Part III — Extremal Graph Theory
Definitions

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Michaelmas 2017

Turán's theorem, giving the maximum size of a graph that contains no complete r-vertex subgraph, is an example of an extremal graph theorem. Extremal graph theory is an umbrella title for the study of how graph and hypergraph properties depend on the values of parameters. This course builds on the material introduced in the Part II Graph Theory course, which includes Turán's theorem and also the Erdös–Stone theorem.

The first few lectures will cover the Erdös–Stone theorem and stability. Then we shall treat Szemerédi's Regularity Lemma, with some applications, such as to hereditary properties. Subsequent material, depending on available time, might include: hypergraph extensions, the flag algebra method of Razborov, graph containers and applications.

Pre-requisites

A knowledge of the basic concepts, techniques and results of graph theory, as afforded by the Part II Graph Theory course, will be assumed. This includes Turán's theorem, Ramsey's theorem, Hall's theorem and so on, together with applications of elementary probability.

Contents

1 The Erdös–Stone theorem

Definition (Turán graph)**.** The *Turán graph* $T_r(n)$ is the complete r-partite graph on n vertices with class sizes $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. We write $t_r(n)$ for the number of edges in $T_r(n)$.

Notation. We denote by $K_r(t)$ the complete r-partite graph with t vertices in each class.

2 Stability

3 Supersaturation

4 Szemerédi's regularity lemma

Definition (Density)**.** Let U, W be disjoint subsets of the vertex set of some graph. The number of edges between U and W is denoted by $e(U, W)$, and the *density* is

$$d(U, W) = \frac{e(U, W)}{|U||W|}.$$

Definition (ε-uniform pair)**.** Let $0 < \varepsilon < 1$. We say a pair (U, W) is *ε-uniform* if

$$|d(U', W') - d(U, W)| < \varepsilon$$

whenever $U' \subseteq U$, $W' \subseteq W$, and $|U'| \geq \varepsilon|U|$, $|W'| \geq \varepsilon|W|$.

Definition (Equipartition)**.** An *equipartition* of $V(G)$ into k parts is a partition into sets $V_1, \ldots, V_k$, where $\lfloor \frac{n}{k} \rfloor \leq V_i \leq \lceil \frac{n}{k} \rceil$, where $n = |G|$.

We say that the partition is ε-uniform if (V_i, V_j) is ε-uniform for all but $\varepsilon\binom{k}{2}$ pairs.

5 Subcontraction, subdivision and linking

Definition. Let G be a graph, $e = xy$ an edge. Then the graph G/e is the graph formed from $G \setminus \{x, y\}$ by adding a new vertex joined to all neighbours of x and y. We say this is the graph formed by *contracting* the edge e.

Definition ((Sub)contraction)**.** A *contraction* of G is a graph obtained by a sequence of edge contractions. A *subcontraction* of G is a contraction of a subgraph. We write $G \succ H$ if H is a subcontraction of G.

Definition (Subdivision)**.** If H is a graph, TH stands for any graph obtained from H by replacing its edges by vertex disjoint paths (i.e. we subdivide edges).

Definition (k-linked graph)**.** We say G is *k-linked* if there exists vertex disjoint s_i-t_i paths for $1 \leq i \leq k$ for any choice of these $2k$ vertices.

Definition (S-cut)**.** Given $S \subseteq V(G)$, an S-cut is a pair (A, B) of subsets of the vertices such that $A \cup B = V(G)$, $S \subseteq A$ and $e(A \setminus B, B \setminus A) = 0$. The *order* of the cut is $|A \cap B|$.

We say (A, B) *avoids* C if $A \cap V(C) = \emptyset$.

6 Extremal hypergraphs