Part III — Extremal Graph Theory
Theorems

Based on lectures by A. G. Thomason
Notes taken by Dexter Chua

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Turán's theorem, giving the maximum size of a graph that contains no complete r-vertex subgraph, is an example of an extremal graph theorem. Extremal graph theory is an umbrella title for the study of how graph and hypergraph properties depend on the values of parameters. This course builds on the material introduced in the Part II Graph Theory course, which includes Turán's theorem and also the Erdös–Stone theorem.

The first few lectures will cover the Erdös–Stone theorem and stability. Then we shall treat Szemerédi's Regularity Lemma, with some applications, such as to hereditary properties. Subsequent material, depending on available time, might include: hypergraph extensions, the flag algebra method of Razborov, graph containers and applications.

Pre-requisites

A knowledge of the basic concepts, techniques and results of graph theory, as afforded by the Part II Graph Theory course, will be assumed. This includes Turán's theorem, Ramsey's theorem, Hall's theorem and so on, together with applications of elementary probability.

Contents

1 The Erdös–Stone theorem

Theorem (Turán's theorem)**.** If G is a graph with $|G| = n$, $e(G) \geq t_r(n)$ and $G \not\supseteq K_{r+1}$. Then $G = T_r(n)$.

Theorem (Erdös–Stone, 1946)**.** Let $r \geq 1$ be an integer and $\varepsilon > 0$. Then there exists $d = d(r, \varepsilon)$ and $n_0 = n_0(r, \varepsilon)$ such that if $|G| = n \geq n_0$ and

$$e(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2},$$

then $G \supseteq K_{r+1}(t)$, where $t = \lfloor d \log n \rfloor$.

Lemma. Let $c, \varepsilon > 0$. Then there exists $n_1 = n_1(c, \varepsilon)$ such that if $|G| = n \geq n_1$ and $e(G) \geq (c + \varepsilon)\binom{n}{2}$, then G has a subgraph H where $\delta(H) \geq c|H|$ and $|H| \geq \sqrt{\varepsilon} n$.

Lemma. Let $r \geq 1$ be an integer and $\varepsilon > 0$. Then there exists a $d_1 = d_1(r, \varepsilon)$ and $n_2 = n_2(r, \varepsilon)$ such that if $|G| = n \geq n_2$ and

$$\delta(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right) n,$$

then $G \supseteq K_{r+1}(t)$, where $t = \lfloor d_1 \log n \rfloor$.

Theorem (Erdös–Simonovits)**.** Let F be a fixed graph with chromatic number $\chi(F) = r + 1$. Then

$$\lim_{n \to \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}} = 1 - \frac{1}{r}.$$

Theorem. Given $r \in \mathbb{N}$, there exists $\varepsilon_r > 0$ such that if $0 < \varepsilon < \varepsilon_r$, then there exists $n_3(r, \varepsilon)$ so that if $n > n_3$, there exists a graph G of order n such that

$$e(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$$

but $K_{r+1}(t) \not\subseteq G$, where

$$t = \left\lceil \frac{3 \log n}{\log 1/\varepsilon} \right\rceil.$$

2 Stability

Theorem. Let $t, r \geq 2$ be fixed, and suppose $|G| = n$ and $G \not\supseteq K_{r+1}(t)$. If

$$e(G) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

Then

(i) There exists $T_r(n)$ on $V(G)$ with $|E(G)\Delta E(T_r(n))| = o(n^2)$.

(ii) G contains an r-partite subgraph with $\left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$ edges.

(iii) G contains an r-partite subgraph with minimum degree $\left(1 - \frac{1}{r} + o(1)\right)n$.

Corollary. Let $\chi(F) = r + 1$, and let G be extremal for F, i.e. $G \not\supseteq F$ and $e(G) = \mathrm{ex}(|G|, F)$. Then $\delta(G) = \left(1 - \frac{1}{r} + o(1)\right)n$.

Theorem (Simonovits)**.** Let F be $(r + 1)$-edge-critical, i.e. $\chi(F) = r + 1$ but $\chi(F) \setminus e) = r$ for every edge e of F. Then for large n,

$$\mathrm{ex}(n, F) = t_r(n).$$

and the only extremal graph is $T_r(n)$.

3 Supersaturation

Theorem (Erdös–Simonovits)**.** Let H be some ℓ-uniform hypergraph. Then for all $\varepsilon > 0$, there exists $\delta(H, \varepsilon)$ such that every ℓ-uniform hypergraph G with $|G| = n$ and

$$e(G) > (\pi(H) + \varepsilon)\binom{n}{\ell}$$

contains $\lfloor \delta n^{|H|} \rfloor$ copies of H.

Theorem (Lorden, 1961)**.** Let G have degree sequence $d_1, \ldots, d_n$. Then

$$k_3(G) + k_3(\bar{G}) = \binom{n}{3} - (n-2)e(G) + \sum_{i=1}^{n} \binom{d_i}{2}.$$

Corollary (Goodman, 1959)**.** We have

$$k_3(G) + k_3(\bar{G}) \geq \frac{1}{24} n(n-1)(n-5).$$

Corollary. For $m = e(G)$ and $n = |G|$, we have

$$k_3(G) \geq \frac{m}{3n}(4m - n^2).$$

Theorem. Let G be a graph. For any graph F, let $i_F(G)$ be the number of induced copies of F in G, i.e. the number of subsets $M \subseteq V(G)$ such that $G[M] \cong F$. So, for example, $i_{K_p}(G) = k_p(G)$.

Define

$$f(G) = \sum_F \alpha_F i_F(G),$$

with the sum being over a finite collection of graphs F, each being complete multipartite, with $\alpha_F \in \mathbb{R}$ and $\alpha_F \geq 0$ if F is not complete. Then amongst graphs of given order, $f(G)$ is maximized on a complete multi-partite graph. Moreover, if $\alpha_{\bar{K}_3} > 0$, then there are no other maxima.

Theorem (Bollobás, 1976)**.** Let $1 \leq p \leq r$, and for $0 \leq x \leq \binom{n}{p}$, let $\psi(x)$ be a maximal convex function lying below the points

$$\{(k_p(T_q(n)), k_r(T_q(n))) : q = r-1, r, \ldots\} \cup \{(0,0)\}.$$

Let G be a graph of order n. Then

$$k_r(G) \geq \psi(k_p(G)).$$

4 Szemerédi's regularity lemma

Lemma. Let (U, W) be an ε-uniform pair with $d(U, W) = d$. Then

$$|\{u \in U : |\Gamma(u) \cap W| > (d - \varepsilon)|W|\}| \geq (1 - \varepsilon)|U|$$
$$|\{u \in U : |\Gamma(u) \cap W| < (d + \varepsilon)|W|\}| \geq (1 - \varepsilon)|U|,$$

where $\Gamma(u)$ is the set of neighbours of u.

Lemma (Graph building lemma)**.** Let G be a graph containing distinct vertex subsets $V_1, \ldots, V_r$ with $|V_i| = u$, such that (V_i, V_j) is ε-uniform and $d(V_i, V_j) \geq \lambda$ for all $1 \leq i \leq j \leq r$.

Let H be a graph with maximum degree $\Delta(H) \leq \Delta$. Suppose H has an r-colouring in which no colour is used more than s times, i.e. $H \subseteq K_r(s)$, and suppose $(\Delta + 1)\varepsilon \leq \lambda^\Delta$ and $s \leq \lfloor \varepsilon u \rfloor$. Then $H \subseteq G$.

Corollary. Let H be a graph with vertex set $\{v_1, \ldots, v_r\}$. Let $0 < \lambda, \varepsilon < 1$ satisfy $r\varepsilon \leq \lambda^{r-1}$.

Let G be a graph with disjoint vertex subsets $V_1, \ldots, V_r$, each of size $u \geq 1$. Suppose each pair (V_i, V_j) is ε uniform, and $d(V_i, V_j) \geq \lambda$ if $v_i v_j \in E(H)$, and $d(V_i, V_j) \leq 1 - \lambda$ if $v_i v_j \notin E(H)$. Then there exists $x_i \in V_i$ so that the map $v_i \to x_i$ is an isomorphism $H \to G[\{x_1, \ldots, x_r\}]$.

Theorem (Szemerédi's regularity lemma)**.** Let $0 < \varepsilon < 1$ and let ℓ be some natural number. Then there exists some $L = L(\ell, \varepsilon)$ such that every graph has an ε-uniform equipartition into m parts for some $\ell \leq m \leq L$, depending on the graph.

Lemma. Let $U' \subseteq U$ and $W' \subseteq W$, where $|U'| \geq (1 - \delta)|U|$ and $|W'| \geq (1 - \delta)|W|$. Then

$$|d(U', W') - d(U, W)| \leq 2\delta.$$

Lemma. Let $x_1, \ldots, x_n$ be real numbers with

$$X = \frac{1}{n}\sum_{i=1}^n x_i,$$

and let

$$x = \frac{1}{m}\sum_{i=1}^m x_i.$$

Then

$$\frac{1}{n}\sum_{i=1}^n x_i^2 \geq X^2 + \frac{m}{n - m}(x - X)^2 \geq X^2 + \frac{m}{n}(x - X)^2.$$

Theorem. Given an integer d, there exists $c(d)$ such that

$$r(G) \leq c|G|$$

for every graph G with $\Delta(G) \leq d$.

Theorem (Triangle removal lemma)**.** Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|G| = n$ and G contains at most δn^3 triangles, then there exists a set of at most εn^2 edges whose removal leaves no triangles.

Corollary (Roth, 1950's)**.** Let $\varepsilon > 0$. Then if n is large enough, and $A \subseteq [n] = \{1, 2, \ldots, n\}$ with $|A| \geq \varepsilon n$, then A contains a 3-term arithmetic progression.

5 Subcontraction, subdivision and linking

Theorem (Menger's theorem)**.** Let G be a graph and $s_1, \ldots, s_k, t_1, \ldots, t_k$ be distinct vertices. If $\kappa(G) \geq k$, then there exists k vertex disjoint paths from $\{s_1, \ldots, s_k\}$ to $\{t_1, \ldots, t_k\}$.

Lemma. If $\kappa(G) \geq 2k$ and $G \supseteq TK_{5k}$, then G is k-linked.

Lemma. If $e(G) \geq k|G|$, then there exists some H with $|H| \leq 2k$ and $\delta(H) \geq k$ such that $G \succ K_1 + H$.

Theorem. If $t \geq 3$ and $e(G) \geq 2^{t-3}|G|$, then $G \succ K_t$.

Lemma. If $\delta(G) \geq 2k$, then G contains vertex disjoint subgraphs H, J with $\delta(H) \geq k$, J connected, and every vertex in H has a neighbour in J.

Theorem. Let F be a graph with n edges and no isolated vertices. If $\delta(G) \geq 2^n$, then $G \supseteq TF$.

Lemma. We have
$$c(t) \geq (\alpha + o(1))t\sqrt{\log t}.$$
To determine α, let $\lambda < 1$ be the root to $1 - \lambda + 2\lambda \log \lambda = 0$. In fact $\lambda \approx 0.284$. Then
$$\alpha = \frac{1 - \lambda}{2\sqrt{\log 1/\lambda}} \approx 0.319.$$

Lemma. Let $k \in \mathbb{N}$ and G be a graph with $e(G) \geq 11k|G|$. Then there exists some H with
$$|H| \leq 11k + 2, \quad 2\delta(H) \geq |H| + 4k - 1$$
such that $G \succ H$.

Theorem. We have
$$c(t) \leq 7t\sqrt{\log t}$$
if t is large.

Lemma. Let $d \geq 0$, $k \geq 2$ and $h \geq d + \lfloor 3k/2 \rfloor$ be integers.

Let G be a graph, $S = \{s_1, \ldots, s_k\} \subseteq V(G)$. Suppose there exists disjoint subgraphs $C_1, \ldots, C_h$ of G such that

- ($*$) Each C_i is either connected, or each of its component meets S. Moreover, each C_i is adjacent to all but at most d of the C_j, $j \neq i$ not meeting S.
- (†) Moreover, no S-cut of order $< k$ avoids $d + 1$ of $C_1, \ldots, C_h$.

Then G contains disjoint non-empty connected subgraphs $D_1, \ldots, D_m$, where
$$m = h - \lfloor k/2 \rfloor,$$
such that for $1 \leq i \leq k$, $s_i \in D_i$, and D_i is adjacent to all but at most d of $D_{k+1}, \ldots, D_m$.

Theorem. Let G be a graph with $\kappa(G) \geq 2k$ and $e(G) \geq 11k|G|$. Then G is k-linked. In particular, $f(k) \leq 22k$.

Lemma. If $\delta(a) \geq t^2$ and G is $\binom{t+2}{3}$-linked, then $G \supseteq TK_t$.

Lemma. Let $k, d \in \mathbb{N}$ with $k \leq \frac{d+1}{2}$, and suppose $e(G) \geq d|G|$. Then G contains a subgraph H with

$$e(H) = d|H| - kd + 1, \quad \delta(H) \geq d + 1, \quad \kappa(H) \geq k.$$

Theorem.

$$\frac{t^2}{16} \leq t(t) \leq 13\binom{t+1}{2}.$$

6 Extremal hypergraphs

Theorem (Erdös)**.** Let G be ℓ-uniform of order n with $p\binom{n}{\ell}$ edges, where $p \geq 2n^{-1/t^{\ell-1}}$. Then G contains $K_\ell^\ell(t)$ (provided n is large).