}{e_{L/K}} - 1 = 0 = \theta(0),$$

since the numerator is the size of the inertia group.

If $s \in [-1, \infty) \setminus \mathbb{Z}$, then

$$\begin{aligned} \eta'_{L/K}(s) &= e_{L/K}^{-1}(|\{\sigma \in G : i_{L/K}(\sigma) \geq s + 1\}|) \\ &= \frac{|G_s(L/K)|}{|G_0(L/K)|} \\ &= \frac{1}{(G_0(L/K) : G_s(L/K))} \\ &= \theta'(s). \end{aligned}$$

So done. □

Lemma. Let $M/L/K$ be a finite extension of local fields, and M/K and L/K be Galois. Then

$$\eta_{M/K} = \eta_{L/K} \circ \eta_{M/L}.$$

Hence

$$\psi_{M/K} = \psi_{M/L} \circ \psi_{L/K}.$$

Proof. Let $s \in [-1, \infty)$, and let $t = \eta_{M/L}(s)$, and $H = \mathrm{Gal}(M/L)$. By Herbrand's theorem, we know

$$G_t(L/K) \cong \frac{G_s(M/K)H}{H} \cong \frac{G_s(M/K)}{H \cap G_s(M/K)} = \frac{G_s(M/K)}{G_s(M/L)}.$$

Thus by multiplicativity of the inertia degree, we have

$$\frac{|G_s(M/K)|}{e_{M/K}} = \frac{|G_t(L/K)|}{e_{L/K}} \frac{|G_s(M/L)|}{e_{M/L}}.$$

By the fundamental theorem of calculus, we know that whenever the derivatives make sense, we have

$$\eta'_{M/K}(s) = \frac{|G_s(M/K)|}{e_{M/K}}.$$

So putting this in, we know

$$\eta'_{M/K}(s) = \eta'_{L/K}(t)\eta'_{M/L}(s) = (\eta_{L/K} \circ \eta_{M/L})'(s).$$

Since $\eta_{M/K}$ and $\eta_{L/K} \circ \eta_{M/L}$ agree at 0 (they both take value 0), we know that the functions must agree everywhere. So done. □

Corollary. Let $M/L/K$ be finite Galois extensions of local fields, and $H = \mathrm{Gal}(M/L)$. Let $t \in [-1, \infty)$. Then

$$\frac{G^t(M/K)H}{H} = G^t(L/K).$$

Proof. Put $s = \eta_{L/K}(t)$. Then by Herbrand's theorem, we have

$$\begin{aligned}\frac{G^t(M/K)H}{H} &= \frac{G_{\psi_{M/K}(t)}(M/K)H}{H}\\ &\cong G_{\eta_{M/L}(\psi_{M/K}(t))}(L/K)\\ &= G_s(L/K)\\ &= G^t(L/K).\end{aligned}$$

□

7 Local class field theory

7.1 Infinite Galois theory

Proposition. Let M/K be a Galois extension. Then $\mathrm{Gal}(M/K)$ is compact and Hausdorff, and if $U \subseteq \mathrm{Gal}(M/K)$ is an open subset such that $1 \in U$, then there is an open normal subgroup $N \subseteq \mathrm{Gal}(M/K)$ such that $N \subseteq U$.

Proof. We will not prove the first part.

For the last part, note that by definition, there is a finite subextension of M/K such that $\mathrm{Gal}(M/L) \subseteq U$. We then let L' be the Galois closure of L over K. Then $\mathrm{Gal}(M/L') \subseteq \mathrm{Gal}(M/L) \subseteq U$, and $\mathrm{Gal}(M/L')$ is open and normal. □

Proposition. Let M/K be a Galois extension. The set I of finite Galois subextensions L/K is a directed system under inclusion. If $L, L' \in I$ and $L \subseteq L'$, then we have a restriction map

$$\cdot|_L^{L'} : \mathrm{Gal}(L'/K) \to \mathrm{Gal}(L/K).$$

Then $(\mathrm{Gal}(L/K), \cdot|_L^{L'})$ is an inverse system, and the map

$$\begin{aligned}\mathrm{Gal}(M/K) &\to \varprojlim_{i\in I} \mathrm{Gal}(L/K)\\ \sigma &\mapsto (\sigma|_L)_{i \in I}\end{aligned}$$

is an isomorphism of topological groups.

Theorem (Fundamental theorem of Galois theory)**.** Let M/K be a Galois extension. Then the map $L \mapsto \mathrm{Gal}(M/L)$ defines a bijection between subextensions L/K of M/K and closed subgroups of $\mathrm{Gal}(M/K)$, with inverse given by sending $H \mapsto M^H$, the fixed field of H.

Moreover, L/K is finite if and only if $\mathrm{Gal}(M/L)$ is open, and L/K is Galois iff $\mathrm{Gal}(M/L)$ is normal, and then

$$\frac{\mathrm{Gal}(L/K)}{\mathrm{Gal}(M/L)} \to \mathrm{Gal}(L/K)$$

is an isomorphism of topological groups.

Proof. This follows easily from the fundamental theorem for finite field extensions. We will only show that $\mathrm{Gal}(M/L)$ is closed and leave the rest as an exercise. We can write

$$L = \bigcup_{\substack{L' \subseteq L\\ L'/K \text{ finite}}} L'.$$

Then we have

$$\mathrm{Gal}(M/L) = \bigcap_{\substack{L' \subseteq L\\ L'/K \text{ finite}}} \mathrm{Gal}(M/L'),$$

and each $\mathrm{Gal}(M/L')$ is open, hence closed. So the whole thing is closed. □

7.2 Unramified extensions and Weil group

Proposition. Let M/K be an unramified extension of local fields. Then M/K is Galois, and

$$\mathrm{Gal}(M/K) \cong \mathrm{Gal}(k_M/k_K)$$

via the reduction map.

Proof. Every finite subextension of M/K is unramified, so in particular is Galois. So M/K is Galois (because normality and separability is checked for each element). Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(M/K) \arrow[r, "\text{reduction}"] \arrow[d, "\sim"] & \mathrm{Gal}(k_M/k_K) \arrow[d, "\sim"] \\
\varprojlim_{L/K} \mathrm{Gal}(L/K) \arrow[r, "\text{reduction}", "\sim"'] & \varprojlim_{L/K} \mathrm{Gal}(k_L/k_K)
\end{tikzcd}$$

The left hand map is an isomorphism by (infinite) Galois theory, and since all finite subextensions of k_M/k_K are of the form k_L/k_K by our finite theory, we know the right-hand map is an isomorphism. The bottom map is an isomorphism since it is an isomorphism in each component. So the top map must be an isomorphism. □

Proposition. Let K be a local field, and M/K Galois. Then $W(M/K)$ is dense in $\mathrm{Gal}(M/K)$. Equivalently, for any finite Galois subextension L/K of M/K, the restriction map $W(M/K) \to \mathrm{Gal}(L/K)$ is surjective.

If L/K is a finite subextension of M/K, then

$$W(M/L) = W(M/K) \cap \mathrm{Gal}(M/L).$$

If L/K is also Galois, then

$$\frac{W(M/K)}{W(M/L)} \cong \mathrm{Gal}(L/K)$$

via restriction.

Proof. We first prove density. To see that density is equivalent to $W(M/K) \to \mathrm{Gal}(L/K)$ being surjective for all finite subextension L/K, note that by the topology on $\mathrm{Gal}(M/K)$, we know density is equivalent to saying that $W(M/K)$ hits every coset of $\mathrm{Gal}(M/L)$, which means that $W(M/K) \to \mathrm{Gal}(L/K)$ is surjective.

Let L/K be a subextension. We let $T = T_{M/K}$. Then $T_{L/K} = T \cap L$. Then we have a diagram

$$\begin{tikzcd}
\mathrm{Gal}(M/T) \arrow[r] \arrow[d, two heads] & W(M/K) \arrow[r] \arrow[d] & \mathrm{Frob}_{T/K}^{\mathbb{Z}} \arrow[d, two heads] \\
\mathrm{Gal}(L/T \cap L) \arrow[r] & \mathrm{Gal}(L/K) \arrow[r] & \mathrm{Gal}(T \cap L/K)
\end{tikzcd}$$

Here the surjectivity of the left vertical arrow comes from field theory, and the right hand vertical map is surjective because $T \cap L/K$ is finite and hence the

Galois group is generated by the Frobenius. Since the top and bottom rows are short exact sequences (top by definition, bottom by Galois theory), by diagram chasing (half of the five lemma), we get surjectivity in the middle.

To prove the second part, we again let L/K be a finite subextension. Then $L \cdot T_{M/K} \subseteq T_{M/L}$. We then have maps

$$\begin{tikzcd}
\mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K} \arrow[r, hook] & \mathrm{Gal}(T_{M/K}/K) \arrow[r, "\cong"] & \mathrm{Gal}(k_M/k_K) \\
\mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L} \arrow[r, hook] \arrow[u] & \mathrm{Gal}(T_{M/L}/L) \arrow[r, "\cong"] \arrow[u, hook] & \mathrm{Gal}(k_M/k_L) \arrow[u, hook]
\end{tikzcd}$$

So the left hand vertical map is an inclusion. So we know

$$\mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L} = \mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K} \cap \mathrm{Gal}(T_{M/L}/L).$$

Now if $\sigma \in \mathrm{Gal}(M/L)$, then we have

$$\begin{aligned}
\sigma \in W(M/L) &\Leftrightarrow \sigma|_{T_{M/L}/L} \in \mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L}\\
&\Leftrightarrow \sigma|_{T_{M/K}/K} \in \mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K}\\
&\Leftrightarrow \sigma \in W(M/K).
\end{aligned}$$

So this gives the second part.

Now L/K is Galois as well. Then $\mathrm{Gal}(M/L)$ is normal in $\mathrm{Gal}(M/K)$. So $W(M/L)$ is normal in $W(M/K)$ by the second part. Then we can compute

$$\begin{aligned}
\frac{W(M/K)}{W(M/L)} &= \frac{W(M/K)}{W(M/K) \cap \mathrm{Gal}(M/L)}\\
&\cong \frac{W(M/K)\,\mathrm{Gal}(M/L)}{\mathrm{Gal}(M/L)}\\
&= \frac{\mathrm{Gal}(M/K)}{\mathrm{Gal}(M/L)}\\
&\cong \mathrm{Gal}(L/K).
\end{aligned}$$

The only non-trivial part in this chain is the assertion that $W(M/K)\,\mathrm{Gal}(M/L) = \mathrm{Gal}(M/K)$, i.e. that $W(M/K)$ hits every coset of $\mathrm{Gal}(M/L)$, which is what density tells us. □

7.3 Main theorems of local class field theory

Theorem (Local Artin reciprocity)**.** There exists a unique topological isomorphism

$$\mathrm{Art}_K : K^\times \to W(K^{\mathrm{ab}}/K)$$

characterized by the properties

(i) $\mathrm{Art}_K(\pi_K)|_{K^{\mathrm{ur}}} = \mathrm{Frob}_K$, where π_K is *any* uniformizer.

(ii) We have
$$\mathrm{Art}_K(N_{L/K}(x))|_L = \mathrm{id}_L$$
for all L/K finite abelian and $x \in L^\times$.

Moreover, if M/K is finite, then for all $x \in M^\times$, we know $\mathrm{Art}_M(x)$ is an automorphism of M^{ab}/M, and restricts to an automorphisms of K^{ab}/K. Then we have
$$\mathrm{Art}_M(x)|_K^{K^{\mathrm{ab}}} = \mathrm{Art}_K(N_{M/K}(x)).$$
Moreover, Art_K induces an isomorphism
$$\frac{K^\times}{N_{M/K}(M^\times)} \to \mathrm{Gal}\left(\frac{M \cap K^{\mathrm{ab}}}{K}\right).$$

Corollary. Let L/K be finite. Then $N(L/K) = N((L \cap K^{ab})/K)$, and
$$(K^\times : N(L/K)) \leq [L : K]$$
with equality iff L/K is abelian.

Proof. To see this, we let $M = L \cap K^{\mathrm{ab}}$. Applying the isomorphism twice gives
$$\frac{K^\times}{N(L/K)} \cong \mathrm{Gal}(M/K) \cong \frac{K^\times}{N(M/K)}.$$
Since $N(L/K) \subseteq N(M/K)$, and $[L : K] \geq [M : K] = |\mathrm{Gal}(M/K)|$, we are done. □

Theorem. Let K be a local field. Then there is an isomorphism of posets
$$\begin{aligned}\left\{\begin{array}{c}\text{open finite index}\\ \text{subgroups of } K^\times\end{array}\right\} &\longleftrightarrow \left\{\begin{array}{c}\text{finite abelian}\\ \text{extensions of } L/K\end{array}\right\}\\ H &\longmapsto (K^{\mathrm{ab}})^{\mathrm{Art}_K(H)}\\ N(L/K) &\longleftarrow\!\shortmid L/K\end{aligned}.$$

In particular, for L/K and M/K finite abelian extensions, we have
$$\begin{aligned}N(LM/K) &= N(L/K) \cap N(M/K),\\ N(L \cap M/K) &= N(L/K)N(M/K).\end{aligned}$$

Theorem. Let L/K be a finite extension, and M/K abelian. Then $N(L/K) \subseteq N(M/K)$ iff $M \subseteq L$.

Proof. By the previous theorem, we may wlog L/K abelian by replacing with $L \cap K^{\mathrm{ab}}$. The $\Leftarrow$ direction is clear by the last part of Artin reciprocity.

For the other direction, we assume that we have $N(L/K) \subseteq N(M/K)$, and let $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/L)$. We want to show that $\sigma|_M = \mathrm{id}_M$. This would then imply that M is a subfield of L by Galois theory.

We know $W(K^{\mathrm{ab}}/L)$ is dense in $\mathrm{Gal}(K^{\mathrm{ab}}/L)$. So it suffices to show this for $\sigma \in W(K^{\mathrm{ab}}/L)$. Then we have
$$W(K^{\mathrm{ab}}/L) \cong \mathrm{Art}_K(N(L/K)) \subseteq \mathrm{Art}_K(N(M/K)).$$
So we can find $x \in M^\times$ such that $\sigma = \mathrm{Art}_K(N_{M/K}(x))$. So $\sigma|_M = \mathrm{id}_M$ by local Artin reciprocity. □

8 Lubin–Tate theory

8.1 Motivating example

Lemma. Let L/K be a finite abelian extension. Then we have

$$e_{L/K} = (\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)).$$

Proof. Pick $x \in L^\times$, and w the valuation on L extending v_K, and $n = [L : K]$. Then by construction of w, we know

$$v_K(N_{L/K}(x)) = nw(x) = f_{L/K} v_L(x).$$

So we have a surjection

$$\begin{tikzcd} \frac{K^\times}{N(L/K)} \arrow[r, two heads, "v_K"] & \frac{\mathbb{Z}}{f_{L/K}\mathbb{Z}} \end{tikzcd} .$$

The kernel of this map is equal to

$$\frac{\mathcal{O}_K^\times N(L/K)}{N(L/K)} \cong \frac{\mathcal{O}_K^\times}{\mathcal{O}_K^\times \cap N(L/K)} = \frac{\mathcal{O}_K^\times}{N_{L/K}(\mathcal{O}_L^\times)}.$$

So by local class field theory, we know

$$n = (K^\times : N(L/K)) = f_{L/K}(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)),$$

and this implies what we want. □

Corollary. Let L/K be a finite abelian extension. Then L/K is unramified if and only if $N_{L/K}(\mathcal{O}_L^\times) = \mathcal{O}_K^\times$.

Lemma. Let K be a local field, and let L_m/K be the extension corresponding to $\langle \pi_K^m \rangle \times \mathcal{O}_K$. Let

$$L = \bigcup_m L_m.$$

Then we have

$$K^{\mathrm{ab}} = K^{\mathrm{ur}} L,$$

Lemma. We have isomorphisms

$$\begin{aligned} W(K^{\mathrm{ab}}/K) &\cong W(K^{\mathrm{ur}}L/K) \\ &\cong W(K^{\mathrm{ur}}/K) \times \mathrm{Gal}(L/K) \\ &\cong \mathrm{Frob}_K^{\mathbb{Z}} \times \mathrm{Gal}(L/K) \end{aligned}$$

Proof. The first isomorphism follows from the previous lemma. The second follows from the fact that $K^{\mathrm{ur}} \cap L = K$ as L is totally ramified. The last isomorphism follows from the fact that $T_{K^{\mathrm{ur}}/K} = K^{\mathrm{ur}}$ trivially, and then by definition $W(K^{\mathrm{ur}/K}) \cong \mathrm{Frob}_K^{\mathbb{Z}}$. □

Theorem (Local Kronecker-Weber theorem)**.**

$$\begin{aligned}\mathbb{Q}_p^{\mathrm{ab}} &= \bigcup_{n\in\mathbb{Z}_{\geq 1}} \mathbb{Q}_p(\zeta_n),\\ \mathbb{Q}_p^{\mathrm{ur}} &= \bigcup_{\substack{n\in\mathbb{Z}_{\geq 1}\\ (n,p)=1}} \mathbb{Q}_p(\zeta_n).\end{aligned}$$

Not a proof. We will comment on the proof of the generalized version later. □

Theorem. We have

$$G^s(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) = \mathrm{Art}_{\mathbb{Q}_p}(1 + p^k\mathbb{Z}_p) = \mathrm{Art}_{\mathbb{Q}_p}(U^{(k)}),$$

where k is chosen such that $k - 1 < s \leq k$, $k \in \mathbb{Z}_{\geq 0}$.

Corollary. If $L/\mathbb{Q}_p$ is a finite abelian extension, then

$$G^s(L/\mathbb{Q}_p) = \mathrm{Art}_{\mathbb{Q}_p}\left(\frac{N(L/\mathbb{Q}_p)(1 + p^n\mathbb{Z}_p)}{N(L/\mathbb{Q}_p)}\right),$$

where $n - 1 < s \leq n$.

8.2 Formal groups

Lemma. Let R be a ring and F a formal group over R. Then

$$F(X, 0) = X.$$

Also, there exists a power series $i(X) \in X \cdot R[[X]]$ such that

$$F(X, i(X)) = 0.$$

Proof. See example sheet 4. □

Lemma. Let $e_1, e_2 \in \mathcal{E}_\pi$ and take a linear form

$$L(x_1, \cdots, x_n) = \sum_{i=1}^n a_i X_i, \quad a_i \in \mathcal{O}_K.$$

Then there is a unique power series $F(x_1, \cdots, x_n) \in \mathcal{O}_K[[x_1, \cdots, x_n]]$ such that

$$F(x_1, \cdots, x_n) \equiv L(x_1, \cdots, x_n) \mod (x_1, \cdots, x_n)^2,$$

and

$$e_1(F(x_1, \cdots, x_n)) = F(e_2(x_1), e_2(x_2), \cdots, e_2(x_n)).$$

Corollary. Let $e \in \mathcal{E}_\pi$ be a Lubin–Tate series. Then there are unique power series $F_e(X, Y) \in \mathcal{O}_K[[X, Y]]$ such that

$$\begin{aligned}F_e(X, Y) &\equiv X + Y \mod (X + Y)^2\\ e(F_e(X, Y)) &= F_e(e(X), e(Y))\end{aligned}$$

Corollary. Let $e_1, e_2 \in \mathcal{E}_\pi$ be Lubin–Tate series and $a \in \mathcal{O}_K$. Then there exists a unique power series $[a]_{e_1,e_2} \in \mathcal{O}_K[[X]]$ such that

$$\begin{aligned}[a]_{e_1,e_2}(X) &\equiv aX \mod X^2\\ e_1([a]_{e_1,e_2}(X)) &= [a]_{e_1,e_2}(e_2(X)).\end{aligned}$$

To simplify notation, if $e_1 = e_2 = e$, we just write $[a]_e = [a]_{e,e}$.

Theorem. The Lubin–Tate $\mathcal{O}_K$ modules for π are precisely the series F_e for $e \in \mathcal{E}_\pi$ with formal $\mathcal{O}_K$-module structure given by

$$a \mapsto [a]_e.$$

Moreover, if $e_1, e_2 \in \mathcal{E}_\pi$ and $a \in \mathcal{O}_K$, then $[a]_{e_1,e_2}$ is a homomorphism from $F_{e_2} \to F_{e_1}$.

If $a \in \mathcal{O}_K^\times$, then it is an isomorphism with inverse $[a^{-1}]_{e_2,e_1}$.

Proof sketch. If F is a Lubin–Tate $\mathcal{O}_K$-module for π, then $e = [\pi]_F \in \mathcal{E}_\pi$ by definition, and F satisfies the properties that characterize the series F_e. So $F = F_e$ by uniqueness.

For the remaining parts, one has to verify the following for all $e, e_1, e_2, e_3 \in \mathcal{E}_\pi$ and $a, b \in \mathcal{O}_K$.

(i) $F_e(X, Y) = F_e(Y, X)$.

(ii) $F_e(X, F_e(Y, Z)) = F_e(F_e(X, Y), Z)$.

(iii) $[a]_{e_1,e_2}(F_e(X, Y)) = F_{e_1}([a]_{e_1,e_2}(X), [a]_{e_1,e_2}(Y))$.

(iv) $[ab]_{e_1,e_3}(X) = [a]_{e_1,e_2}([b]_{e_2,e_3}(X))$.

(v) $[a + b]_{e_1,e_2}(X) = [a]_{e_1,e_2}(X) + [b]_{e_1,e_2}(X)$.

(vi) $[\pi]_e(X) = e(X)$.

The proof is just repeating the word "uniqueness" ten times. □

8.3 Lubin–Tate extensions

Proposition. If F is a formal $\mathcal{O}_K$-module, then $\bar{\mathfrak{m}}$ becomes a (genuine) $\mathcal{O}_K$ module under the operations $+_F$ and $\cdot$

$$\begin{aligned}x +_F y &= F(x, y)\\ a \cdot x &= [a]_F(x)\end{aligned}$$

for all $x, y \in \bar{\mathfrak{m}}$ and $a \in \mathcal{O}_K$.

We denote this $\bar{\mathfrak{m}}_F$.

Proof. If $x, y \in \bar{\mathfrak{m}}$, then $F(x, y)$ is a series in $K(x, y) \subseteq \bar{K}$. Since $K(x, y)$ is a finite extension, we know it is complete. Since the terms in the sum have absolute value < 1 and $\to 0$, we know it converges to an element in $\mathfrak{m}_{K(x,y)} \subseteq \bar{\mathfrak{m}}$. The rest then essentially follows from definition. □

Lemma. Let $e(X) = X^q + \pi X$. We let

$$f_n(X) = \underbrace{(e \circ \cdots \circ e)}_{n \text{ times}}(X).$$

Then f_n has no repeated roots. Here we take f_0 to be the identity function.

Proof. Let $x \in \bar{K}$. We claim that if $|f_i(x)| < 1$ for $i = 0, \cdots, n-1$, then $f_n'(X) \neq 0$.

We proceed by induction on n.

(i) When $n = 1$, we assume $|x| < 1$. Then

$$f_1'(x) = e'(x) = qx^{q-1} + \pi = \pi\left(1 + \frac{q}{\pi}x^{q-1}\right) \neq 0,$$

since we know $\frac{q}{\pi}$ has absolute value ≤ 1 (q vanishes in k_K, so q/π lives in $\mathcal{O}_K$), and x^{q-1} has absolute value < 1.

(ii) in the induction step, we have

$$f_{n+1}'(x) = (qf_n(x)^{q-1} + \pi)f_n'(x) = \pi\left(1 + \frac{q}{\pi}f_n(x)^{q-1}\right)f_n'(x).$$

By induction hypothesis, we know $f_n'(x) \neq 0$, and by assumption $|f_n(x)| < 1$. So the same argument works.

We now prove the lemma. We assume that $f_n(x) = 0$. We want to show that $|f_i(x)| < 1$ for all $i = 0, \cdots, n-1$. By induction, we have

$$f_n(x) = x^{q^n} + \pi g_n(x)$$

for some $g_n(x) \in \mathcal{O}_K[x]$. It follows that if $f_n(x) = 0$, then $|x| < 1$. So $|f_i(x)| < 1$ for all i. So $f_n'(x) \neq 0$. □

Proposition. $F(n)$ is a free $\mathcal{O}_K/\pi^n\mathcal{O}_K$ module of rank 1. In particular, it has q^n elements.

Proof. By definition, we know

$$\pi^n \cdot F(n) = 0.$$

So $F(n)$ is indeed an $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module.

To prove that it is free of rank 1, we note that all Lubin–Tate modules for π are isomorphic. This implies that all the honest $\mathcal{O}_K$ modules $F(n)$ are isomorphic. We choose $F = F_e$, where $e = X^q + \pi X$. Then $F(n)$ consists of the roots of the polynomial $f_n = e^n(X)$, which is of degree q^n and has no repeated roots. So $|F(n)| = q^n$. To show that it is actually the right thing, if $\lambda_n \in F(n) \setminus F(n-1)$, then we have a homomorphism

$$\mathcal{O}_K \to F(n)$$

given by $A \mapsto a \cdot \lambda_n$. Its kernel is $\pi^n\mathcal{O}_K$ by our choice of λ_n. By counting, we get an $\mathcal{O}_K$-module isomorphism

$$\frac{\mathcal{O}_K}{\pi^n\mathcal{O}_K} \to F(n)$$

as desired. □

Corollary. We have isomorphisms

$$\begin{aligned}\frac{\mathcal{O}_K}{\pi^n \mathcal{O}_K} &\cong \mathrm{End}_{\mathcal{O}_K}(F(n))\\ \frac{U_K}{U_K^{(n)}} &\cong \mathrm{Aut}_{\mathcal{O}_K}(F(n)).\end{aligned}$$

Theorem. L_n/K is a totally ramified abelian extension of degree $q^{n-1}(q-1)$ with Galois group

$$\mathrm{Gal}(L_n/K) \cong \mathrm{Aut}_{\mathcal{O}_K}(F(n)) \cong \frac{U_K}{U_K^{(n)}}.$$

Explicitly, for any $\sigma \in \mathrm{Gal}(L_n/K)$, there is a unique $u \in U_K/U_K^{(n)}$ such that

$$\sigma(\lambda) = [u]_F(\lambda)$$

for all $\lambda \in F(n)$. Under this isomorphism, for $m \geq n$, we have

$$\mathrm{Gal}(L_m/L_n) \cong \frac{U_K^{(n)}}{U_K^{(m)}}.$$

Moreover, if $F = F_e$, where

$$e(X) = X^q + \pi(a_{q-1}\pi^{q-1} + \cdots + a_2 X^2) + \pi X,$$

and $\lambda_n \in F(n) \setminus F(n-1)$, then λ_n is a uniformizer of L_n and

$$\phi_n(X) = \frac{e^n(X)}{e^{n-1}(X)} = X^{q^{n-1}(q-1)} + \cdots + \pi$$

is the minimal polynomial of λ_n. In particular,

$$N_{L_n/K}(-\lambda_n) = \pi.$$

Proof. Consider a Lubin–Tate polynomial

$$e(X) = x^q + \pi(a_{q-1}X^{q-1} + \cdots + a_2 X^2) + \pi X.$$

We set $F = F_e$. Then

$$\phi_n(X) = \frac{e^n(X)}{e^{n-1}(X)} = (e^{n-1}(X))^{q-1} + \pi(a_{q_1}e^{n-1}(X)^{q-2} + \cdots + a_2 e^{n-1}(X)) + \pi$$

is an Eisenstein polynomial of degree $q^{n-1}(q-1)$ by starting at it long enough. So if $\lambda_n \in F(n) \setminus F(n-1)$, then λ_n is a root of $\phi_n(x)$, so $K(\lambda_n)/K$ is totally ramified of degree $q^{n-1}(q-1)$, and λ_n is a uniformizer, and

$$N_{K(\lambda_n)/K}(-\lambda_n) = \pi$$

as the norm is just the constant coefficient of the minimal polynomial.

Now let $\sigma \in \mathrm{Gal}(L_n/K)$. Then σ induces a permutation of $F(n)$, as these are the roots of $e^n(X)$, which is in fact $\mathcal{O}_K$-linear, i.e.

$$\sigma(x) +_F \sigma(y) = F(\sigma(x), \sigma(y)) = \sigma(F(x, y)) = \sigma(x +_F y)$$
$$\sigma(a \cdot x) = \sigma([a]_F(x)) = [a]_F(\sigma(x)) = a \cdot \sigma(x)$$

for all $x, y \in \mathfrak{m}_{L_n}$ and $a \in \mathcal{O}_K$.

So we have an injection of groups

$$\mathrm{Gal}(L_n/K) \hookrightarrow \mathrm{Aut}_{\mathcal{O}_K}(F(n)) = \frac{U_K}{U_K^{(n)}}$$

But we know

$$\left|\frac{U_K}{U_K^{(n)}}\right| = q^{n-1}(q-1) = [K(\lambda_n) : K] \leq [L_n : K] = |\mathrm{Gal}(L_n/K)|.$$

So we must have equality throughout, the above map is an isomorphism, and $K(\lambda_n) = L_n$.

It is clear from the construction of the isomorphism that for $m \geq n$, the diagram

$$\begin{tikzcd} \mathrm{Gal}(L_m/K) \arrow[r, "\sim"] \arrow[d, "\text{restriction}"] & U_K/U_K^{(m)} \arrow[d, "\text{quotient}"] \\ \mathrm{Gal}(L_n/K) \arrow[r, "\sim"] & U_K/U_K^{(n)} \end{tikzcd}$$

commutes. So the isomorphism

$$\mathrm{Gal}(L_m/L_n) \cong \frac{U_K^{(m)}}{U_K^{(n)}}$$

follows by looking at the kernels. □

Theorem (Generalized local Kronecker-Weber theorem)**.** We have

$$K^{\mathrm{ab}} = K^{\mathrm{ur}} L_\infty$$

(for any π).

Comments on the proof. One can prove this from the *Hasse-Arf theorem*, which states that in an abelian extension, the jumps in the upper ramification groups occur only at integer values. This, together with the calculation of ramification groups done later, easily implies the theorem. Essentially, L_∞ maxed out all possible jumps of the upper ramification groups. However, the Hasse-Arf theorem is difficult to prove.

Another approach is to prove the existence of the Artin map using other techniques (e.g. Galois cohomology). Consideration of the norm group (cf. the next theorem) then implies the theorem. The content of this section then becomes an explicit construction of a certain family of abelian extensions. □

Theorem. We have

$$N(L_n/K) = \langle \pi \rangle \times U_k^{(n)}.$$

Comments on the proof. This can be done by defining *Coleman operators*, which are power series representations of the norm. Alternatively, assuming the description of the local Artin map given below and local Artin reciprocity, $U_k^{(n)}$ is in the kernel of $\mathrm{Art}|_{L_n}$, so $\langle \pi \rangle \times U_k^{(n)} \subseteq N(L_n/K)$. The result follows by comparing order. □

Theorem. Let K be a local field. Then we have an isomorphism $\mathrm{Art} : K^\times \to W(K^{\mathrm{ab}}/K)$ given by the composition

$$\begin{tikzcd}
K^\times \arrow[r, dashed, "\mathrm{Art}"] \arrow[d, "\sim"] & W(K^{\mathrm{ab}}/K) \arrow[d, "\sim"] \\
\langle \pi \rangle \times U_K \arrow[r] & \mathrm{Frob}_K^{\mathbb{Z}} \times \mathrm{Gal}(L_\infty/K)
\end{tikzcd}$$

where the bottom map is given by $(\pi^m, u) \mapsto (\mathrm{Frob}_K^m, \sigma_{u^{-1}})$, where

$$\sigma_u(\lambda) = [u]_F(\lambda)$$

for all $\lambda \in \bigcup_{n=1}^\infty F(n)$.

Theorem. We have

$$G_s(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/K) & -1 \leq s \leq 0 \\ \mathrm{Gal}(L_n/L_k) & q^{k-1} - 1 < s \leq q^k - 1,\ 1 \leq k \leq n-1 \\ 1 & s > q^{n-1} \end{cases}$$

Proof. The case for $-1 \leq s \leq 0$ is clear.

For $0 \leq s \leq 1$ (which we may wlog is actually 1), we know that

$$\mathrm{Gal}(L_n/L_k) \cong U_K^{(k)}/U_K^{(n)}$$

under the isomorphism $\mathrm{Gal}(L_n/K) \cong U_K/U_K^{(n)}$. On the other hand, we know $G_1(L_n/K)$ is the Sylow p-subgroup of $\mathrm{Gal}(L_n/K)$. So we must have

$$G_1(L_n/K) \cong U_K^{(1)}/U_K^{(n)}.$$

So we know that $G_1(L_n/K) = \mathrm{Gal}(L_n/L_1)$. Thus we know that $G_s(L_n/K) = \mathrm{Gal}(L_n/K)$ for $0 < s \leq 1$.

We now let $\sigma = \sigma_u \in G_1(L_n/K)$ and $u \in U_K^{(1)}/U_K^{(n)}$. We write

$$u = 1 + \varepsilon \pi^k$$

for some $\varepsilon \in U_K$ and some $k = k(u) \geq 1$. Since σ is not the identity, we know $k < n$. We claim that

$$i_{L_n/K}(\sigma) = v_{L_n}(\sigma(\lambda) - \lambda) = q^k.$$

Indeed, we let $\lambda \in F(n) \setminus F(n-1)$, where F is a choice of Lubin–Tate module for π. Then λ is a uniformizer of L_n and $\mathcal{O}_{L_n} = \mathcal{O}_K[\lambda]$. We can compute

$$\begin{aligned} \sigma_u(\lambda) &= [u]_F(\lambda) \\ &= [1 + \varepsilon\pi^k]_F(\lambda) \\ &= F(\lambda, [\varepsilon\pi^k]_F(\lambda)) \end{aligned}$$

Now we can write

$$[\varepsilon\pi^k]_F(\lambda) = [\varepsilon]_F([\pi^k]_F(\lambda)) \in F(n-k) \setminus F(n-k-1),$$

since $[\varepsilon]_F$ is invertible, and applying $[\pi^{n-k}]_F$ to $[\pi^k]_F(\lambda)$ kills it, but applying $[\pi^{n-k-1}]_F$ gives $[\pi^{n-1}]_F$, which does not kill.

So we know $[\varepsilon\pi^k]_F(\lambda)$ is a uniformizer of L_{n-k}. Since L_n/L_{n-k} is totally ramified of degree q^k, we can find $\varepsilon_0 \in \mathcal{O}_{L_n}^\times$ such that

$$[\varepsilon\pi^k]_F(\lambda) = \varepsilon_0\lambda^{q^k}$$

Recall that $F(X,0) = X$ and $F(0,Y) = Y$. So we can write

$$F(X,Y) = X + Y + XYG(X,Y),$$

where $G(X,Y) \in \mathcal{O}_K[[X,Y]]$. So we have

$$\begin{aligned}\sigma(\lambda) - \lambda &= F(\lambda, [\varepsilon\pi^k]_F(\lambda)) - \lambda\\ &= F(\lambda, \varepsilon_0\lambda^{q^k}) - \lambda\\ &= \lambda + \varepsilon_0\lambda^{q^k} + \varepsilon_0\lambda^{q^k+1}G(\lambda, \varepsilon_0\lambda^{q^k}) - \lambda\\ &= \varepsilon_0\lambda^{q^k} + \varepsilon_0\lambda^{q^k+1}G(\lambda, \varepsilon_0\lambda^{q^k}).\end{aligned}$$

In terms of valuation, the first term is the dominating term, and

$$i_{L_n/K}(\sigma) = v_{L_n}(\sigma(\lambda) - \lambda) = q^k$$

So we know

$$i_{L_n/K}(\sigma_k) \geq s + 1 \Leftrightarrow q^{k(u)} - 1 \geq s.$$

So we know

$$G_s(L_n/K) = \{\sigma_K \in G_1(L_n/K) : q^{k(u)} - 1 \geq s\} = \mathrm{Gal}(L_n/L_k),$$

where $q^{k-1} - 1 < s \leq q^k - 1$ for $k = 1, \cdots, n-1$, and 1 if $s > q^{n-1} = 1$. □

Corollary. We have

$$G^t(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/K) & -1 \leq t \leq 0\\ \mathrm{Gal}(L_n/L_k) & k-1 < t \leq k, \quad k = 1, \cdots, n-1\\ 1 & t > n-1\end{cases}$$

In other words, we have

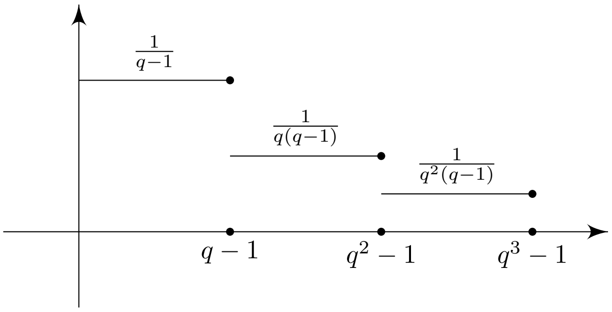
$$G^t(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/L_{\lceil t\rceil}) & -1 \leq t \leq n-1\\ 1 & t > n-1\end{cases},$$

where we set $L_0 = K$.

Proof. We have to compute the integral of

$$\frac{1}{(G_0(L_n/K) : G_x(L_n/K))}.$$

We again plot this out



So by the same computation as the ones we did last time, we find that

$$\eta_{L_n/K}(s) = \begin{cases} s & -1 \leq s \leq 0 \\ (k-1) + \frac{s-(q^{k-1}-1)}{q^{k-1}(q-1)} & q^{k-1} - 1 \leq s \leq q^k - 1, \quad k = 1, \cdots, n-1 \\ (n-1) + \frac{s-(q^{n-1}-1)}{q^{n-1}(q-1)} & s > q^{n-1} - 1. \end{cases}$$

Inverting this, we find that

$$\psi_{L_n/K} = \begin{cases} t & -1 \leq t \leq 0 \\ q^{\lceil t \rceil - 1}(q-1)(t - (\lceil t \rceil - 1)) + q^{\lceil t \rceil - 1} - 1 & 1 < t \leq n-1 \\ q^{n-1}(q-1)(t-(n-1)) + q^{n-1} - 1 & t > n-1 \end{cases}.$$

Then we have

$$G^t(L_n/K) = G_{\psi(L_n/K)(t)}(L_n/K),$$

which gives the desired by the previous theorem. □

Corollary. When $t > -1$, we have

$$G^t(K^{\mathrm{ab}/K}) = \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}}L_{\lceil t \rceil}),$$

and

$$\mathrm{Art}_K^{-1}(G^t(K^{\mathrm{ab}}/K)) = U^{(\lceil t \rceil)}.$$

Proof. Recall the following fact from the examples class: If L/K is finite unramified and M/K is finite totally ramified, then LM/L is totally ramified, and $\mathrm{Gal}(LM/L) \cong \mathrm{Gal}(M/K)$ by restriction, and

$$G^t(LM/K) \cong G^t(M/K).$$

via this isomorphism (for $t > -1$).

Now let K_m/K be the unramified extension of degree m. By the lemma and the previous corollary, we have

$$\begin{aligned} G^t(K_mL_n/K) \cong G^t(L_n/K) &= \begin{cases} \mathrm{Gal}(L_n/L_{\lceil t \rceil}) & -1 < t \leq n \\ 1 & t \geq n \end{cases} \\ &= \begin{cases} \mathrm{Gal}(K_mL_n/K_mL_{\lceil t \rceil}) & -1 < t \leq n \\ 1 & t \geq n \end{cases} \end{aligned}$$

So we have

$$\begin{aligned} G^t(K^{\mathrm{ab}}/K) &= G^t(K^{\mathrm{ur}}L_\infty/K) \\ &= \varprojlim_{m,n} G^t(K_m L_n/K) \\ &= \varprojlim_{\substack{m,n \\ n \geq \lceil t \rceil}} \mathrm{Gal}(K_m L_n / K_m L_{\lceil t \rceil}) \\ &= \mathrm{Gal}(K^{\mathrm{ur}} L_\infty / K^{\mathrm{ur}} L_{\lceil t \rceil}) \\ &= \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}} L_{\lceil t \rceil}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Art}_K^{-1}(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}}L_{\lceil t \rceil})) &= \mathrm{Art}_K^{-1}\left(\varprojlim_{\substack{m,n \\ n \geq \lceil t \rceil}} \mathrm{Gal}(K_m L_n/K_m L_{\lceil t \rceil})\right) \\ &= \varprojlim_{\substack{m,n \\ n \geq \lceil t \rceil}} \mathrm{Art}_K^{-1}\left(\mathrm{Gal}(K_m L_n / K_m L_{\lceil t \rceil})\right) \\ &= \varprojlim_{\substack{m,n \\ n \geq \lceil t \rceil}} \frac{U_K^{(\lceil t \rceil)}}{U_K^{(n)}} \\ &= U^{\lceil t \rceil}. \end{aligned}$$

□

Corollary. Let M/K be a finite abelian extension. Then we have an isomorphism

$$\mathrm{Art}_K : \frac{K^\times}{N(M/K)} \cong \mathrm{Gal}(M/K).$$

Moreover, for $t > -1$, we have

$$G^t(M/K) = \mathrm{Art}_K\left(\frac{U_K^{(\lceil t \rceil)} N(M/K)}{N(M/K)}\right).$$

Proof. We have

$$G^t(M/K) = \frac{G^t(K^{\mathrm{ab}}/K) G(K^{\mathrm{ab}}/M)}{G(K^{\mathrm{ab}}/M)} = \mathrm{Art}\left(\frac{U_K^{(\lceil t \rceil)} N(M/K)}{N(M/K)}\right).$$

□