Part III: Differential Geometry
(Version 2: October 14, 2016)

1. (Fundamentals about Smooth Maps) Verify the following:
   (*i*) If $U \subset M$ is an open set of a manifold then U has inherits the structure of a manifold such that the inclusion map is smooth.
   (*ii*) A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth as a map between manifolds (where $\mathbb{R}^n$ and $\mathbb{R}^m$ are given the standard smooth structure) if and only if it is smooth in the usual sense.
   (*iii*) Let (U, φ) be a chart on a manifold M. Then $\varphi\colon U \to \varphi(U)$ is a diffeomorphism.
   (*iv*) Compositions of smooth maps are smooth.
   (*v*) A smooth map between manifolds is continuous (with respect to the topology defined in lectures). [*Throughout this course you are expected to use standard results from analysis without proof.*]
2. (Products) Let M_1 and M_2 be smooth manifolds of dimension m_1 and m_2 respectively. Show that $M_1 \times M_2$ is naturally a manifold of dimension $m_1 + m_2$ and the projections $p_i\colon M_1 \times M_2 \to M_i$ are smooth maps. Show also that if N is another manifold then a map $f\colon N \to M_1 \times M_2$ is smooth if and only if $p_i \circ f$ is smooth for $i = 1, 2$
3. (Topology of manifolds)
   (*i*) Show any manifold is locally path connected and locally compact. That is if $p \in V$ where V is open, then $p \in U \subset V$ for some open U that is path connected, and whose closure is compact.
   (*ii*) Show that the open set U in (i) can be taken to be a chart given by a bijection $\varphi : U \to B_1$ where B_1 is the open unit ball in $\mathbb{R}^n$ and $\varphi(p) = 0$. Prove the same with B_1 replaced by $\mathbb{R}^n$. [*We will need very little material from the theory of topological spaces, but the statement of this exercise can be useful*]
4. (Dimension) Suppose a manifold M is connected. Prove that it has the same dimension at every point.
5. (Differentiable structure on $\mathbb{R}$) Do the charts $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$ $(x \in \mathbb{R})$ belong to the same atlas on the set $\mathbb{R}$? Let R_j, $j = 1, 2$, be the manifold defined by using the chart φ_j on the topological space $\mathbb{R}$. Are R_1 and R_2 diffeomorphic?
6. (Projective Space) Let $\mathbb{R}P^n$ be the set of lines in $\mathbb{R}^{n+1}$. Show how $\mathbb{R}P^n$ can be made into a manifold in such a way that the natural map $\pi\colon S^n \to \mathbb{R}P^n$ taking $v \in S^n$ to the line spanned by v is smooth. [*Hint: Any line in $\mathbb{R}^{n+1}$ is spanned by some non-zero vector $v = (v_0, \ldots, v_n)$. Start by defining charts on the set $U_i = \{v : v_i = 1\}$.* Now do the same to show $\mathbb{C}P^n$, the space of (complex) lines in $\mathbb{C}^{n+1}$, is a real manifold of dimension $2n$.

7. (*i*) Prove that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2.
   (*ii*) The natural map $(\mathbb{C}^2 \setminus \{\mathbf{0}\}) \to \mathbb{C}P^1$ induces a smooth map of manifolds $\pi\colon S^3 \to S^2$, called the *Poincaré map.* Show that the derivative of this map induces surjections on the tangent spaces, and $\pi^{-1}(p)$ is diffeomorphic to S^1 for all $p \in S^2$.

8. (Tangent Space) Let $p \in U$ where $U \subset M$ is open. Prove that for all sufficiently small open $V \subset U$ containing p and $f \in C^\infty(V, \mathbb{R})$ there exists a $g \in C^\infty(U, \mathbb{R})$ and an open $p \in W \subset U \cap V$ with $f|_W = g|_W$ *[Hint: bump functions].* Next show that the restriction map

$$C^\infty(U, \mathbb{R}) \to C^\infty(V, \mathbb{R}) \quad f \mapsto f|_V$$

induces a map

$$Der_p(C^\infty(V, \mathbb{R})) \to Der_p(C^\infty(U, \mathbb{R}))$$

which is an isomorphism for sufficiently small open $V \subset U$ containing p. Deduce that the definition of the tangent space given in lectures does not depend on any choice of open set.

9. Show that
   (*i*) TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$;
   (*ii*) TS^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$. [*More generally if G is a Lie group, then TG is diffeomorphic to $G \times \mathbb{R}^d$,where $d = \dim G$.*]

10. Let $f : M \to N$ be a diffeomorphism of manifolds. If X, Y denote smooth vector fields on M, define the corresponding vector fields f_*X, f_*Y on N. Show that f_* respects the relevant Lie brackets, i.e. that $f_*([X, Y]_M) = [f_*X, f_*Y]_N$ as vector fields on N.

11. Let $x_1, \ldots, x_n$ be local coordinates on a manifold, and suppose that vector fields X and Y are vector fields given locally by $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$. By constructing a suitable flow or otherwise, show from the definition of the Lie derivative that

$$L_X Y = \sum_{i=1}^n \sum_{j=1}^n \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

[*It may be helpful to use linearity*]. Use this to verify that $L_X Y = [X, Y]$.

12. For which values of $c \in \mathbb{R}$ is the zero locus in $\mathbb{R}^3$ of the polynomial

$$z^2 - (x^2 + y^2)^2 + c$$

an embedded manifold in $\mathbb{R}^3$, and for which values is it an immersed manifold?

13. Show a compact manifold M of dimension n can be embedded in $\mathbb{R}^N$ for some N. [*Hint: Start by using bump functions to prove the existence of a cover of M by finitely many charts $(U_\alpha, \varphi_\alpha)$ for $1 \leq \alpha \leq m$ and smooth functions ψ_α such that (1) ψ_α is supported in U_α and (2) $\psi_\alpha \equiv 1$ on some open set V_α and (3) the V_α cover M. Then*

define $f\colon M \to \mathbb{R}^{m(n+1)}$ *by* $f(p) = (\psi_1\varphi_1(p), \ldots, \psi_m\varphi_m(p), \psi_1, \ldots, \psi_m)$*. A harder theorem due to Whitney states it is possible to have* $N = 2n+1$*.]*

14. Prove that the map

$$\rho(x : y : z) = \frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, xy, yz, zx)$$

gives a well-defined *embedding* of $\mathbb{R}P^2$ into $\mathbb{R}^6$. Find on $\mathbb{R}^6$ a finite system of polynomials, of degree ≤ 2, whose common zero locus is precisely the image of ρ. Construct an embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$. [*Hint: Compose* ρ *with a suitable map.*]

15. Show that the following groups are Lie groups (in particular, smooth manifolds):

   (*i*) special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$;
   (*ii*) The special unitary group $SU(n) = \{A \in SL(n, \mathbb{C}) : AA^* = I\}$, where A^* denotes the conjugate transpose of A and I is the $n \times n$ identity matrix;
   (*iii*) $Sp(m) = \{A \in U(2m) : AJA^t = J\}$, where A^t denotes the transpose of A (no conjugation!) and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

   In each case, find the corresponding Lie algebra.

16. (Alternative definition of a manifold) It is said that the physicist's definition of a manifold is a "Lie group without the group structure". Discuss.

The last two exercises go slightly beyond the course and connect with some concepts with other courses which you may be taking. They are not necessarily hard, but will not be examined.

17. (Intrinsic/better definition of the tangent space). Given $p \in M$ define a relation on smooth functions f, g defined on open sets around p by $f \simeq g$ if there is an open nhood W around p such that $f|_W = g|_W$. This gives an equivalence relation. The equivalence class of a function f is called the *germ* of f at p and the set C_p^∞ of germs at p is an algebra. We define T_pM to be the space of derivation $C_p^\infty \to \mathbb{R}$ which is a vector space.

    Show that this definition is naturally isomorphic to the definition of the tangent space given in lectures for any chart U [*Hint: Given a function* f *defined on an nhood* W *of* p *let* φ *be a bump function supported on* W *that takes the value* 1 *on some* $W' \subset W$*. Then* φf *and* f *have the same germ at* p*, and start by showing that if* v *is a derivation of* C_p^∞ *then* $v(\varphi f) = v(f)$*.*]

18. (Connection with algebraic geometry) Let M be a manifold and $p \in M$. Show that the evaluation map $ev_p\colon C_p^\infty(M) \to \mathbb{R}$ given by $ev_p(f) = f(p)$ is a well-defined ring homomorphism, and that its kernel is the unique maximal ideal m in $C_p^\infty(M)$ (this says that $C_p^\infty(M)$ is a local ring). Given $v \in T_p(M)$ and $f \in m$ let $(v, f) := ev_p(v(f))$. Show this is a well-defined pairing and use it to show $T_p(M) = (m/m^2)^*$ where the star denotes the dual vector space.

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Part III Differential geometry (Version 3: November 16, 2016)

Example Sheet 2

1. (*i*) (Lie Derivative and Lie Bracket) Let $U \subset M$ be an subset such that $\overline{U}$ is compact. Using bump functions show that there exists a $\chi \in C^\infty(M, \mathbb{R})$ such that $supp(\chi)$ is compact and $\chi \equiv 1$ on $\overline{U}$. Deduce that $\tilde{X} := \chi X$ has compact support and is equal to X on U.

   (*ii*) Prove that $\tilde{X}$ is complete (or more generally any vector field with compact support is complete). What is the relationship between the flows of X and $\tilde{X}$?

   (*iii*) Show that $[\tilde{X}, Y] = [X, Y]$ and $\mathcal{L}_{\tilde{X}}(Y) = \mathcal{L}_X(Y)$ on U. Deduce from this that

   $$\mathcal{L}_X Y = [X, Y]$$

   for all $X, Y \in Vect(M)$. *[The idea is that from lectures we have proved this last statement under the assumption that X is complete, and from this you can deduce it generally. Alternatively you could go through the proof in lectures using the local flow.]*

2. (Commuting Vector Fields) Given a diffeomorphism $f : M \to M$ and $X \in Vect(M)$ we say that $f_* X = Y$ if

   $$Df_p(X_p) = Y_{f(p)} \text{ for all } p \in M.$$

   (*i*) Suppose $X \in Vect(M)$ and $Y \in Vect(M)$ with flows φ_t and ψ_t respectively. Show that $f_* X = Y$ if and only if $\psi_t \circ f = f \circ \varphi_t$ for all t [*as always this is understood to hold whenever both sides are defined, Hint: for $p \in M$ differentiate the curve $\gamma(t) = f(\varphi_t(p))$*]

   (*ii*) Now consider the curve in $T_p(M)$ given by

   $$X(t) := D\varphi_{-t}|_{\varphi_t(p)}(Y_{\varphi_t(p)})$$

   Explain why $\mathcal{L}_X Y = 0$ implies $X'(0) = 0$. Now show that if $\mathcal{L}_X Y = 0$ then $X(t)$ is constant with respect to t [*Hint: show that $X'(t_0) = 0$ for any given t_0 by making a change of variables.*]

   (*iii*) Use the above to prove that that $[X, Y] = 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all s, t.

3. (Properties of Tensor Product I) Let V and W be finite dimensional vector spaces. Give a construction of a linear isomorphism $\varphi_{VW} : V \otimes W \to W \otimes V$ in two ways (a) using the defining property of the tensor product given in lectures and (b) by picking a basis for V and W. In both cases show your definition of φ_{VW} is natural in the following sense: if $f : V \to V'$ and $g : W \to W'$ are linear then $(g \otimes f) \circ \varphi_{VW} = \varphi_{V'W'} \circ (f \otimes g)$. [*If you are feeling energetic then do the same for associativity*]

4. (Properties of Tensor Product II) Verify $V^* \otimes W$ is (naturally) isomorphic to $\mathrm{Hom}(V, W)$.

5. (Construction of Tensor Product) Let U, V, W be finite dimensional vector spaces and set $B := \mathrm{Bilinear}(V \times W, \mathbb{R})$. Given a vector space U and a bilinear $\alpha : V \times W \to U$

define $\tilde{\alpha}\colon B^* \to U^{**}$ as follows: If $\psi \in B^*$ and $\sigma \in U^*$ then $\tilde{\alpha}(\psi)(\sigma) = \psi(\sigma \circ \alpha)$. Use this to show that B^* satisfies the defining property of the tensor product as claimed in lectures. [*Use the natural isomorphism between a finite dimensional vector space and its double dual*].

6. (Exterior Product) Let V be a finite dimensional vector space of dimension n Prove that $\Lambda^0 V \simeq \mathbb{R}$ and $\Lambda^p V = \{0\}$ for $p > n$. Now suppose that $v_1, \ldots, v_n \in V$ is a basis and let $\alpha_1, \ldots, \alpha_n \in V^*$. Show that $v_1 \wedge \cdots \wedge v_n \in \Lambda^n V$ thought of as a multilinear map $M : V^* \times \cdots V^* \to \mathbb{R}$ is given by

$$M(\alpha_1, \ldots, \alpha_n) = \frac{1}{n!} \det A$$

where A is the matrix with entries $A_{ij} = (\alpha_i(v_j))$ Conclude that $v_1 \wedge \cdots \wedge v_n \neq 0$ is non-zero and that $\Lambda^n V$ has dimension 1.

7. (Bundle Maps) Let E, E' be vector bundles over M and suppose that $\alpha\colon C^\infty(M, E) \to C^\infty(M, E')$ is a map that is linear over $C^\infty(M)$. In this exercise you will verify that α is induced by a bundle map $F\colon E \to E'$.
   (*i*) Let $p \in M$ and $v \in E_p$. Show there is an $s \in C^\infty(M, E)$ such that $s(p) = v$ [*First work in a trivialisation then use a bump function*]
   (*ii*) Now let $s \in C^\infty(M, E)$. Show that if s vanishes in a neighbourhood of p then the same is true for $\alpha(s)$. Using this or otherwise, show that if $s(p) = 0$ then $\alpha(s)(p) = 0$ [*For the first use bump functions, for the second work in a local trivialisation*]
   (*iii*) Set $F(v) = \alpha(s)(p)$ where $s \in C^\infty(M, E)$ such that $s(p) = v$. Show that F is a well-defined bundle map (i.e. independent of s) and moreover it is smooth and finally show $\alpha(s) = F \circ s$.

8. (Contractions) Let V be a vector space, $X \in V$ and $\omega \in \Lambda^k V^*$. Define $i_X(\omega)$ by

$$i_X(\omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$$

Prove that $i_X(\omega) \in \Lambda^{k-1} V$. Prove also that if $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$ then

$$i_X(\omega \wedge \eta) = i(X)\omega \wedge \eta + (-1)^k \omega \wedge (i_X \eta)$$

Now on a smooth manifold M suppose $X \in Vect(M)$ and $\omega \in \Omega^p(M)$. Show how the formula

$$i_X \omega|_p := i_{X_p}(\omega_p)$$

gives an element of $\Omega^{p-1}(M)$.

9. (Differential Form Identity) Prove the identity $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, for a 1-form ω and vector fields X, Y. *Can you generalize this result to the case when ω is a p-form?

10. (Alternative definition of exterior derivative) Show directly that if ω is a 1-form and $X, Y \in \mathrm{Vect}(M)$ then $X\omega(Y) - Y\omega(X) - \omega([X, Y])$ defines a 2-form. [*Hint: show it is linear over smooth function. The point is one *could* use this expression to given*

an invariant definition *of $d\omega$, and the previous exercise shows this agrees with the definition from lectures*]

11. Let G be Lie group that is a subgroup of $GL_n(\mathbb{R})$ and X_i, $i = 1, \dots, d = \dim G$, be linearly independent left-invariant vector fields on G induced by a basis of $T_I G$. Show that the condition that $\omega^i(X_j) = \delta^i_j$ identically on G defines a system of pointwise linearly independent smooth 1-forms ω^i on G. Show further that the 1-forms ω^i are *left-invariant* in the sense that

$$L_g^*(\omega^i) = \omega^i, \qquad \text{for every } g \in G.$$

Let C^k_{ij} be a set of real constants determined by $[X_i, X_j] = \sum_k C^k_{ij} X_k$. Deduce from the identity of the previous question the formula

$$d\omega^k = -\frac{1}{2} \sum_{i,j} C^k_{ij} \omega^i \wedge \omega^j.$$

12. (de-Rham cohomology of S^1) Show that

$$d\omega = 0, \qquad \text{where} \quad \omega = \frac{-y dx + x dy}{x^2 + y^2},$$

but ω cannot be written as *df* for any smooth function f on $\mathbb{R}^2 \setminus \{0\}$. [*Consider an appropriate embedding of S^1 in $\mathbb{R}^2$ and integrate the pull-back of ω over S^1. We will soon see that the obstruction to writing an ω with $d\omega = 0$ as $\omega = df$ is captured by de-Rham cohomology*]

13. Given a form ω of degree $r > 0$ and a vector field $X \in Vect(M)$ define the Lie derivative $\mathcal{L}_X(\omega)$, and verify from your definition that this is again a form of degree r. If we let $i(X)\omega$ denote the interior product of X with ω prove that

$$\mathcal{L}_X \omega = i_X d\omega + d i_X \omega.$$

If ω is a closed 2-form with $\mathcal{L}_X \omega = 0$ on a manifold M with $H^1_{DR}(M) = 0$, deduce that $i(X)\omega = dH$ for some smooth function H on M. If $i_X \omega$ is non-zero at a point P, show that the level set of H through P is locally near P a codimension one submanifold of M, and that its tangent space at P is the codimension one subspace of $T_P M$ defined by $\{v \in T_P M \ : \ (i_X \omega)(v) = 0\}$.

The following exercises are aimed to connect with other courses you may be taking, and cover material that will not be examined.

14. (For those with some knowledge of algebraic topology) Let $\pi \colon G' \to G$ be a homomorphism between connected Lie groups which is a finite cover. Show that π induces an isomorphism of Lie algebras $\mathfrak{g}' \to \mathfrak{g}$. Deduce that if G is not simply connected then there exists a Lie group H and a Lie algebra morphism $\mathfrak{g} \to \mathfrak{h}$ that is not induced by a homormorphism of Lie groups between G and H [*Hint: Let $\pi \colon G' \to G$ be the universal cover. Make G' into a Lie group in a natural way so π is a homomorphism of Lie groups. Now set $H = G'$ and consider the identity map*

on Lie algebras. The point of this question is to illustrate that really the Lie-algebra is insensitive to finite covers, and thus the translation between statements about Lie groups and Lie algebras is most powerful when one assumes the group to be simply connected.]

15. (Again, for those with some knowledge of algebraic topology) Let G, H be a Lie groups, and suppose U is an open nhood of the identity in G. Let $\varphi\colon U \to H$ be such that $\varphi(ab) = \varphi(a)\varphi(b)$ whenever $a, b, ab \in U$.
    - (i) For each $g \in G$ consider pairs (V, ψ) where V is an open neighbourhood of g with $V.V^{-1} \subset U$ and $\psi\colon V \to H$ satisfies $\psi(a)\psi(b)^{-1} = \psi(ab^{-1})$ for $a, b \in V$. Define $(V, \psi) \sim (V', \psi')$ if $\psi = \psi'$ on some smaller nhood of g. Show how the set of equivalence classes as g ranges over G is naturally a Lie group that is a covering space of G.
    - (ii) Deduce that if G is simply connected that it is possible to extend φ to a Lie-group homomorphism $G \to H$. [*So the point of this exercise is to illustrate that for simply connected groups G knowing behaviour of some map on a neighbourhood of the identity gives a lot of information*]

16. (For those who like sheaves) Let E be a smooth vector bundle. For open sets $U \subset M$ let $\Gamma(U, E)$ be the space of sections of $E|_U$. Show that the assignment $U \to \Gamma(U, E)$ is a sheaf of vector spaces (in fact it is a sheaf of $C^\infty(M)$-modules by which we mean that $U \mapsto C^\infty(U)$ is a sheaf, and $\Gamma(U, E)$ is a $C^\infty(U)$-module and the restriction maps preserve this structure). Show also that this sheaf is fine. Prove that this sheaf is "flabby" in the sense that if $U \subset V$ are open sets then the restriction map $\Gamma(V, E) \to \Gamma(U, E)$ is surjective. [*look up on Wikipedia if you have not come across the definition of fine or flabby sheaves. The point of this question is that although there is some sheaf theory here, the sheaves have strong properties that mean that using sheaf-theory does not really help. Also for this question you may need the notion of "partition of unity" that will come soon in the course*]

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Part III 2016: Differential geometry (Version 1: November 11, 2016)

Example Sheet 3

1. (Hopf Line Bundle) For $p \in \mathbb{RP}^n$ let L_p denote the subspace $\mathbb{R}^{n+1}$ spanned by p. Show that the set
$$\{(p,w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : w \in L_p\}$$
is the total space of a rank 1 vector bundle on $\mathbb{RP}^n$ with projection map $\pi(p,w) = p$. [*This is called the Hopf line bundle*]

2. (Transition functions for vector bundles) Suppose $\{U_\alpha\}$ is an open cover of a manifold M and that for all α, β we have a smooth map $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_r(\mathbb{R})$ that satisfy the cocycle conditions. Prove that there exists a rank r vector bundle E on M whose transition functions are $\varphi_{\alpha\beta}$. *[For thought: how does this statement differ from the vector bundle construction theorem given in lectures? Also, as an extension for those interested, when do two sets of such data determine isomorphic vector bundles?.*]

3. (Metrics on vector bundles) Define what it means for g to be a smooth metric on a vector bundle E. Prove that any vector bundle E admits at least one smooth metric, and so in particular any manifold admits at least one Riemannian metric *[Hint: start locally in some trivialization*]. Prove that if g and g' are metrics on vector bundles E and F over the same manifold then there are induced metrics on E^* and $E \otimes F$. Show that a metric g on E induces a smooth bundle isomorphism between E and E^*.

4. (Cup Product) Show that the map
$$H^p_{dR}(M) \times H^q_{dR}(M) \to H^{p+q}_{dR}(M)$$
given by
$$([\omega], [\sigma]) \mapsto [\omega \wedge \sigma]$$
is well-defined.

5. Complete the proof from lectures that if U, V are open in M then the sequence
$$0 \to \Omega^p(U \cup V) \to \Omega^p(U) \oplus \Omega^p(V) \to \Omega^p(U \cap V) \to 0$$
is exact for all p.

6. (Naturality of Mayer-Vietoris) Let $f : M \to N$ be a smooth map between manifolds. Let U', V' be open in N and set $U = f^{-1}(U')$ and $V = f^{-1}(V')$. Show that the linear map defined in the statement of the Mayer-Vietoris Theorem is natural, in the sense that for all p the diagram
$$\begin{array}{ccc} H^p_{dR}(U' \cap V') & \xrightarrow{\delta} & H^{p+1}_{dR}(U' \cup V') \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^p_{dR}(U \cap V) & \xrightarrow{\delta} & H^{p+1}_{dR}(U \cup V) \end{array}$$

commutes.[*The most direct way to do this is to look carefully at the definition of* δ. *Another (almost equivalent) way is to first show the corresponding result for the diagram in the previous question. In any case, this is an expected result but is important as we will use it later on*]

7. (deRham Cohomology of Spheres)
   (i) Show that $\mathbb{R}^n \setminus 0$ is smoothly homotopy equivalent to S^{n-1}.
   (ii) Use the Mayer-Vietoris Theorem to compute the de-Rham cohomology of the sphere S^n. [*Hint: Use induction on* n *and cover the sphere by two pieces slightly larger than a hemisphere.*]

8. (i) Is $\alpha \wedge \alpha = 0$ true for every differential form α of positive degree?
   (ii) Let α be a nowhere-zero 1-form. Prove that for a $(p+1)$-form β $(p \geq 0)$, one has $\alpha \wedge \beta = 0$ if and only if $\beta = \alpha \wedge \gamma$ for some p-form γ. [*You might like to do it on* $\mathbb{R}^n$ *first. Partitions of unity are useful in the general case.*]

9. (Well-definedness of integration) Prove the following exercises set in lectures
   (i) Let U, V be open subsets of $\mathbb{R}^n$ and $G : U \to V$ an orientation preserving diffeomorphism. Show that for any compactly supported n-form ω on V we have
   $$\int_U G^*\omega = \int_V \omega.$$
   (ii) Now suppose that ω is an n-form on a manifold M with compact support in a chart (U, φ). Show that the definition
   $$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^*\omega$$
   does not depend on choice of such (U, φ).
   (iii) Finally use this to show that the definition of integration given in lectures using an open cover and partition of unity does not depend on the choice of cover (or partition of unity). *[Hint: first do the case that* ω *is supported in some chart]*

10. (Orientability)
    (i) Show that any Lie Group is orientable
    (ii) Show that $\mathbb{RP}^n$ is orientable if and only if n is odd.

11. (Induced Volume forms on boundaries)
    (i) Let M be an oriented manifold-with-boundary of dimension n. Show that the boundary ∂M is a manifold of dimension $n-1$ and the inclusion $\iota : \partial M \to M$ is smooth. If g is a Riemannian metric on M show that g induces a Riemannian metric $\tilde{g}$ on ∂M and that the induced volume forms satisfy
    $$dV_{\tilde{g}} = \iota_N(dV_g)|_{\partial M}$$
    where N is an outward normal vector field on ∂M normalised so $g(N, N) = 1$.

(ii) Now prove the identity

$$\iota_X(dV_g)|_{\partial M} = g(X,N)dV_{\tilde{g}}$$

where X is any vector field along ∂M.*[Hint: decompose X into a part normal to ∂M and a part tangential to ∂M]*

12. (Divergence Theorem) Let M be an oriented manifold-with-boundary of dimension $n$and g a Riemannian metric on M. Show that the map

$$* : \Omega^0(M) \to \Omega^n(M)$$

defined by

$$*f := f dV_g$$

is an isomorphism. For any $X \in Vect(M)$ let $div(X)$ be defined by

$$div(X)dV_g = d(\iota_X(dV_g))$$

(which is well defined by the first part of the question). Prove that if M is compact

$$\int_M (divX)dV_g = \int_{\partial M} g(X,N)dV_{\tilde{g}}.$$

where N is the outward pointing unit normal vector along ∂M and $\tilde{g}$ is the induced Riemannian metric on ∂M from the previous question.

13. (Hamiltonian Vector Fields) Let ω be a symplectic form on M. Verify that ω induces an isomorphism

$$\alpha : TM \to T^*M$$

Given $f \in C^\infty(M)$ the *Hamiltonian vector field* X_f associated to f is defined by

$$X_f = \alpha^{-1}(df)$$

Now consider $\mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Check that

$$\omega := \sum_i dx_i \wedge dy_i$$

is a symplectic form, and compute X_f for a smooth function $f(x_1, \ldots x_n, y_1, \ldots, y_n)$ in terms of the partial derivatives of f.

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Part III 2016: Differential geometry (Version 1: December 6, 2016)

Example Sheet 4

1. Let ∇ and ∇' be linear connections and set

$$A(X,Y) = \nabla_X Y - \nabla'_X Y$$

for $X, Y \in Vect(M)$. Show that ∇ and ∇' have the same geodesics if and only if $A(X,Y) = -A(Y,X)$ for all X, Y.

2. (*i*) Given a connection d_E on a vector bundle E we have the curvature $\mathcal{F} := d_E \circ d_E : \Omega^0(E) \to \Omega^2(E)$. As said in lectures this map can be considered as an element $\Omega^2(End(E))$ which for this question we shall denote by F. Show that, under the various identifications and abuse of notations involved, we have

$$\mathcal{F}(\sigma) = F \wedge \sigma \text{ for } \sigma \in \Omega^0(E).$$

(*ii*) Show also that for all $p \geq 0$, the map $d_E \circ d_E : \Omega^p(E) \to \Omega^{p+2}(E)$ is also given by $\sigma \mapsto F \wedge \sigma$.

3. Suppose that E and F are vector bundles on the same manifold, with connections ∇_E and ∇_F respectively. Verify the claim made in lectures that these induce connections on $E \otimes F$ and E^*. Now suppose that g is a Riemannian metric on M and that ∇ is a linear connection on M that is compatible with g. Show that the induced connection on the tensor bundle $T^k_l M$ is compatible with the induced metric.

4. (Connections in terms of parallel transport) Let ∇ be a linear connection and $\gamma : I \to M$ be a smooth curve. Assume $0 \in I$.

(*i*) Suppose that $e_1, \ldots, e_n$ is a basis for $T_{\gamma(0)}M$. Show that the parallel transport along γ gives vector fields $E_1, \ldots, E_n$ along γ such that $E_i(0) = e_i$ and such that each E_i is parallel along γ (such a collection is called a *parallel frame* along γ).

(*ii*) Now denote by $P_t \colon E_{\gamma(0)} \to E_{\gamma(t)}$ the parallel transport map along γ. Prove that if Y is a vector field then

$$\nabla_{\dot\gamma(0)} Y = \lim_{t \to 0} \frac{P_{-t}(Y(\gamma(t)) - Y(p)}{t}.$$

[*This justifies the terminology "connection" as from this it is possible to define connections in terms of parallel transport maps*]

5. (Compatible connections in terms of parallel transport) Let g be a Riemannian metric on M and ∇ be a linear connection. Prove that ∇ is compatible with g if and only if parallel transport along any curve is an isometry.

6. (The Bianchi Identity) Let d_E is a connection on a vector bundle E, and $d_{End(E)}$ denote the induced connection on $\mathrm{End}(E) = E \otimes E^*$. With $F \in \Omega^2(\mathrm{End}(E))$ denoting the curvature of d_E, prove that

$$d_{\mathrm{End(E)}}(F) = 0.$$

7. (Explicit formula for the Levi Cvita Connection) Let g be a Riemannian metric on M. Suppose we have local coordinates $x_1, \ldots, x_n$ and write $g = g_{ij} dx_i \otimes dx_j$ for smooth functions g_{ij}. If ∇ is a connection on M we defined the Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Show that ∇ defines the Levi-Civita connection if and only if

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

where g^{uv} is the inverse of g_{ij} (so $\sum_j g_{ij} g^{jv} = \delta_{iv}$).

8. Let S be an embedded submanifold of $\mathbb{R}^3$. By a regular parameterization of S we mean a diffeomorphism $\mathbf{r} : U \to V$ where $U \subset \mathbb{R}^2$ is open and V is an open subset of S. Let u, v be standard coordinates on $\mathbb{R}^2$. There is then a standard choice of a 'moving frame' (a basis of the tangent space $T_{\mathbf{r}}\mathbb{R}^3$) given by $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$ at every point $\mathbf{r}$ in V, where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ is a unit normal vector to S. (Here the subscripts u and v at $\mathbf{r}$ are used to denote the respective partial derivatives.) Show there is a unique way to write the second derivatives of $\mathbf{r}$ as

$$\begin{aligned} \mathbf{r}_{uu} &= \Gamma_{11}^1 \mathbf{r}_u + \Gamma_{11}^2 \mathbf{r}_v + L\mathbf{n} \\ \mathbf{r}_{uv} &= \Gamma_{12}^1 \mathbf{r}_u + \Gamma_{12}^2 \mathbf{r}_v + M\mathbf{n} \\ \mathbf{r}_{vv} &= \Gamma_{22}^1 \mathbf{r}_u + \Gamma_{22}^2 \mathbf{r}_v + N\mathbf{n}, \end{aligned}$$

for some functions Γ_{jk}^i, L, M, N on S. By deducing the expressions for Γ_{jk}^i in terms of the first fundamental form of S (i.e. the expression $Edu^2 + 2Fdudv + Gdv^2$ for the metric in terms of the coordinates u, v), or otherwise, show that the Γ_{jk}^i are the Christoffel symbols for the Levi-Cvita connection of the metric induced on S by restriction from the ambient $\mathbb{R}^3$.

9. Define hyperbolic space $\mathbb{H}$ to be the upper half plane $\{(x, y) : y > 0\}$ and set

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

Verify that g defines a Riemannian metric on $\mathbb{H}$, and compute the Christoffel symbols of the Levi-Civita connecion. Show that $\gamma = (u, v)$ is a geodesic if and only if $u''v = 2u'v'$ and $v''v = v'^2 - u'^2$ and use this to find the geodesics.

10. Let ∇ be a linear connection on a manifold M. We say ∇ is flat if its curvature F vanishes. Show the following are equivalent
   - (i) ∇ is flat
   - (ii) Around any point in M there exists a local frame for TM consisting of parallel vector fields
   - (iii) For any sufficiently small closed curve γ the holonomy map H is the identity. That is, for any $p \in M$ there exists an open $p \in U \subset M$ such that for any closed smooth curve γ in U starting and ending at p the parallel transport $P : T_pM \to T_pM$ is the identity. *[Hint: For (iii) implies (ii) start with a basis for T_pM and extend it to a basis for any point in U by parallel transport along some curve*]

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