Part III — Differential Geometry
Theorems with proof

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This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are, for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, smooth maps. Tangent vectors and vector fields, tangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Lie Groups.
- *Vector Bundles.* Structure group. The example of Hopf bundle. Bundle morphisms and automorphisms. Exterior algebra of differential forms. Tensors. Symplectic forms. Orientability of manifolds. Partitions of unity and integration on manifolds, Stokes Theorem; de Rham cohomology. Lie derivative of tensors. Connections on vector bundles and covariant derivatives: covariant exterior derivative, curvature. Bianchi identity.
- *Riemannian Geometry.* Connections on the tangent bundle, torsion. Bianchi's identities for torsion free connections. Riemannian metrics, Levi-Civita connection, Christoffel symbols, geodesics. Riemannian curvature tensor and its symmetries, second Bianchi identity, sectional curvatures.

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor's theorem in several variables). Exposure to some of the ideas of classical differential geometry might also be useful.

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0 Introduction

1 Manifolds

1.1 Manifolds

Lemma. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in some atlas, and $f : M \to \mathbb{R}$, then $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi_\alpha(p)$ if and only if $f \circ \varphi_\beta^{-1}$ is smooth at $\varphi_\beta(p)$ for all $p \in U_\alpha \cap U_\beta$.

Proof. We have
$$f \circ \varphi_\beta^{-1} = f \circ \varphi_\alpha^{-1} \circ (\varphi_\alpha \circ \varphi_\beta^{-1}).$$
□

Lemma. Let M be a manifold, and $\varphi_1 : U_1 \to \mathbb{R}^n$ and $\varphi_2 : U_2 \to \mathbb{R}^m$ be charts. If $U_1 \cap U_2 \neq \emptyset$, then $n = m$.

Proof. We know
$$\varphi_1 \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$$
is a smooth map with inverse $\varphi_2 \varphi_1^{-1}$. So the derivative
$$D(\varphi_1 \varphi_2^{-1})(\varphi_2(p)) : \mathbb{R}^m \to \mathbb{R}^n$$
is a linear isomorphism, whenever $p \in U_1 \cap U_2$. So $n = m$. □

1.2 Smooth functions and derivatives

Lemma. $\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$ is a basis of $T_p\mathbb{R}^n$. So these are all the derivations.

Proof. Independence is clear as
$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}.$$
We need to show spanning. For notational convenience, we wlog take $p = 0$. Let $X \in T_0\mathbb{R}^n$.

We first show that if $g \in C^\infty(U)$ is the constant function $g = 1$, then $X(g) = 0$. Indeed, we have
$$X(g) = X(g^2) = g(0)X(g) + X(g)g(0) = 2X(g).$$
Thus, if h is any constant function, say, c, then $X(h) = X(cg) = cX(g)$. So the derivative of any constant function vanishes.

In general, let $f \in C^\infty(U)$. By Taylor's theorem, we have
$$f(x_1, \cdots, x_n) = f(0) + \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_0 x_i + \varepsilon,$$
where ε is a sum of terms of the form $x_i x_j h$ with $h \in C^\infty(U)$.

We set $\lambda_i = X(x_i) \in \mathbb{R}$. We first claim that $X(\varepsilon) = 0$. Indeed, we have
$$X(x_i x_j h) = x_i(0)X(x_j h) + (x_j h)(0)X(x_i) = 0.$$
So we have
$$X(f) = \sum_{i=1}^n \lambda_i \left.\frac{\partial f}{\partial x_i}\right|_0.$$
So we have
$$X = \sum_{i=1}^n \lambda_i \left.\frac{\partial}{\partial x_i}\right|_0.$$
□

Proposition (Chain rule)**.** Let M, N, P be manifolds, and $F \in C^\infty(M, N)$, $G \in C^\infty(N, P)$, and $p \in M, q = F(p)$. Then we have

$$\mathrm{D}(G \circ F)|_p = \mathrm{D}G|_q \circ \mathrm{D}F|_p.$$

Proof. Let $h \in C^\infty(P)$ and $X \in T_pM$. We have

$$\mathrm{D}G|_q(\mathrm{D}F|_p(X))(h) = \mathrm{D}F|_p(X)(h \circ G) = X(h \circ G \circ F) = \mathrm{D}(G \circ F)|_p(X)(h). \qquad \square$$

Corollary. If F is a diffeomorphism, then $\mathrm{D}F|_p$ is a linear isomorphism, and $(\mathrm{D}F|_p)^{-1} = \mathrm{D}(F^{-1})|_{F(p)}$.

Lemma. We have

$$\mathrm{D}F|_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right) = \sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p)\left.\frac{\partial}{\partial y_j}\right|_q.$$

In other words, $\mathrm{D}F|_p$ has matrix representation

$$\left(\frac{\partial F_j}{\partial x_i}(p)\right)_{ij}.$$

Proof. We let

$$\mathrm{D}F|_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right) = \sum_{j=1}^m \lambda_j \left.\frac{\partial}{\partial y_j}\right|_q.$$

for some λ_j. We apply this to the local function y_k to obtain

$$\begin{aligned}
\lambda_k &= \left(\sum_{j=1}^m \lambda_j \left.\frac{\partial}{\partial y_j}\right|_q\right)(y_k)\\
&= \mathrm{D}F_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)(y_k)\\
&= \left.\frac{\partial}{\partial x_i}\right|_p (y_k \circ F)\\
&= \left.\frac{\partial}{\partial x_i}\right|_p (F_k)\\
&= \frac{\partial F_k}{\partial x_i}(p).
\end{aligned} \qquad \square$$

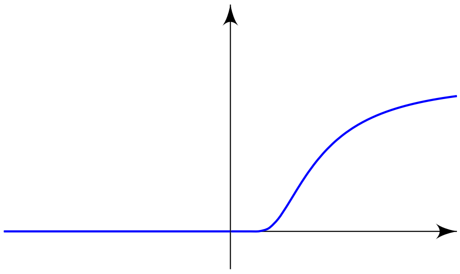
1.3 Bump functions and partitions of unity

Lemma. Suppose $W \subseteq M$ is a coordinate chart with $p \in W$. Then there is an open neighbourhood V of p such that $\bar{V} \subseteq W$ and an $X \in C^\infty(M, \mathbb{R})$ such that $X = 1$ on V and $X = 0$ on $M \setminus W$.

Proof. Suppose we have coordinates $x_1, \cdots, x_n$ on W. We wlog suppose these are defined for all $|x| < 3$.

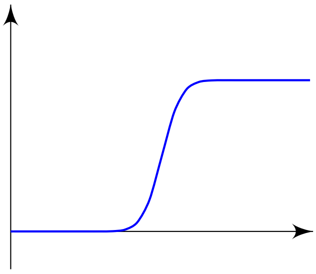
We define $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ by

$$\alpha(t) = \begin{cases} e^{-t^{-2}} & t > 0\\ 0 & t \leq 0\end{cases}.$$



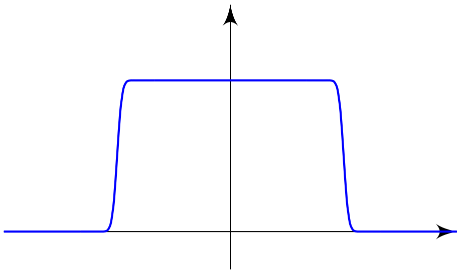
We now let

$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}.$$



Then we let

$$\gamma(t) = \beta(t+2)\beta(2-t).$$



Finally, we let

$$X(x_1, \cdots, x_n) = \gamma(x_1) \cdots \gamma(x_n).$$

on W. We let

$$V = \{\mathbf{x} : |x_i| < 1\}.$$

Extending X to be identically 0 on $M \setminus W$ to get the desired smooth function (up to some constant). □

Lemma. Let $p \in W \subseteq U$ and W, U open. Let $f_1, f_2 \in C^\infty(U)$ be such that $f_1 = f_2$ on W. If $X \in \mathrm{Der}_p(C^\infty(U))$, then we have $X(f_1) = X(f_2)$

Proof. Set $h = f_1 - f_2$. We can wlog assume that W is a coordinate chart. We pick a bump function $\chi \in C^\infty(U)$ that vanishes outside W. Then $\chi h = 0$. Then we have

$$0 = X(\chi h) = \chi(p) X(h) + h(p) X(\chi) = X(h) + 0 = X(f_1) - X(f_2). \qquad \square$$

Theorem. Given any $\{U_\alpha\}$ open cover, there exists a partition of unity subordinate to $\{U_\alpha\}$.

Proof. We will only do the case where M is compact. Given $p \in M$, there exists a coordinate chart $p \in V_p$ and $\alpha(p)$ such that $V_p \subseteq U_{\alpha(p)}$. We pick a bump function $\chi_p \in C^\infty(M, \mathbb{R})$ such that $\chi_p = 1$ on a neighbourhood $W_p \subseteq V_p$ of p. Then $\operatorname{supp}(\chi_p) \subseteq U_{\alpha(p)}$.

Now by compactness, there are some $p_1, \cdots, p_N$ such that M is covered by $W_{p_1} \cup \cdots \cup W_{p_N}$. Now let

$$\tilde{\varphi}_\alpha = \sum_{i : \alpha(p_i) = \alpha} \chi_{p_i}.$$

Then by construction, we have

$$\operatorname{supp}(\tilde{\varphi}_\alpha) \subseteq U_\alpha.$$

Also, by construction, we know $\sum_\alpha \tilde{\varphi}_\alpha > 0$. Finally, we let

$$\varphi_\alpha = \frac{\tilde{\varphi}_\alpha}{\sum_\beta \tilde{\varphi}_\beta}.$$ □

1.4 Submanifolds

Lemma. If S is an embedded submanifold of M, then there exists a unique differential structure on S such that the inclusion map $\iota : S \hookrightarrow M$ is smooth and S inherits the subspace topology.

Proof. Basically if $x_1, \cdots, x_n$ is a slice chart for S in M, then $x_1, \cdots, x_k$ will be coordinates on S.

More precisely, let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection map

$$\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_k).$$

Given a slice chart (U, φ) for S in M, consider $\tilde{\varphi} : S \cap U \to \mathbb{R}^k$ by $\tilde{\varphi} = \pi \circ \varphi$. This is smooth and bijective, and is so a chart on S. These cover S by assumption. So we only have to check that the transition functions are smooth.

Given another slice chart (V, ξ) for S in M, we let $\tilde{\xi} = \pi \circ \xi$, and check that

$$\tilde{\xi} \circ \tilde{\varphi}^{-1} = \pi \circ \xi \circ \varphi^{-1} \circ j,$$

where $j : \mathbb{R}^k \to \mathbb{R}^n$ is given by $j(x_1, \cdots, x_k) = (x_1, \cdots, x_k, 0, \cdots, 0)$.

From this characterization, by looking at local charts, it is clear that S has the subspace topology. It is then easy to see that the embedded submanifold is Hausdorff and second-countable, since these properties are preserved by taking subspaces.

We can also check easily that $\iota : S \hookrightarrow M$ is smooth, and this is the only differential structure with this property. □

Proposition. Let S be an embedded submanifold. Then the derivative of the inclusion map $\iota : S \hookrightarrow M$ is injective.

Proposition. Let $F \in C^\infty(M, N)$, and let $c \in N$. Suppose c is a regular value. Then $S = F^{-1}(c)$ is an embedded submanifold of dimension $\dim M - \dim N$.

Proof. We let $n = \dim M$ and $m = \dim N$. Notice that for the map DF to be surjective, we must have $n \geq m$.

Let $p \in S$, so $F(p) = c$. We want to find a slice coordinate for S near p. Since the problem is local, by restricting to local coordinate charts, we may wlog assume $N = \mathbb{R}^m$, $M = \mathbb{R}^n$ and $c = p = 0$.

Thus, we have a smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$ with surjective derivative at 0. Then the derivative is

$$\left(\left.\frac{\partial F_j}{\partial x_i}\right|_0 \right)_{i=1,\ldots,n;\, j=1,\ldots,m},$$

which by assumption has rank m. We reorder the x_i so that the first m columns are independent. Then the $m \times m$ matrix

$$R = \left(\left.\frac{\partial F_j}{\partial x_i}\right|_0 \right)_{i,j=1,\ldots,m}$$

is non-singular. We consider the map

$$\alpha(x_1, \cdots, x_n) = (F_1, \cdots, F_m, x_{m+1}, \cdots, x_n).$$

We then obtain

$$\mathrm{D}\alpha|_0 = \begin{pmatrix} R & * \\ 0 & I \end{pmatrix},$$

and this is non-singular. By the inverse function theorem, α is a local diffeomorphism. So there is an open $W \subseteq \mathbb{R}^n$ containing 0 such that $\alpha|_W : W \to \alpha(W)$ is smooth with smooth inverse. We claim that α is a slice chart of S in $\mathbb{R}^n$.

Since it is a smooth diffeomorphism, it is certainly a chart. Moreover, by construction, the points in S are exactly those whose image under F have the first m coordinates vanish. So this is the desired slice chart. □

2 Vector fields

2.1 The tangent bundle

Lemma. The charts actually make TM into a manifold.

Proof. If (V, ξ) is another chart on M with coordinates $y_1, \cdots, y_n$, then

$$\left.\frac{\partial}{\partial x_i}\right|_p = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p) \left.\frac{\partial}{\partial y_j}\right|_p.$$

So we have $\tilde{\xi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \xi(U \cap V) \times \mathbb{R}^n$ given by

$$\tilde{\xi} \circ \tilde{\varphi}^{-1}(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_n) = \left(y_1, \cdots, y_n, \sum_{i=1}^n \alpha_i \frac{\partial y_1}{\partial x_i}, \cdots, \sum_{i=1}^n \alpha_i \frac{\partial y_n}{\partial x_i}\right),$$

and is smooth (and in fact fiberwise linear).

It is easy to check that TM is Hausdorff and second countable as M is. □

Lemma. The map $X \mapsto \mathcal{X}$ is an $\mathbb{R}$-linear isomorphism

$$\Gamma : \mathrm{Vect}(M) \to \mathrm{Der}(C^\infty(M)).$$

Proof. Suppose that α is a derivation. If $p \in M$, we define

$$X_p(f) = \alpha(f)(p)$$

for all $f \in C^\infty(M)$. This is certainly a linear map, and we have

$$X_p(fg) = \alpha(fg)(p) = (f\alpha(g) + g\alpha(f))(p) = f(p)X_p(g) + g(p)X_p(f).$$

So $X_p \in T_pM$. We just need to check that the map $M \to TM$ sending $p \mapsto X_p$ is smooth. Locally on M, we have coordinates $x_1, \cdots, x_n$, and we can write

$$X_p = \sum_{i=1}^n \alpha_i(p) \left.\frac{\partial}{\partial x_i}\right|_p.$$

We want to show that $\alpha_i : U \to \mathbb{R}$ are smooth.

We pick a bump function φ that is identically 1 near p, with $\operatorname{supp} \varphi \subseteq U$. Consider the function $\varphi x_j \in C^\infty(M)$. We can then consider

$$\alpha(\varphi x_j)(p) = X_p(\varphi x_j).$$

As φx_j is just x_j near p, by properties of derivations, we know this is just equal to α_j. So we have

$$\alpha(\varphi x_j) = \alpha_j.$$

So α_j is smooth. □

2.2 Flows

Theorem (Fundamental theorem on ODEs)**.** Let $U \subseteq \mathbb{R}^n$ be open and $\alpha : U \to \mathbb{R}^n$ smooth. Pick $t_0 \in \mathbb{R}$.

Consider the ODE

$$\begin{aligned} \dot{\gamma}_i(t) &= \alpha_i(\gamma(t)) \\ \gamma_i(t_0) &= c_i, \end{aligned}$$

where $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{R}^n$.

Then there exists an open interval I containing t_0 and an open $U_0 \subseteq U$ such that for every $\mathbf{c} \in U_0$, there is a smooth solution $\gamma_{\mathbf{c}} : I \to U$ satisfying the ODE.

Moreover, any two solutions agree on a common domain, and the function $\Theta : I \times U_0 \to U$ defined by $\Theta(t, \mathbf{c}) = \gamma_{\mathbf{c}}(t)$ is smooth (in both variables).

Theorem (Existence of integral curves)**.** Let $X \in \mathrm{Vect}(M)$ and $p \in M$. Then there exists some open interval $I \subseteq \mathbb{R}$ with $0 \in I$ and an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.

Moreover, if $\tilde{\gamma} : \tilde{I} \to M$ is another integral curve for X, and $\tilde{\gamma}(0) = p$, then $\tilde{\gamma} = \gamma$ on $I \cap \tilde{I}$.

Proof. Pick local coordinates for M centered at p in an open neighbourhood U. So locally we write

$$X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i},$$

where $\alpha_i \in C^\infty(U)$. We want to find $\gamma = (\gamma_1, \cdots, \gamma_n) : I \to U$ such that

$$\sum_{i=1}^n \gamma_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)} = \sum_{i=1}^n \alpha_i(\gamma(t)) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}, \quad \gamma_i(0) = 0.$$

Since the $\frac{\partial}{\partial x_i}$ form a basis, this is equivalent to saying

$$\gamma_i(t) = \alpha_i(\gamma(t)), \quad \gamma_i(0) = 0$$

for all i and $t \in I$.

By the general theory of ordinary differential equations, there is an interval I and a solution γ, and any two solutions agree on their common domain.

However, we need to do a bit more for uniqueness, since all we know is that there is a unique integral curve lying in this particular chart. It might be that there are integral curves that do wild things when they leave the chart.

So suppose $\gamma : I \to M$ and $\tilde{\gamma} : \tilde{I} \to M$ are both integral curves passing through the same point, i.e. $\gamma(0) = \tilde{\gamma}(0) = p$.

We let

$$J = \{t \in I \cap \tilde{I} : \gamma(t) = \tilde{\gamma}(t)\}.$$

This is non-empty since $0 \in J$, and J is closed since γ and $\tilde{\gamma}$ are continuous. To show it is all of $I \cap \tilde{I}$, we only have to show it is open, since $I \cap \tilde{I}$ is connected.

So let $t_0 \in J$, and consider $q = \gamma(t_0)$. Then γ and $\tilde{\gamma}$ are integral curves of X passing through q. So by the first part, they agree on some neighbourhood of t_0. So J is open. So done. □

Theorem. Let M be a manifold and X a complete vector field on M. Define $\Theta_t : \mathbb{R} \times M \to M$ by

$$\Theta_t(p) = \gamma_p(t),$$

where γ_p is the maximal integral curve of X through p with $\gamma(0) = p$. Then Θ is a function smooth in p and t, and

$$\Theta_0 = \mathrm{id}, \quad \Theta_t \circ \Theta_s = \Theta_{s+t}$$

Proof. This follows from uniqueness of integral curves and smooth dependence on initial conditions of ODEs. □

Theorem. Let M be a manifold, and $X \in \mathrm{Vect}(M)$. Define

$$D = \{(t, p) \in \mathbb{R} \times M : t \in I_p\}.$$

In other words, this is the set of all (t, p) such that $\gamma_p(t)$ exists. We set

$$\Theta_t(p) = \Theta(t, p) = \gamma_p(t)$$

for all $(t, p) \in D$. Then

(i) D is open and $\Theta : D \to M$ is smooth

(ii) $\Theta(0, p) = p$ for all $p \in M$.

(iii) If $(t, p) \in D$ and $(t, \Theta(s, p)) \in D$, then $(s + t, p) \in D$ and $\Theta(t, \Theta(s, p)) = \Theta(t + s, p)$.

(iv) For any $t \in \mathbb{R}$, the set $M_t : \{p \in M : (t, p) \in D\}$ is open in M, and

$$\Theta_t : M_t \to M_{-t}$$

is a diffeomorphism with inverse Θ_{-t}.

Proposition. Let M be a compact manifold. Then any $X \in \mathrm{Vect}(M)$ is complete.

Proof. Recall that

$$D = \{(t, p) : \Theta_t(p) \text{ is defined}\}$$

is open. So given $p \in M$, there is some open neighbourhood $U \subseteq M$ of p and an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times U \subseteq D$. By compactness, we can find finitely many such U that cover M, and find a small ε such that $(-\varepsilon, \varepsilon) \times M \subseteq D$.

In other words, we know $\Theta_t(p)$ exists and $p \in M$ and $|t| < \varepsilon$. Also, we know $\Theta_t \circ \Theta_s = \Theta_{t+s}$ whenever $|t|, |s| < \varepsilon$, and in particular Θ_{t+s} is defined. So $\Theta_{Nt} = (\Theta_t)^N$ is defined for all N and $|t| < \varepsilon$, so Θ_t is defined for all t. □

2.3 Lie derivative

Lemma. $\mathcal{L}_X(g) = X(g)$. In particular, $\mathcal{L}_X(g) \in C^\infty(M, \mathbb{R})$.

Proof.

$$\begin{aligned}
\mathcal{L}_X(g)(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_t^*(g)(p) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(\Theta_t(p)) \\
&= \mathrm{d}g|_p(X(p)) \\
&= X(g)(p).
\end{aligned}$$

□

Lemma. We have

$$\mathcal{L}_X Y = [X, Y].$$

Proof. Let $g \in C^\infty(M, \mathbb{R})$. Then we have

$$\Theta_t^*(Y)(g \circ \Theta_t) = Y(g) \circ \Theta_t.$$

We now look at

$$\frac{\Theta_t^*(Y)(g) - Y(g)}{t} = \underbrace{\frac{\Theta_t^*(Y)(g) - \Theta_t^*(Y)(g \circ \Theta_t)}{t}}_{\alpha_t} + \underbrace{\frac{Y(g) \circ \Theta_t - Y(g)}{t}}_{\beta_t}.$$

We have

$$\lim_{t \to 0} \beta_t = \mathcal{L}_X(Y(g)) = XY(g)$$

by the previous lemma, and we have

$$\lim_{t \to 0} \alpha_t = \lim_{t \to 0} (\Theta_t^*(Y)) \left(\frac{g - g \circ \Theta_t}{t}\right) = Y(-\mathcal{L}_X(g)) = -YX(g).$$

□

Corollary. Let $X, Y \in \mathrm{Vect}(M)$ and $f \in C^\infty(M, \mathbb{R})$. Then

(i) $\mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X Y = X(f)Y + f\mathcal{L}_X Y$

(ii) $\mathcal{L}_X Y = -\mathcal{L}_Y X$

(iii) $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

Proof. Immediate from the properties of the Lie bracket. □

3 Lie groups

Lemma. Given $\xi \in T_e G$, we let

$$X_\xi|_g = \mathrm{D}L_g|_e(\xi) \in T_g(G).$$

Then the map $T_e G \to \mathrm{Vect}^L(G)$ by $\xi \mapsto X_\xi$ is an isomorphism of vector spaces.

Proof. The inverse is given by $X \mapsto X|_e$. The only thing to check is that X_ξ actually is a left invariant vector field. The left invariant part follows from

$$\mathrm{D}L_h|_g(X_\xi|_g) = \mathrm{D}L_h|_g(\mathrm{D}L_g|_e(\xi)) = \mathrm{D}L_{hg}|_e(\xi) = X_\xi|_{hg}.$$

To check that X_ξ is smooth, suppose $f \in C^\infty(U, \mathbb{R})$, where U is open and contains e. We let $\gamma : (-\varepsilon, \varepsilon) \to U$ be smooth with $\dot{\gamma}(0) = \xi$. So

$$X_\xi f|_g = \mathrm{D}L_g(\xi)(f) = \xi(f \circ L_g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ L_g \circ \gamma)$$

But as $(t, g) \mapsto f \circ L_g \circ \gamma(t)$ is smooth, it follows that $X_\xi f$ is smooth. So $X_\xi \in \mathrm{Vect}^L(G)$. □

Lemma. Let G be an abelian Lie group. Then the bracket of $\mathfrak{g}$ vanishes.

Proposition. Let G be a Lie group and $\xi \in \mathfrak{g}$. Then the integral curve γ for X_ξ through $e \in G$ exists for all time, and $\gamma : \mathbb{R} \to G$ is a Lie group homomorphism.

Proof. Let $\gamma : I \to G$ be a maximal integral curve of X_ξ, say $(-\varepsilon, \varepsilon) \in I$. We fix a t_0 with $|t_0| < \varepsilon$. Consider $g_0 = \gamma(t_0)$.

We let

$$\tilde{\gamma}(t) = L_{g_0}(\gamma(t))$$

for $|t| < \varepsilon$.

We claim that $\tilde{\gamma}$ is an integral curve of X_ξ with $\tilde{\gamma}(0) = g_0$. Indeed, we have

$$\dot{\tilde{\gamma}}|_t = \frac{\mathrm{d}}{\mathrm{d}t} L_{g_0}\gamma(t) = \mathrm{D}L_{g_0}\dot{\gamma}(t) = \mathrm{D}L_{g_0} X_\xi|_{\gamma(t)} = X_\xi|_{g_0 \cdot \gamma(t)} = X_\xi|_{\tilde{\gamma}(t)}.$$

By patching these together, we know $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq I$. Since we have a fixed ε that works for all t_0, it follows that $I = \mathbb{R}$.

The fact that this is a Lie group homomorphism follows from general properties of flow maps. □

Proposition.

(i) exp is a smooth map.

(ii) If $F(t) = \exp(t\xi)$, then $F : \mathbb{R} \to G$ is a Lie group homomorphism and $\mathrm{D}F|_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \xi$.

(iii) The derivative

$$\mathrm{D}\exp : T_0\mathfrak{g} \cong \mathfrak{g} \to T_e G \cong \mathfrak{g}$$

is the identity map.

(iv) exp is a local diffeomorphism around $0 \in \mathfrak{g}$, i.e. there exists an open $U \subseteq \mathfrak{g}$ containing 0 such that $\exp : U \to \exp(U)$ is a diffeomorphism.

(v) exp is natural, i.e. if $f : G \to H$ is a Lie group homomorphism, then the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \\ \downarrow{\scriptstyle \mathrm{D}f|_e} & & \downarrow{\scriptstyle f} \\ \mathfrak{h} & \xrightarrow{\exp} & H \end{array}$$

commutes.

Proof.

(i) This is the smoothness of ODEs with respect to parameters

(ii) Exercise.

(iii) If $\xi \in \mathfrak{g}$, we let $\sigma(t) = t\xi$. So $\dot{\sigma}(0) = \xi \in T_0\mathfrak{g} \cong \mathfrak{g}$. So

$$\mathrm{D}\exp|_0(\xi) = \mathrm{D}\exp|_0(\dot{\sigma}(0)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(\sigma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) = X_\xi|_e = \xi.$$

(iv) Follows from above by inverse function theorem.

(v) Exercise. □

Theorem. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then there exists a unique connected Lie subgroup $H \subseteq G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.

Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a (unique) simply-connected Lie group G with Lie algebra $\mathfrak{g}$.

Theorem. Let G, H be Lie groups with G simply connected. Then every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ lifts to a Lie group homomorphism $G \to H$.

4 Vector bundles

4.1 Tensors

Lemma. Tensor products exist (and are unique up to isomorphism) for all pairs of finite-dimensional vector spaces.

Proof. We can construct $V \otimes W = \text{Bilin}(V \times W, \mathbb{R})^*$. The verification is left as an exercise on the example sheet. □

Proposition. Given maps $f : V \to W$ and $g : V' \to W'$, we obtain a map $f \otimes g : V \otimes V' \to W \otimes W'$ given by the bilinear map

$$(f \otimes g)(v, w) = f(v) \otimes g(w).$$

Lemma. Given $v, v_i \in V$ and $w, w_i \in W$ and $\lambda_i \in \mathbb{R}$, we have

$$\begin{aligned}(\lambda_1 v_1 + \lambda_2 v_2) \otimes w &= \lambda_1(v_1 \otimes w) + \lambda_2(v_2 \otimes w)\\ v \otimes (\lambda_1 w_1 + \lambda_2 w_2) &= \lambda_1(v \otimes w_1) + \lambda_2(v \otimes w_2).\end{aligned}$$

Proof. Immediate from the definition of bilinear map. □

Lemma. If $v_1, \cdots, v_n$ is a basis for V, and $w_1, \cdots, w_m$ is a basis for W, then

$$\{v_i \otimes w_j : i = 1, \cdots, n; j = 1, \cdots, m\}$$

is a basis for $V \otimes W$. In particular, $\dim V \otimes W = \dim V \times \dim W$.

Proof. We have $V \otimes W = \text{Bilin}(V \times W, \mathbb{R})^*$. We let $\alpha_{pq} : V \times W \to \mathbb{R}$ be given by

$$\alpha_{pq}\left(\sum a_i v_i, \sum b_j w_j\right) = a_p b_q.$$

Then $\alpha_{pq} \in \text{Bilin}(V \times W, \mathbb{R})$, and $(v_i \otimes w_j)$ are dual to α_{pq}. So it suffices to show that α_{pq} are a basis. It is clear that they are independent, and any bilinear map can be written as

$$\alpha = \sum c_{pq} \alpha_{pq},$$

where

$$c_{pq} = \alpha(v_p, w_q).$$

So done. □

Proposition. For any vector spaces V, W, U, we have (natural) isomorphisms

(i) $V \otimes W \cong W \otimes V$

(ii) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$

(iii) $(V \otimes W)^* \cong V^* \otimes W^*$

Lemma.

(i) If $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

(ii) If $\dim V = n$ and $p > n$, then we have

$$\dim \Lambda^0 V = 1, \quad \dim \Lambda^n V = 1, \quad \Lambda^p V = \{0\}.$$

(iii) The multilinear map $\det : V \times \cdots \times V \to \mathbb{R}$ spans $\Lambda^n V$.

(iv) If $v_1, \cdots, v_n$ is a basis for V, then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : i_1 < \cdots < i_p\}$$

is a basis for $\Lambda^p V$.

Proof.

(i) We clearly have $v \wedge v = 0$. So

$$v \wedge w = -w \wedge v$$

Then

$$(v_1 \wedge \cdots \wedge v_p) \wedge (w_1 \wedge \cdots \wedge w_q) = (-1)^{pq} w_1 \wedge \cdots \wedge w_q \wedge v_1 \wedge \cdots \wedge v_p$$

since we have pq swaps. Since

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : i_1, \cdots, i_p \in \{1, \cdots, n\}\} \subseteq \Lambda^p V$$

spans $\Lambda^p V$ (by the corresponding result for tensor products), the result follows from linearity.

(ii) Exercise.

(iii) The det map is non-zero. So it follows from the above.

(iv) We know that

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : i_1, \cdots, i_p \in \{1, \cdots, n\}\} \subseteq \Lambda^p V$$

spans, but they are not independent since there is a lot of redundancy (e.g. $v_1 \wedge v_2 = -v_2 \wedge v_1$). By requiring $i_1 < \cdots < i_p$, then we obtain a unique copy for combination.

To check independence, we write $I = (i_1, \cdots, i_p)$ and let $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$. Then suppose

$$\sum_I a_I v_I = 0$$

for $a_I \in \mathbb{R}$. For each I, we let J be the multi-index $J = \{1, \cdots, n\} \setminus I$. So if $I \neq I'$, then $v_{I'} \wedge v_J = 0$. So wedging with v_J gives

$$\sum_{I'} \alpha_{I'} v_{I'} \wedge v_J = a_I v_I \wedge v_J = 0.$$

So $a_I = 0$. So done by (ii). □

Lemma. Let $F : V \to V$ be a linear map. Then $\Lambda^n F : \Lambda^n V \to \Lambda^n V$ is multiplication by $\det F$.

Proof. Let $v_1, \cdots, v_n$ be a basis. Then $\Lambda^n V$ is spanned by $v_1 \wedge \cdots \wedge v_n$. So we have

$$(\Lambda^n F)(v_1 \wedge \cdots \wedge v_n) = \lambda\, v_1 \wedge \cdots \wedge v_n$$

for some λ. Write

$$F(v_i) = \sum_j A_{ji} v_j$$

for some $A_{ji} \in \mathbb{R}$, i.e. A is the matrix representation of F. Then we have

$$(\Lambda^n F)(v_1 \wedge \cdots \wedge v_n) = \left(\sum_j A_{j1} v_j\right) \wedge \cdots \wedge \left(\sum_j A_{jn} v_j\right).$$

If we expand the thing on the right, a lot of things die. The only things that live are those where we get each of v_i once in the wedges in some order. Then this becomes

$$\sum_{\sigma \in S_n} \varepsilon(\sigma)(A_{\sigma(1),1} \cdots A_{\sigma(n),n}) v_1 \wedge \cdots \wedge v_n = \det(F)\, v_1 \wedge \cdots \wedge v_n,$$

where $\varepsilon(\sigma)$ is the sign of the permutation, which comes from rearranging the v_i to the right order. □

4.2 Vector bundles

Proposition. We have the following equalities whenever everything is defined:

(i) $\varphi_{\alpha\alpha} = \mathrm{id}$

(ii) $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$

(iii) $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, where $\varphi_{\alpha\beta}\varphi_{\beta\gamma}$ is pointwise matrix multiplication.

These are known as the *cocycle conditions*.

Proposition (Vector bundle construction)**.** Suppose that for each $p \in M$, we have a vector space E_p. We set

$$E = \bigcup_p E_p$$

We let $\pi : E \to M$ be given by $\pi(v_p) = p$ for $v_p \in E_p$. Suppose there is an open cover $\{U_\alpha\}$ of open sets of M such that for each α, we have maps

$$t_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

over U_α that induce fiberwise linear isomorphisms. Suppose the transition functions $\varphi_{\alpha\beta}$ are smooth. Then there exists a unique smooth structure on E making $\pi : E \to M$ a vector bundle such that the t_α are trivializations for E.

Proof. The same as the case for the tangent bundle. □

5 Differential forms and de Rham cohomology

5.1 Differential forms

Theorem (Exterior derivative)**.** There exists a unique linear map

$$\mathrm{d} = \mathrm{d}_{M,p} : \Omega^p(M) \to \Omega^{p+1}(M)$$

such that

(i) On $\Omega^0(M)$ this is as previously defined, i.e.

$$\mathrm{d}f(X) = X(f) \text{ for all } X \in \mathrm{Vect}(M).$$

(ii) We have

$$\mathrm{d} \circ \mathrm{d} = 0 : \Omega^p(M) \to \Omega^{p+2}(M).$$

(iii) It satisfies the *Leibniz rule*

$$\mathrm{d}(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^p \omega \wedge \mathrm{d}\sigma.$$

It follows from these assumptions that

(iv) d acts locally, i.e. if $\omega, \omega' \in \Omega^p(M)$ satisfy $\omega|_U = \omega'|_U$ for some $U \subseteq M$ open, then $\mathrm{d}\omega|_U = \mathrm{d}\omega'|_U$.

(v) We have

$$\mathrm{d}(\omega|_U) = (\mathrm{d}\omega)|_U$$

for all $U \subseteq M$.

Proof. The above computations suggest that in local coordinates, the axioms already tell use completely how d works. So we just work locally and see that they match up globally.

Suppose M is covered by a single chart with coordinates $x_1, \cdots, x_n$. We define $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ as required by (i). For $p > 0$, we define

$$\mathrm{d}\left(\sum_{i_1 < \ldots < i_p} \omega_{i_1, \ldots, i_p} \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}\right) = \sum \mathrm{d}\omega_{i_1, \ldots, i_p} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

Then (i) is clear. For (iii), we suppose

$$\begin{aligned} \omega &= f \, \mathrm{d}x_I \in \Omega^p(M) \\ \sigma &= g \, \mathrm{d}x_J \in \Omega^q(M). \end{aligned}$$

We then have

$$\begin{aligned} \mathrm{d}(\omega \wedge \sigma) &= \mathrm{d}(fg \, \mathrm{d}x_I \wedge \mathrm{d}x_J) \\ &= \mathrm{d}(fg) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J \\ &= g \, \mathrm{d}f \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + f \, \mathrm{d}g \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J \\ &= g \, \mathrm{d}f \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + f(-1)^p \, \mathrm{d}x_I \wedge (\mathrm{d}g \wedge \mathrm{d}x_J) \\ &= (\mathrm{d}\omega) \wedge \sigma + (-1)^p \omega \wedge \mathrm{d}\sigma. \end{aligned}$$

So done. Finally, for (ii), if $f \in \Omega^0(M)$, then

$$\mathrm{d}^2 f = \mathrm{d}\left(\sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\,\mathrm{d}x_j \wedge \mathrm{d}x_i = 0,$$

since partial derivatives commute. Then for general forms, we have

$$\begin{aligned}\mathrm{d}^2\omega = \mathrm{d}^2\left(\sum \omega_I\,\mathrm{d}x_I\right) &= \mathrm{d}\left(\sum \mathrm{d}\omega_I \wedge \mathrm{d}x_I\right)\\ &= \mathrm{d}\left(\sum \mathrm{d}\omega_I \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}\right)\\ &= 0\end{aligned}$$

using Leibniz rule. So this works.

Certainly this has the extra properties. To claim uniqueness, if $\partial : \Omega^p(M) \to \Omega^{p+1}(M)$ satisfies the above properties, then

$$\begin{aligned}\partial\omega &= \partial\left(\sum \omega_I \mathrm{d}x_I\right)\\ &= \sum \partial\omega_I \wedge \mathrm{d}x_I + \omega_I \wedge \partial \mathrm{d}x_I\\ &= \sum \mathrm{d}\omega_I \wedge \mathrm{d}x_I,\end{aligned}$$

using the fact that $\partial = \mathrm{d}$ on $\Omega^0(M)$ and induction.

Finally, if M is covered by charts, we can define $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ by defining it to be the d above on any single chart. Then uniqueness implies this is well-defined. This gives existence of d, but doesn't immediately give uniqueness, since we only proved local uniqueness.

So suppose $\partial : \Omega^p(M) \to \Omega^{p+1}(M)$ again satisfies the three properties. We claim that ∂ is local. We let $\omega, \omega' \in \Omega^p(M)$ be such that $\omega|_U = \omega'|_U$ for some $U \subseteq M$ open. Let $x \in U$, and pick a bump function $\chi \in C^\infty(M)$ such that $\chi \equiv 1$ on some neighbourhood W of x, and $\mathrm{supp}(\chi) \subseteq U$. Then we have

$$\chi \cdot (\omega - \omega') = 0.$$

We then apply ∂ to get

$$0 = \partial(\chi \cdot (\omega - \omega')) = \mathrm{d}\chi \wedge (\omega - \omega') + \chi(\partial\omega - \partial\omega').$$

But $\chi \equiv 1$ on W. So $\mathrm{d}\chi$ vanishes on W. So we must have

$$\partial\omega|_W - \partial\omega'|_W = 0.$$

So $\partial\omega = \partial\omega'$ on W.

Finally, to show that $\partial = d$, if $\omega \in \Omega^p(M)$, we take the same χ as before, and then on x, we have

$$\begin{aligned}\partial\omega &= \partial\left(\chi \sum \omega_I\,\mathrm{d}x_I\right)\\ &= \partial\chi \sum \omega_I\,\mathrm{d}x_I + \chi \sum \partial\omega_I \wedge \mathrm{d}x_I\\ &= \chi \sum \mathrm{d}\omega_I \wedge \mathrm{d}x_I\\ &= \mathrm{d}\omega.\end{aligned}$$

So we get uniqueness. Since x was arbitrary, we have $\partial = \mathrm{d}$. □

Lemma. Let $F \in C^\infty(M, N)$. Let F^* be the associated pullback map. Then

(i) F^* is a linear map $\Omega^p(N) \to \Omega^p(M)$.

(ii) $F^*(\omega \wedge \sigma) = F^*\omega \wedge F^*\sigma$.

(iii) If $G \in C^\infty(N, P)$, then $(G \circ F)^* = F^* \circ G^*$.

(iv) We have $\mathrm{d}F^* = F^*\mathrm{d}$.

Proof. All but (iv) are clear. We first check that this holds for 0 forms. If $g \in \Omega^0(N)$, then we have

$$\begin{aligned}(F^*\mathrm{d}g)|_x(v) &= \mathrm{d}g|_{F(x)}(\mathrm{D}F|_x(v))\\ &= \mathrm{D}F|_x(v)(g)\\ &= v(g \circ F)\\ &= \mathrm{d}(g \circ F)(v)\\ &= \mathrm{d}(F^*g)(v).\end{aligned}$$

So we are done.

Then the general result follows from (i) and (ii). Indeed, in local coordinates $y_1, \cdots, y_n$, if

$$\omega = \sum \omega_{i_1, \ldots, i_p}\, \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_p},$$

then we have

$$F^*\omega = \sum (F^*\omega_{i_1, \ldots, i_p})(F^*\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_p}).$$

Then we have

$$\mathrm{d}F^*\omega = F^*\mathrm{d}\omega = \sum (F^*\mathrm{d}\omega_{i_1, \ldots, i_p})(F^*\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_p}).$$ □

5.2 De Rham cohomology

Proposition.

(i) Let M have k connected components. Then

$$H^0_{\mathrm{dR}}(M) = \mathbb{R}^k.$$

(ii) If $p > \dim M$, then $H^p_{\mathrm{dR}}(M) = 0$.

(iii) If $F \in C^\infty(M, N)$, then this induces a map $F^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ given by

$$F^*[\omega] = [F^*\omega].$$

(iv) $(F \circ G)^* = G^* \circ F^*$.

(v) If $F : M \to N$ is a diffeomorphism, then $F^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ is an isomorphism.

Proof.

(i) We have

$$\begin{aligned} H^0_{\mathrm{dR}}(M) &= \{f \in C^\infty(M, \mathbb{R}) : \mathrm{d}f = 0\} \\ &= \{\text{locally constant functions } f\} \\ &= \mathbb{R}^{\text{number of connected components}}. \end{aligned}$$

(ii) If $p > \dim M$, then all p-forms are trivial.

(iii) We first show that $F^*\omega$ indeed represents some member of $H^p_{\mathrm{dR}}(M)$. Let $[\omega] \in H^p_{\mathrm{dR}}(N)$. Then $\mathrm{d}\omega = 0$. So

$$\mathrm{d}(F^*\omega) = F^*(\mathrm{d}\omega) = 0.$$

So $[F^*\omega] \in H^p_{\mathrm{dR}}(M)$. So this map makes sense.

To see it is well-defined, if $[\omega] = [\omega']$, then $\omega - \omega' = \mathrm{d}\sigma$ for some σ. So $F^*\omega - F^*\omega' = \mathrm{d}(F^*\sigma)$. So $[F^*\omega] = [F^*\omega']$.

(iv) Follows from the corresponding fact for pullback of differential forms.

(v) If F^{-1} is an inverse to F, then $(F^{-1})^*$ is an inverse to F^* by above. □

Theorem (Homotopy invariance)**.** Let F_0, F_1 be homotopic maps. Then $F_0^* = F_1^* : H^p_{dR}(N) \to H^p_{\mathrm{dR}}(M)$.

Proof. Let $F : [0, 1] \times M \to N$ be the homotopy, and

$$F_t(x) = F(t, x).$$

We denote the exterior derivative on M by d_M (and similarly d_N), and that on $[0, 1] \times M$ by d.

Let $\omega \in \Omega^p(N)$ be such that $\mathrm{d}_N\omega = 0$. We let t be the coordinate on $[0, 1]$. We write

$$F^*\omega = \sigma + \mathrm{d}t \wedge \gamma,$$

where $\sigma = \sigma(t) \in \Omega^p(M)$ and $\gamma = \gamma(t) \in \Omega^{p-1}(M)$. We claim that

$$\sigma(t) = F_t^*\omega.$$

Indeed, we let $\iota : \{t\} \times M \to [0, 1] \times M$ be the inclusion. Then we have

$$\begin{aligned} F_t^*\omega|_{\{t\}\times M} &= (F \circ \iota)^*\omega = \iota^*F^*\omega \\ &= \iota^*(\sigma + \mathrm{d}t \wedge \gamma) \\ &= \iota^*\sigma + \iota^*\mathrm{d}t \wedge \iota^*\gamma \\ &= \iota^*\sigma, \end{aligned}$$

using the fact that $\iota^*\mathrm{d}t = 0$. As $\mathrm{d}_N\omega = 0$, we have

$$\begin{aligned} 0 &= F^*\mathrm{d}_N\omega \\ &= \mathrm{d}F^*\omega \\ &= \mathrm{d}(\sigma + \mathrm{d}t \wedge \gamma) \\ &= \mathrm{d}_M(\sigma) + (-1)^p\frac{\partial\sigma}{\partial t} \wedge \mathrm{d}t + \mathrm{d}t \wedge \mathrm{d}_M\gamma \\ &= \mathrm{d}_M\sigma + (-1)^p\frac{\partial\sigma}{\partial t} \wedge \mathrm{d}t + (-1)^{p-1}\mathrm{d}_M\gamma \wedge \mathrm{d}t. \end{aligned}$$

Looking at the dt components, we have

$$\frac{\partial \sigma}{\partial t} = \mathrm{d}_M \gamma.$$

So we have

$$F_1^* \omega - F_0^* \omega = \sigma(1) - \sigma(0) = \int_0^1 \frac{\partial \sigma}{\partial t}\,\mathrm{d}t = \int_0^1 \mathrm{d}_M \gamma \;\mathrm{d}t = \mathrm{d}_M \int_0^1 \gamma(t)\,\mathrm{d}t.$$

So we know that

$$[F_1^* \omega] = [F_0^* \omega].$$

So done. □

Corollary (Poincaré lemma)**.** Let $U \subseteq \mathbb{R}^n$ be open and star-shaped. Suppose $\omega \in \Omega^p(U)$ is such that $\mathrm{d}\omega = 0$. Then there is some $\sigma \in \Omega^{p-1}(M)$ such that $\omega = \mathrm{d}\sigma$.

Proof. $H^p_{\mathrm{dR}}(U) = 0$ for $p \geq 1$. □

Corollary. If M and N are smoothly homotopy equivalent, then $H^p_{\mathrm{dR}}(M) \cong H^p_{\mathrm{dR}}(N)$.

5.3 Homological algebra and Mayer-Vietoris theorem

Proposition. A cochain map induces a well-defined homomorphism on the cohomology groups.

Theorem (Snake lemma)**.** Suppose we have a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{\;i\;} B^\bullet \xrightarrow{\;q\;} C^\bullet \longrightarrow 0\ ,$$

i.e. the i, q are cochain maps and we have a short exact sequence

$$0 \longrightarrow A^p \xrightarrow{\;i^p\;} B^p \xrightarrow{\;q^p\;} C^p \longrightarrow 0\ ,$$

for each p.

Then there are maps

$$\delta : H^p(C^\bullet) \to H^{p+1}(A^\bullet)$$

such that there is a long exact sequence

$$\cdots \longrightarrow H^p(A^\bullet) \xrightarrow{\;i^*\;} H^p(B^\bullet) \xrightarrow{\;q^*\;} H^p(C^\bullet) \xrightarrow{\;\delta\;} H^{p+1}(A^\bullet) \xrightarrow{\;i^*\;} H^{p+1}(B^\bullet) \xrightarrow{\;q^*\;} H^{p+1}(C^\bullet) \longrightarrow \cdots.$$

Theorem (Mayer-Vietoris theorem)**.** Let M be a manifold, and $M = U \cup V$, where U, V are open. We denote the inclusion maps as follows:

$$\begin{array}{ccc} U \cap V & \overset{i_1}{\hookrightarrow} & U \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ V & \overset{j_2}{\hookrightarrow} & M \end{array}$$

Then there exists a natural linear map

$$\delta : H^p_{\mathrm{dR}}(U \cap V) \to H^{p+1}_{\mathrm{dR}}(M)$$

such that the following sequence is exact:

$$H^p_{\mathrm{dR}}(M) \xrightarrow{j_1^* \oplus j_2^*} H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \xrightarrow{i_1^* - i_2^*} H^p_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{p+1}_{\mathrm{dR}}(M) \xrightarrow{j_1^* \oplus j_2^*} H^{p+1}_{\mathrm{dR}}(U) \oplus H^{p+1}_{\mathrm{dR}}(V) \xrightarrow{i_1^* - i_2^*} \cdots$$

Proof of Mayer-Vietoris. By the snake lemma, it suffices to prove that the following sequence is exact for all p:

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{j_1^* \oplus j_2^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i_1^* - i_2^*} \Omega^p(U \cap V) \longrightarrow 0$$

It is clear that the two maps compose to 0, and the first map is injective. By counting dimensions, it suffices to show that $i_1^* - i_2^*$ is surjective.

Indeed, let $\{\varphi_U, \varphi_V\}$ be partitions of unity subordinate to $\{U, V\}$. Let $\omega \in \Omega^p(U \cap V)$. We set $\sigma_U \in \Omega^p(U)$ to be

$$\sigma_U = \begin{cases} \varphi_V \omega & \text{on } U \cap V \\ 0 & \text{on } U \setminus \operatorname{supp} \varphi_V \end{cases}.$$

Similarly, we define $\sigma_V \in \Omega^p(V)$ by

$$\sigma_V = \begin{cases} -\varphi_U \omega & \text{on } U \cap V \\ 0 & \text{on } V \setminus \operatorname{supp} \varphi_U \end{cases}.$$

Then we have

$$i_1^* \sigma_U - i_2^* \sigma_V = (\varphi_V \omega + \varphi_U \omega)|_{U \cap V} = \omega.$$

So $i_1^* - i_2^*$ is surjective. □

6 Integration

6.1 Orientation

6.2 Integration

Lemma. Let $F : D \to E$ be a smooth map between domains of integration in $\mathbb{R}^n$, and assume that $F|_{\mathring{D}} : \mathring{D} \to \mathring{E}$ is an orientation-preserving diffeomorphism. Then

$$\int_E \omega = \int_D F^* \omega.$$

Proof. Suppose we have coordinates $x_1, \cdots, x_n$ on D and $y_1, \cdots, y_n$ on E. Write

$$\omega = f \; \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n.$$

Then we have

$$\begin{aligned}\int_E \omega &= \int_E f \; \mathrm{d}y_1 \cdots \mathrm{d}y_n \\ &= \int_D (f \circ F) \, |\det \mathrm{D}F| \; \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_D (f \circ F) \det \mathrm{D}F \; \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_D F^* \omega.\end{aligned}$$

Here we used the fact that $|\det \mathrm{D}F| = \det \mathrm{D}F$ because F is orientation-preserving. □

Lemma. This is well-defined, i.e. it is independent of cover and partition of unity.

Theorem. Given a parametrization $\{S_i\}$ of M and an $\omega \in \Omega^n(M)$ with compact support, we have

$$\int_M \omega = \sum_i \int_{D_i} F_i^* \omega.$$

Proof. By using partitions of unity, we may consider the case where ω has support in a single chart, and thus we may wlog assume we are working on $\mathbb{R}^n$, and then the result is obvious. □

Lemma. Let M be an oriented manifold, and g a Riemannian metric on M. Then there is a unique $\omega \in \Omega^n(M)$ such that for all p, if $e_1, \cdots, e_n$ is an oriented orthonormal basis of T_pM, then

$$\omega(e_1, \cdots, e_n) = 1.$$

We call this the *Riemannian volume form*, written $\mathrm{d}V_g$.

Proof. Uniqueness is clear, since if ω' is another, then $\omega_p = \lambda \omega'_p$ for some λ, and evaluating on an orthonormal basis shows that $\lambda = 1$.

To see existence, let σ be any nowhere vanishing n-form giving the orientation of M. On a small set U, pick a frame $s_1, \cdots, s_n$ for $TM|_U$ and apply the Gram-Schmidt process to obtain an orthonormal frame $e_1, \cdots, e_n$, which we may wlog assume is oriented. Then we set

$$f = \sigma(e_1, \cdots, e_n),$$

which is non-vanishing because σ is nowhere vanishing. Then set

$$\omega = \frac{\sigma}{f}.$$

This proves existence locally, and can be patched together globally by uniqueness. □

6.3 Stokes Theorem

Proposition. Let M be a manifold with boundary. Then $\mathrm{Int}(M)$ and ∂M are naturally manifolds, with

$$\dim \partial M = \dim \mathrm{Int}\, M - 1.$$

Lemma. Let $p \in \partial M$, say $p \in U \subseteq M$ where (U, φ) is a chart (with boundary). Then

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis for T_pM. In particular, $\dim T_pM = n$.

Proof. Since this is a local thing, it suffices to prove it for $M = \mathbb{H}^n$. We write $C^\infty(\mathbb{H}, \mathbb{R})$ for the functions $f : \mathbb{H}^n \to \mathbb{R}^n$ that extend smoothly to an open neighbourhood of $\mathbb{H}^n$. We fix $a \in \partial\mathbb{H}^n$. Then by definition, we have

$$T_a\mathbb{H}^n = \mathrm{Der}_a(C^\infty(\mathbb{H}^n, \mathbb{R})).$$

We let $i_* : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ be given by

$$i_*(X)(g) = X(g|_{\mathbb{H}^n})$$

We claim that i_* is an isomorphism. For injectivity, suppose $i_*(X) = 0$. If $f \in C^\infty(\mathbb{H}^n)$, then f extends to a smooth g on some neighbourhood U of $\mathbb{H}^n$. Then

$$X(f) = X(g|_{\mathbb{H}^n}) = i_*(X)(g) = 0.$$

So $X(f) = 0$ for all f. Then $X = 0$. So i_* is injective.

To see surjectivity, let $Y \in T_a\mathbb{R}^n$, and let $X \in T_a\mathbb{H}^n$ be defined by

$$X(f) = Y(g),$$

where $g \in C^\infty(\mathbb{H}^n, \mathbb{R})$ is any extension of f to U. To see this is well-defined, we let

$$Y = \sum_{i=1}^n \alpha_i \left.\frac{\partial}{\partial x_i}\right|_a.$$

Then

$$Y(g) = \sum_{i=1}^n \alpha_i \frac{\partial g}{\partial x_i}(a),$$

which only depends on $g|_{\mathbb{H}^n}$, i.e. f. So X is a well-defined element of $T_a\mathbb{H}^n$, and $i_*(X) = Y$ by construction. So done. □

Theorem (Stokes' theorem)**.** Let M be an oriented manifold with boundary of dimension n. Then if $\omega \in \Omega^{n-1}(M)$ has compact support, then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

In particular, if M has no boundary, then

$$\int_M \mathrm{d}\omega = 0$$

Proof. We first do the case where $M = \mathbb{H}^n$. Then we have

$$\omega = \sum_{i=1}^n \omega_i \; \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n,$$

where ω_i is compactly supported, and the hat denotes omission. So we have

$$\begin{aligned}
\mathrm{d}\omega &= \sum_i \mathrm{d}\omega_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\\
&= \sum_i \frac{\partial \omega_i}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\\
&= \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

Let's say

$$\mathrm{supp}(\omega) = \{x_j \in [-R, R] : j = 1, \cdots, n-1; x_n \in [0, R]\} = A.$$

Then suppose $i \neq n$. Then we have

$$\begin{aligned}
&\int_{\mathbb{H}^n} \frac{\partial \omega_i}{\partial x_i} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_i \wedge \cdots \wedge \mathrm{d}x_n\\
&= \int_A \frac{\partial \omega_i}{\partial x_i} \mathrm{d}x_1 \cdots \mathrm{d}x_n\\
&= \int_{-R}^R \int_{-R}^R \cdots \int_{-R}^R \int_0^R \frac{\partial \omega_i}{\partial x_i} \; \mathrm{d}x_1 \cdots \mathrm{d}x_n
\end{aligned}$$

By Fubini's theorem, we can integrate this in any order. We integrate with respect to $\mathrm{d}x_i$ first. So this is

$$= \pm \int_{-R}^R \cdots \int_{-R}^R \int_0^R \left(\int_{-R}^R \frac{\partial \omega_i}{\partial x_i} \; \mathrm{d}x_i\right) \mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_n$$

By the fundamental theorem of calculus, the inner integral is

$$\omega(x_1, \cdots, x_{i-1}, R, x_{i+1}, \cdots, x_n) - \omega(x_1, \cdots, x_{i-1}, -R, x_{i+1}, \cdots, x_n) = 0 - 0 = 0.$$

So the integral vanishes. So we are only left with the $i = n$ term. So we have

$$\begin{aligned}\int_{\mathbb{H}^n} \mathrm{d}\omega &= (-1)^{n-1} \int_A \frac{\partial \omega_n}{\partial x_n}\,\mathrm{d}x_1 \cdots \mathrm{d}x_n\\ &= (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \left(\int_0^R \frac{\partial \omega_n}{\partial x_n}\,\mathrm{d}x_n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}\end{aligned}$$

Now that integral is just

$$\omega_n(x_1, \cdots, x_{n-1}, R) - \omega_n(x_1, \cdots, x_{n-1}, 0) = -\omega_n(x_1, \cdots, x_{n-1}, 0).$$

So this becomes

$$= (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x_1, \cdots, x_{n-1}, 0)\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}.$$

Next we see that

$$i^*\omega = \omega_n \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1},$$

as $i^*(\mathrm{d}x_n) = 0$. So we have

$$\int_{\partial\mathbb{H}^n} i^*\omega = \pm \int_{A \cap \partial\mathbb{H}^n} \omega(x_1, \cdots, x_{n-1}, 0)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Here the sign is a plus iff $x_1, \cdots, x_{n-1}$ are an oriented coordinate for $\partial\mathbb{H}^n$, i.e. n is even. So this is

$$\int_{\partial\mathbb{H}^n} \omega = (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x_1, \cdots, x_{n-1}, 0)\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = \int_{\mathbb{H}^n} \mathrm{d}\omega.$$

Now for a general manifold M, suppose first that $\omega \in \Omega^{n-1}(M)$ is compactly supported in a single oriented chart (U, φ). Then the result is true by working in local coordinates. More explicitly, we have

$$\int_M \mathrm{d}\omega = \int_{\mathbb{H}^n} (\varphi^{-1})^*\mathrm{d}\omega = \int_{\mathbb{H}^n} \mathrm{d}((\varphi^{-1})^*\omega) = \int_{\partial\mathbb{H}^n} (\varphi^{-1})^*\omega = \int_{\partial M} \omega.$$

Finally, for a general ω, we just cover M by oriented charts (U, φ_α), and use a partition of unity χ_α subordinate to $\{U_\alpha\}$. So we have

$$\omega = \sum \chi_\alpha \omega.$$

Then

$$\mathrm{d}\omega = \sum (\mathrm{d}\chi_\alpha)\omega + \sum \chi_\alpha \mathrm{d}\omega = \mathrm{d}\left(\sum \chi_\alpha\right)\omega + \sum \chi_\alpha \mathrm{d}\omega = \sum \chi_\alpha \mathrm{d}\omega,$$

using the fact that $\sum \chi_\alpha$ is constant, hence its derivative vanishes. So we have

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_M \chi_\alpha \mathrm{d}\omega = \sum_\alpha \int_{\partial M} \chi_\alpha \omega = \int_{\partial M} \omega. \qquad \square$$

7 De Rham's theorem*

Theorem (de Rham's theorem)**.** There exists a natural isomorphism

$$H^p_{\mathrm{dR}}(M) \cong H^p(M, \mathbb{R}),$$

where $H^p(M, \mathbb{R})$ is the singular cohomology of M, and this is in fact an isomorphism of rings, where $H^p_{\mathrm{dR}}(M)$ has the product given by the wedge, and $H^p(M, \mathbb{R})$ has the cup product.

Theorem. The map $i_* : H_p^\infty(M) \to H_p(M)$ is an isomorphism.

Lemma. I is a well-defined map $H^p_{\mathrm{dR}}(M) \to H^p_\infty(M, \mathbb{R})$.

Proof. If $[\omega] = [\omega']$, then $\omega - \omega' = \mathrm{d}\alpha$. Then let $\sigma \in H^p_\infty(M, \mathbb{R})$. Then

$$\int_\sigma (\omega - \omega') = \int_\sigma \mathrm{d}\alpha = \int_{\partial\sigma} \alpha = 0,$$

since $\partial\sigma = 0$.

On the other hand, if $[\sigma] = [\sigma']$, then $\sigma - \sigma = \partial\beta$ for some β. Then we have

$$\int_{\sigma - \sigma'} \omega = \int_{\partial\beta} \omega = \int_\beta \mathrm{d}\omega = 0.$$

So this is well-defined. □

Lemma. I is functorial and commutes with the boundary map of Mayer-Vietoris. In other words, if $F : M \to N$ is smooth, then the diagram

$$\begin{array}{ccc} H^p_{\mathrm{dR}}(M) & \xrightarrow{F^*} & H^p_{\mathrm{dR}}(N) \\ \downarrow I & & \downarrow I \\ H^p_\infty(M) & \xrightarrow{F^*} & H^p_\infty(N) \end{array}.$$

And if $M = U \cup V$ and U, V are open, then the diagram

$$\begin{array}{ccc} H^p_{\mathrm{dR}}(U \cap V) & \xrightarrow{\delta} & H^{p+1}_{\mathrm{dR}}(U \cup V) \\ \downarrow I & & \downarrow I \\ H^p_\infty(U \cap V, \mathbb{R}) & \xrightarrow{\delta} & H^p(U \cup V, \mathbb{R}) \end{array}$$

also commutes. Note that the other parts of the Mayer-Vietoris sequence commute because they are induced by maps of manifolds.

Proof. Trace through the definitions. □

Proposition. Let $U \subseteq \mathbb{R}^n$ is convex, then

$$U : H^p_{\mathrm{dR}}(U) \to H^p_\infty(U, \mathbb{R})$$

is an isomorphism for all p.

Proof. If $p > 0$, then both sides vanish. Otherwise, we check manually that $I : H^0_{\mathrm{dR}}(U) \to H^0_\infty(U, \mathbb{R})$ is an isomorphism. □

Proposition. Suppose $\{U, V\}$ is a de Rham cover of $U \cup V$. Then $U \cup V$ is de Rham.

Proof. We use the five lemma! We write the Mayer-Vietoris sequence that is impossible to fit within the margins:

$$\begin{array}{ccccccccc} H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) & \longrightarrow & H^p_{\mathrm{dR}}(U \cup V) & \longrightarrow & H^{p+1}_{\mathrm{dR}}(U \cap V) & \longrightarrow & H^p_{\mathrm{dR}}(U) \oplus H^{p+1}_{\mathrm{dR}}(V) & \longrightarrow & H^{p+1}_{\mathrm{dR}}(U \cup V) \\ \downarrow{\scriptstyle I \oplus I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I \oplus I} & & \downarrow{\scriptstyle I} \\ H^p_\infty(U) \oplus H^p_\infty(V) & \longrightarrow & H^p_\infty(U \cup V) & \longrightarrow & H^{p+1}_\infty(U \cap V) & \longrightarrow & H^p_\infty(U) \oplus H^{p+1}_\infty(V) & \longrightarrow & H^{p+1}_\infty(U \cup V) \end{array}$$

This huge thing commutes, and all but the middle map are isomorphisms. So by the five lemma, the middle map is also an isomorphism. So done. □

Corollary. If $U_1, \cdots, U_k$ is a finite de Rham cover of $U_1 \cup \cdots \cup U_k = N$, then M is de Rham.

Proof. By induction on k. □

Proposition. The disjoint union of de Rham spaces is de Rham.

Proof. Let A_i be de Rham. Then we have

$$H^p_{\mathrm{dR}}\left(\coprod A_i\right) \cong \prod H^p_{\mathrm{dR}}(A_i) \cong \prod H^p_\infty(A_i) \cong H^p_\infty\left(\coprod A_i\right). \qquad \square$$

Lemma. Let M be a manifold. If it has a de Rham basis, then it is de Rham.

Proof sketch. Let $f : M \to \mathbb{R}$ be an "exhaustion function", i.e. $f^{-1}([-\infty, c])$ for all $c \in \mathbb{R}$. This is guaranteed to exist for any manifold. We let

$$A_m = \{q \in M : f(q) \in [m, m+1]\}.$$

We let

$$A'_m = \left\{q \in M : f(q) \in \left[m - \frac{1}{2}, m + \frac{3}{2}\right]\right\}.$$

Given any $q \in A_m$, there is some $U_{\alpha(q)} \subseteq A'_m$ in the de Rham basis containing q. As A_m is compact, we can cover it by a finite number of such U_{α_i}, with each $U_{\alpha_i} \subseteq A'_m$. Let

$$B_m = U_{\alpha_1} \cup \cdots \cup U_{\alpha_r}.$$

Since B_m has a finite de Rham cover, so it is de Rham. Observe that if $B_m \cap B_{\tilde{m}} \neq \emptyset$, then $\tilde{M} \in \{m, m-1, m+1\}$. We let

$$U = \bigcup_{m \text{ even}} B_m, \quad V = \bigcup_{m \text{ odd}} B_m.$$

Then this is a countable union of de Rham spaces, and is thus de Rham. Similarly, $U \cap V$ is de Rham. So $M = U \cup V$ is de Rham. □

Theorem. Any manifold has a de Rham basis.

Proof. If $U \subseteq \mathbb{R}^n$ is open, then it is de Rham, since there is a basis of convex sets $\{U_\alpha\}$ (e.g. take open balls). So they form a de Rham basis.

Finally, M has a basis of subsets diffeomorphic to open subsets of $\mathbb{R}^n$. So it is de Rham. □

8 Connections

8.1 Basic properties of connections

Proposition. For any X, ∇_X is linear in s over $\mathbb{R}$, and linear in X over $C^\infty(M)$. Moreover,

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s$$

for $f \in C^\infty(M)$ and $s \in \Omega^0(E)$.

Lemma. Given a linear connection ∇ and a path $\gamma : I \to M$, there exists a unique map $\mathrm{D}_t : J(\gamma) \to J(\gamma)$ such that

(i) $\mathrm{D}_t(fV) = \dot{f}V + f\mathrm{D}_tV$ for all $f \in C^\infty(I)$

(ii) If U is an open neighbourhood of $\mathrm{im}(\gamma)$ and $\tilde{V}$ is a vector field on U such that $\tilde{V}|_{\gamma(t)} = V_t$ for all $t \in I$, then

$$\mathrm{D}_t(V)|_t = \nabla_{\dot{\gamma}(0)}\tilde{V}.$$

We call D_t the *covariant derivative* along γ.

Lemma. Given a connection ∇ and vector fields $X, Y \in \mathrm{Vect}(M)$, the quantity $\nabla_X Y|_p$ depends only on the values of Y near p and the value of X at p.

Proof. It is clear from definition that this only depends on the value of X at p.

To show that it only depends on the values of Y near p, by linearity, we just have to show that if $Y = 0$ in a neighbourhood U of p, then $\nabla_X Y|_p = 0$. To do so, we pick a bump function χ that is identically 1 near p, then $\mathrm{supp}(X) \subseteq U$. Then $\chi Y = 0$. So we have

$$0 = \nabla_X(\chi Y) = \chi\nabla_X(Y) + X(\chi)Y.$$

Evaluating at p, we find that $X(\chi)Y$ vanishes since χ is constant near p. So $\nabla_X(Y) = 0$. □

Proof of previous lemma. We first prove uniqueness.

By a similar bump function argument, we know that $\mathrm{D}_tV|_{t_0}$ depends only on values of $V(t)$ near t_0. Suppose that locally on a chart, we have

$$V(t) = \sum_j V_j(t) \left.\frac{\partial}{\partial x_j}\right|_{\gamma(t)}$$

for some $V_j : I \to \mathbb{R}$. Then we must have

$$\mathrm{D}_tV|_{t_0} = \sum_j \dot{V}_j(t) \left.\frac{\partial}{\partial x_j}\right|_{\gamma(t_0)} + \sum_j V_j(t_0)\nabla_{\dot{\gamma}(t_0)}\frac{\partial}{\partial x_j}$$

by the Leibniz rule and the second property. But every term above is uniquely determined. So it follows that D_tV must be given by this formula.

To show existence, note that the above formula works locally, and then they patch because of uniqueness. □

Proposition. Any vector bundle admits a connection.

Proof. Cover M by U_α such that $E|_{U_\alpha}$ is trivial. This is easy to do locally, and then we can patch them up with partitions of unity. □

Proposition. The map d_E extends uniquely to $\mathrm{d}_E : \Omega^p(E) \to \Omega^{p+1}(E)$ such that d_E is linear and

$$\mathrm{d}_E(w \otimes s) = \mathrm{d}\omega \otimes s + (-1)^p \omega \wedge \mathrm{d}_E s,$$

for $s \in \Omega^0(E)$ and $\omega \in \Omega^p(M)$. Here $\omega \wedge \mathrm{d}_E s$ means we take the wedge on the form part of $\mathrm{d}_E s$. More generally, we have a wedge product

$$\begin{aligned} \Omega^p(M) \times \Omega^q(E) &\to \Omega^{p+q}(E) \\ (\alpha, \beta \otimes s) &\mapsto (\alpha \wedge \beta) \otimes s. \end{aligned}$$

More generally, the extension satisfies

$$\mathrm{d}_E(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^q \omega \wedge \mathrm{d}_E \xi,$$

where $\xi \in \Omega^p(E)$ and $\omega \in \Omega^q(M)$.

Proof. The formula given already uniquely specifies the extension, since every form is locally a sum of things of the form $\omega \otimes s$. To see this is well-defined, we need to check that

$$\mathrm{d}_E((f\omega) \otimes s) = \mathrm{d}_E(\omega \otimes (fs)),$$

and this follows from just writing the terms out using the Leibniz rule. The second part follows similarly by writing things out for $\xi = \eta \otimes s$. □

8.2 Geodesics and parallel transport

Theorem. Let ∇ be a linear connection on M, and let $W \in T_pM$. Then there exists a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ for some $\varepsilon > 0$ such that

$$\dot{\gamma}(0) = W.$$

Any two such geodesics agree on their common domain.

Lemma (Parallel transport). Let $t_0 \in I$ and $\xi \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field $V \in J(\gamma)$ such that $V(t_0) = \xi$. We call V the *parallel transport* of ξ along γ.

Proof. Suppose first that $\gamma(I) \subseteq U$ for some coordinate chart U with coordinates $x_1, \cdots, x_n$. Then $V \in J(\gamma)$ is parallel iff $\mathrm{D}_t V = 0$. We put

$$V = \sum V^j(t) \frac{\partial}{\partial x^j}.$$

Then we need

$$\dot{V}^k + V^j \dot{\gamma}^i \Gamma^k_{ij} = 0.$$

This is a first-order linear ODE in V with initial condition given by $V(t_0) = \xi$, which has a unique solution.

The general result then follows by patching, since by compactness, the image of γ can be covered by finitely many charts. □

8.3 Riemannian connections

Lemma. Let ∇ be a connection. Then ∇ is compatible with g if and only if for all $\gamma : I \to M$ and $V, W \in J(\gamma)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(V(t), W(t)) = g(\mathrm{D}_t V(t), W(t)) + g(V(t), \mathrm{D}_t W(t)). \tag{$*$}$$

Proof. Write it out explicitly in local coordinates. □

Corollary. If V, W are parallel along γ, then $g(V(t), W(t))$ is constant with respect to t.

Corollary. If γ is a geodesic, then $|\dot{\gamma}|$ is constant.

Corollary. Parallel transport is an isometry.

Proposition. τ is a tensor of type $(2, 1)$.

Proof. We have

$$\begin{aligned}
\tau(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\
&= f\nabla_X Y - Y(f)X - f\nabla_Y X - fXY + Y(fX) \\
&= f(\nabla_X Y - \nabla_Y X - [X, Y]) \\
&= f\tau(X, Y).
\end{aligned}$$

So it is linear.

We also have $\tau(X, Y) = -\tau(Y, X)$ by inspection. □

Theorem. Let M be a manifold with Riemannian metric g. Then there exists a unique torsion-free linear connection ∇ compatible with g.

Proof. In local coordinates, we write

$$g = \sum g_{ij}\ \mathrm{d}x_i \otimes \mathrm{d}x_j.$$

Then the connection is explicitly given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}),$$

where $g^{k\ell}$ is the inverse of g_{ij}.

We then check that it works. □

8.4 Curvature

Lemma. F_E is a tensor. In particular, $F_E \in \Omega^2(\mathrm{End}(E))$.

Proof. We have to show that F_E is linear over $C^\infty(M)$. We let $f \in C^\infty(M)$ and $s \in \Omega^0(E)$. Then we have

$$\begin{aligned}
F_E(fs) &= \mathrm{d}_E \mathrm{d}_E (fs) \\
&= \mathrm{d}_E(\mathrm{d}f \otimes s + f \mathrm{d}_E s) \\
&= \mathrm{d}^2 f \otimes s - \mathrm{d}f \wedge \mathrm{d}_E s + \mathrm{d}f \wedge \mathrm{d}_E s + f \mathrm{d}_E^2 s \\
&= f F_E(s)
\end{aligned}$$

□

Lemma. We have

$$F_E(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

In other words, we have

$$F_E(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Proof. We claim that if $\mu \in \Omega^1(E)$, then we have

$$(\mathrm{d}_E \mu)(X, Y) = \nabla_X(\mu(Y)) - \nabla_Y(\mu(X)) - \mu([X, Y]).$$

To see this, we let $\mu = \omega \otimes s$, where $\omega \in \Omega^1(M)$ and $s \in \Omega^0(E)$. Then we have

$$\mathrm{d}_E \mu = \mathrm{d}\omega \otimes s - \omega \wedge \mathrm{d}_E s.$$

So we know

$$(\mathrm{d}_E \mu)(X, Y) = \mathrm{d}\omega(X, Y) \otimes s - (\omega \wedge \mathrm{d}_E s)(X, Y)$$

By a result in the example sheet, this is equal to

$$\begin{aligned} &= (X\omega(Y) - Y\omega(X) - \omega([X, Y])) \otimes s \\ &\quad\quad - \omega(X)\nabla_Y(s) + \omega(Y)\nabla_X(s) \\ &= X\omega(Y) \otimes s + \omega(Y)\nabla_X s \\ &\quad\quad - (Y\omega(X) \otimes s + \omega(X)\nabla_Y s) - \omega([X, Y]) \otimes s \end{aligned}$$

Then the claim follows, since

$$\begin{aligned} \mu([X, Y]) &= \omega([X, Y]) \otimes s \\ \nabla_X(\mu(Y)) &= \nabla_X(\omega(Y)s) \\ &= X\omega(Y) \otimes s + \omega(Y)\nabla_X s. \end{aligned}$$

Now to prove the lemma, we have

$$\begin{aligned} (F_E s)(X, Y) &= \mathrm{d}_E(\mathrm{d}_E s)(X, Y) \\ &= \nabla_X((\mathrm{d}_E s)(Y)) - \nabla_Y((\mathrm{d}_E s)(X)) - (\mathrm{d}_E s)([X, Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s. \end{aligned}$$

□

Theorem. Let M be a manifold with Riemannian metric g .Then M is flat iff it is locally isometric to $\mathbb{R}^n$.

Proposition. Let $\dim M = n$ and $U \subseteq M$ open. Let $V_1, \cdots, V_n \in \mathrm{Vect}(U)$ be such that

(i) For all $p \in U$, we know $V_1(p), \cdots, V_n(p)$ is a basis for T_pM, i.e. the V_i are a frame.

(ii) $[V_i, V_j] = 0$, i.e. the V_i form a frame that pairwise commutes.

Then for all $p \in U$, there exists coordinates $x_1, \cdots, x_n$ on a chart $p \in U_p$ such that

$$V_i = \frac{\partial}{\partial x_i}.$$

Suppose that g is a Riemannian metric on M and the V_i are orthonormal in T_pM. Then the map defined above is an isometry.

Proof. We fix $p \in U$. Let Θ_i be the flow of V_i. From example sheet 2, we know that since the Lie brackets vanish, the Θ_i commute.

Recall that $(\Theta_i)_t(q) = \gamma(t)$, where γ is the maximal integral curve of V_i through q. Consider

$$\alpha(t_1, \cdots, t_n) = (\Theta_n)_{t_n} \circ (\Theta_{n-1})_{t_{n-1}} \circ \cdots \circ (\Theta_1)_{t_1}(p).$$

Now since each of Θ_i is defined on some small neighbourhood of p, so if we just move a bit in each direction, we know that α will be defined for $(t_0, \cdots, t_n) \in B = \{|t_i| < \varepsilon\}$ for some small ε.

Our next claim is that

$$\mathrm{D}\alpha\left(\frac{\partial}{\partial t_i}\right) = V_i$$

whenever this is defined. Indeed, for $\mathbf{t} \in B$ and $f \in C^\infty(M, \mathbb{R})$. Then we have

$$\begin{aligned}\mathrm{D}\alpha\left(\left.\frac{\partial}{\partial t_i}\right|_t\right)(f) &= \left.\frac{\partial}{\partial t_i}\right|_t f(\alpha(t_1, \cdots, t_n))\\ &= \left.\frac{\partial}{\partial t_i}\right|_{\mathbf{t}} f((\Theta_i)_t \circ (\Theta_n)_{t_n} \circ \cdots \circ \widehat{(\Theta_i)_{t_i}} \circ \cdots \circ (\Theta_1)_{t_1}(p))\\ &= V_i|_{\alpha(\mathbf{t})}(f).\end{aligned}$$

So done. In particular, we have

$$\mathrm{D}\alpha|_0\left(\left.\frac{\partial}{\partial t_i}\right|_0\right) = V_i(p),$$

and this is a basis for T_pM. So $\mathrm{D}\alpha|_0 : T_0\mathbb{R}^n \to T_pM$ is an isomorphism. By the inverse function theorem, this is a local diffeomorphism, and in this chart, the claim tells us that

$$V_i = \frac{\partial}{\partial x_i}.$$

The second part with a Riemannian metric is clear. □

Proof of theorem. Let (M, g) be a flat manifold. We fix $p \in M$. We let $x_1, \cdots, x_n$ be coordinates centered at p_1, say defined for $|x_i| < 1$. We need to construct orthonormal vector fields. To do this, we pick an orthonormal basis at a point, and parallel transport it around.

We let $e_1, \cdots, e_n$ be an orthonormal basis for T_pM. We construct vector fields $E_1, \cdots, E_n \in \mathrm{Vect}(U)$ by parallel transport. We first parallel transport along $(x_1, 0, \cdots, 0)$ which defines $E_i(x_1, 0, \cdots, 0)$, then parallel transport along the x_2 direction to cover all $E_i(x_1, x_2, 0, \cdots, 0)$ etc, until we define on all of U. By construction, we have

$$\nabla_k E_i = 0 \tag{$*$}$$

on $\{x_{k+1} = \cdots = x_n = 0\}$.

We will show that the $\{E_i\}$ are orthonormal and $[E_i, E_j] = 0$ for all i, j. We claim that each E_i is parallel, i.e. for any curve γ, we have

$$\mathrm{D}_\gamma E_i = 0.$$

It is sufficient to prove that

$$\nabla_j E_i = 0$$

for all i, j.

By induction on k, we show

$$\nabla_j E_i = 0$$

for $j \leq k$ on $\{x_{k+1} = \cdots = x_n = 0\}$. The statement for $k = 1$ is already given by $(*)$. We assume the statement for k, so

$$\nabla_j E_i = 0 \tag{A}$$

for $j \leq k$ and $\{x_{k+1} = \cdots = x_n = 0\}$. For $j = k + 1$, we know that $\nabla_{k+1} E_i = 0$ on $\{x_{k+2} = \cdots = x_n = 0\}$ by $(*)$. So the only problem we have is for $j = k$ and $\{x_{k+2} = \cdots = x_n = 0\}$.

By flatness of the Levi-Civita connection, we have

$$[\nabla_{k+1}, \nabla_k] = \nabla_{[\partial_{k+1}, \partial_k]} = 0.$$

So we know

$$\nabla_{k+1} \nabla_k E_i = \nabla_k \nabla_{k+1} E_i = 0 \tag{B}$$

on $\{x_{k+2} = \cdots = x_n = 0\}$. Now at $x_{k+1} = 0$, we know $\nabla_k E_i$ vanishes. So it follows from parallel transport that $\nabla_k E_i$ vanishes on $\{x_{k+2} = \cdots = x_n = 0\}$.

As the Levi-Civita connection is compatible with g, we know that parallel transport is an isometry. So the inner product product $g(E_i, E_j) = g(e_i, e_j) = \delta_{ij}$. So this gives an orthonormal frame at all points.

Finally, since the torsion vanishes, we know

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0,$$

as the E_i are parallel. So we are done by the proposition. $\square$