Part III — Differential Geometry
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

This course is intended as an introduction to modern differential geometry. It can be taken with a view to further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are, for instance in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, smooth maps. Tangent vectors and vector fields, tangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Lie Groups.
- *Vector Bundles.* Structure group. The example of Hopf bundle. Bundle morphisms and automorphisms. Exterior algebra of differential forms. Tensors. Symplectic forms. Orientability of manifolds. Partitions of unity and integration on manifolds, Stokes Theorem; de Rham cohomology. Lie derivative of tensors. Connections on vector bundles and covariant derivatives: covariant exterior derivative, curvature. Bianchi identity.
- *Riemannian Geometry.* Connections on the tangent bundle, torsion. Bianchi's identities for torsion free connections. Riemannian metrics, Levi-Civita connection, Christoffel symbols, geodesics. Riemannian curvature tensor and its symmetries, second Bianchi identity, sectional curvatures.

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including bilinear forms) and multivariate calculus (e.g. differentiation and Taylor's theorem in several variables). Exposure to some of the ideas of classical differential geometry might also be useful.

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0 Introduction

1 Manifolds

1.1 Manifolds

Definition (Chart)**.** A *chart* (U, φ) on a set M is a bijection $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, where $U \subseteq M$ and $\varphi(U)$ is open.

A chart (U, φ) is *centered at* p for $p \in U$ if $\varphi(p) = 0$.

Definition (Smooth function)**.** Let (U, φ) be a chart on M and $f : M \to \mathbb{R}$. We say f is *smooth* or C^∞ at $p \in U$ if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth at $\varphi(p)$ in the usual sense.

$$\mathbb{R}^n \supseteq \varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f} \mathbb{R}$$

Definition (Atlas)**.** An *atlas* on a set M is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ on M such that

(i) $M = \bigcup_\alpha U_\alpha$.

(ii) For all α, β, we have $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$, and the transition function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth (in the usual sense).

Definition (Equivalent atlases)**.** Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas.

Definition (Differentiable structure)**.** A *differentiable structure* on M is a choice of equivalence class of atlases.

Definition (Manifold)**.** A *manifold* is a set M with a choice of differentiable structure whose topology is

(i) Hausdorff, i.e. for all $x, y \in M$, there are open neighbourhoods $U_x, U_y \subseteq M$ with $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.

(ii) Second countable, i.e. there exists a countable collection $(U_n)_{n \in \mathbb{N}}$ of open sets in M such that for all $V \subseteq M$ open, and $p \in V$, there is some n such that $p \in U_n \subseteq V$.

Definition (Local coordinates)**.** Let M be a manifold, and $\varphi : U \to \varphi(U)$ a chart of M. We can write

$$\varphi = (x_1, \cdots, x_n)$$

where each $x_i : U \to \mathbb{R}$. We call these the *local coordinates*.

Definition (Dimension)**.** If $p \in M$, we say M has *dimension* n at p if for one (thus all) charts $\varphi : U \to \mathbb{R}^m$ with $p \in U$, we have $m = n$. We say M has dimension n if it has dimension n at all points.

1.2 Smooth functions and derivatives

Definition (Smooth function)**.** A function $f : M \to N$ is *smooth at a point* $p \in M$ if there are charts (U, φ) for M and (V, ξ) for N with $p \in U$ and $f(p) \in V$ such that $\xi \circ f \circ \varphi^{-1} : \varphi(U) \to \xi(V)$ is smooth at $\varphi(p)$.

A function is *smooth* if it is smooth at all points $p \in M$.

A *diffeomorphism* is a smooth f with a smooth inverse.

We write $C^\infty(M, N)$ for the space of smooth maps $f : M \to N$. We write $C^\infty(M)$ for $C^\infty(M, \mathbb{R})$, and this has the additional structure of an algebra, i.e. a vector space with multiplication.

Definition (Curve)**.** A *curve* is a smooth map $I \to M$, where I is a non-empty open interval.

Definition (Derivation)**.** A *derivation* on an open subset $U \subseteq M$ at $p \in U$ is a linear map $X : C^\infty(U) \to \mathbb{R}$ satisfying the Leibniz rule

$$X(fg) = f(p)X(g) + g(p)X(f).$$

Definition (Tangent space)**.** Let $p \in U \subseteq M$, where U is open. The *tangent space* of M at p is the vector space

$$T_pM = \{\text{ derivations on } U \text{ at } p\,\} \equiv \mathrm{Der}_p(C^\infty(U)).$$

The subscript p tells us the point at which we are taking the tangent space.

Definition (Derivative)**.** Suppose $F \in C^\infty(M, N)$, say $F(p) = q$. We define $\mathrm{D}F|_p : T_pM \to T_qN$ by

$$\mathrm{D}F|_p(X)(g) = X(g \circ F)$$

for $X \in T_pM$ and $g \in C^\infty(V)$ with $q \in V \subseteq N$.

This is a linear map called the *derivative* of F at p.

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ & \underset{g\circ F}{\searrow} & \downarrow g \\ & & \mathbb{R} \end{array}$$

Definition (Derivative)**.** Let $\gamma : \mathbb{R} \to M$ be a smooth function. Then we write

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \dot{\gamma}(t) = \mathrm{D}\gamma|_t(1).$$

Definition ($\frac{\partial}{\partial x_i}$)**.** Given a chart $\varphi : U \to \mathbb{R}^n$ with $\varphi = (x_1, \cdots, x_n)$, we define

$$\left.\frac{\partial}{\partial x_i}\right|_p = (\mathrm{D}\varphi|_p)^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right) \in T_pM.$$

1.3 Bump functions and partitions of unity

Definition (Partition of unity)**.** Let $\{U_\alpha\}$ be an open cover of a manifold M. A *partition of unity* subordinate to $\{U_\alpha\}$ is a collection $\varphi_\alpha \in C^\infty(M, \mathbb{R})$ such that

(i) $0 \leq \varphi_\alpha \leq 1$

(ii) $\operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$

(iii) For all $p \in M$, all but finitely many $\varphi_\alpha(p)$ are zero.

(iv) $\sum_\alpha \varphi_\alpha = 1$.

1.4 Submanifolds

Definition (Embedded submanifold)**.** Let M be a manifold with $\dim M = n$, and S be a submanifold of M. We say S is an *embedded submanifold* if for all $p \in S$, there are coordinates $x_1, \cdots, x_n$ on some chart $U \subseteq M$ containing p such that

$$S \cap U = \{x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$$

for some k. Such coordinates are known as *slice coordinates* for S.

Definition (Immersed submanifold)**.** Let S, M be manifolds, and $\iota : S \hookrightarrow M$ be a smooth injective map with $\mathrm{D}\iota|_p : T_pS \to T_pM$ injective for all $p \in S$. Then we call (ι, S) an *immersed submanifold*. By abuse of notation, we identify S and $\iota(S)$.

Definition (Regular value)**.** Let $F \in C^\infty(M, N)$ and $c \in N$. Let $S = F^{-1}(c)$. We say c is a *regular value* if for all $p \in S$, the map $\mathrm{D}F|_p : T_pM \to T_cN$ is surjective.

2 Vector fields

2.1 The tangent bundle

Definition (Vector field)**.** A *vector field* on some $U \subseteq M$ is a smooth map $X : U \to TM$ such that for all $p \in U$, we have

$$X(p) \in T_pM.$$

In other words, we have $\pi \circ X = \mathrm{id}$.

Definition ($\mathrm{Vect}(U)$)**.** Let $\mathrm{Vect}(U)$ denote the set of all vector fields on U. Let $X, Y \in \mathrm{Vect}(U)$, and $f \in C^\infty(U)$. Then we can define

$$(X + Y)(p) = X(p) + Y(p), \quad (fX)(p) = f(p)X(p).$$

Then we have $X + Y, fX \in \mathrm{Vect}(U)$. So $\mathrm{Vect}(U)$ is a $C^\infty(U)$ module.

Moreover, if $V \subseteq U \subseteq M$ and $X \in \mathrm{Vect}(U)$, then $X|_V \in \mathrm{Vect}(V)$.

Conversely, if $\{V_i\}$ is a cover of U, and $X_i \in \mathrm{Vect}(V_i)$ are such that they agree on intersections, then they patch together to give an element of $\mathrm{Vect}(U)$. So we say that Vect is a *sheaf of $C^\infty(M)$ modules*.

Definition (Tangent bundle)**.** Let M be a manifold, and

$$TM = \bigcup_{p \in M} T_pM.$$

There is a natural projection map $\pi : TM \to M$ sending $v_p \in T_pM$ to p.

Let $x_1, \cdots, x_n$ be coordinates on a chart (U, φ). Then for any $p \in U$ and $v_p \in T_pM$, there are some $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ such that

$$v_p = \sum_{i=1}^n \alpha_i \left.\frac{\partial}{\partial x_i}\right|_p.$$

This gives a bijection

$$\begin{aligned} \pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^n \\ v_p &\mapsto (x_1(p), \cdots, x_n(p), \alpha_1, \cdots, \alpha_n), \end{aligned}$$

These charts make TM into a manifold of dimension $2 \dim M$, called the *tangent bundle* of M.

Definition (F-related)**.** Let M, N be manifolds, and $X \in \mathrm{Vect}(M)$, $Y \in \mathrm{Vect}(N)$ and $F \in C^\infty(M, N)$. We say they are *F-related* if

$$Y_q = \mathrm{D}F|_p(X_p)$$

for all $p \in M$ and $F(p) = q$. In other words, if the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{D}F} & TN \\ {\scriptstyle X}\uparrow & & {\scriptstyle Y}\uparrow \\ M & \xrightarrow{F} & N \end{array}.$$

Definition ($\mathrm{Der}(C^\infty(M))$)**.** Let $\mathrm{Der}(C^\infty(M))$ be the set of all $\mathbb{R}$-linear maps $\mathcal{X} : C^\infty(M) \to C^\infty(M)$ that satisfy

$$\mathcal{X}(fg) = f\mathcal{X}(g) + \mathcal{X}(f)g.$$

This is an $\mathbb{R}$-vector space, and in fact a $C^\infty(M)$ module.

Definition (Lie bracket)**.** If $X, Y \in \mathrm{Vect}(M)$, then the *Lie bracket* $[X, Y]$ is (the vector field corresponding to) the derivation $XY - YX \in \mathrm{Vect}(M)$.

Definition (Lie algebra)**.** A *Lie algebra* is a vector space V with a bracket $[\,\cdot\,,\,\cdot\,] : V \times V \to V$ such that

(i) $[\,\cdot\,,\,\cdot\,]$ is bilinear.

(ii) $[\,\cdot\,,\,\cdot\,]$ is antisymmetric, i.e. $[X, Y] = -[Y, X]$.

(iii) The *Jacobi identity* holds

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

2.2 Flows

Definition (Integral curve)**.** Let $X \in \mathrm{Vect}(M)$. An *integral curve* of X is a smooth $\gamma : I \to M$ such that I is an open interval in $\mathbb{R}$ and

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

Definition (Maximal integral curve)**.** Let $p \in M$, and $X \in \mathrm{Vect}(M)$. Let I_p be the union of all I such that there is an integral curve $\gamma : I \to M$ with $\gamma(0) = p$. Then there exists a unique integral curve $\gamma : I_p \to M$, known as the *maximal integral curve*.

Definition (Complete vector field)**.** A vector field is *complete* if $I_p = \mathbb{R}$ for all $p \in M$.

2.3 Lie derivative

Notation. Let $F : M \to M$ be a diffeomorphism, and $g \in C^\infty(M)$. We write

$$F^* g = g \circ F \in C^\infty(M).$$

Definition (Lie derivative of a function)**.** Let X be a complete vector field, and Θ be its flow. We define the *Lie derivative* of g along X by

$$\mathcal{L}_X(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_t^* g.$$

Here this is defined pointwise, i.e. for all $p \in M$, we define

$$\mathcal{L}_X(g)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_t^*(g)(p).$$

Notation. Let $Y \in \mathrm{Vect}(M)$, and $F : M \to M$ be a diffeomorphism. Then $\mathrm{D}F^{-1}|_{F(p)} : T_{F(p)}M \to T_pM$. So we can write

$$F^*(Y)|_p = \mathrm{D}F^{-1}|_{F(p)}(Y_{F(p)}) \in T_pM.$$

Then $F^*(Y) \in \mathrm{Vect}(M)$. If $g \in C^\infty(M)$, then

$$F^*(Y)|_p(g) = Y_{F(p)}(g \circ F^{-1}).$$

Alternatively, we have

$$F^*(Y)|_p(g \circ F) = Y_{F(p)}(g).$$

Removing the p's, we have

$$F^*(Y)(g \circ F) = (Y(g)) \circ F.$$

Definition (Lie derivative of a vector field)**.** Let $X \in \mathrm{Vect}(M)$ be complete, and $Y \in \mathrm{Vect}(M)$ be a vector field. Then the *Lie derivative* is given pointwise by

$$\mathcal{L}_X(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Theta_t^*(Y).$$

3 Lie groups

Definition (Lie group)**.** A *Lie group* is a manifold G with a group structure such that multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are smooth maps.

Notation. Let G be a Lie group and $g \in G$. We write $L_g : G \to G$ for the diffeomorphism

$$L_g(h) = gh.$$

Definition (Left invariant vector field)**.** Let $X \in \mathrm{Vect}(G)$ be a vector field. This is *left invariant* if

$$\mathrm{D}L_g|_h(X_h) = X_{gh}$$

for all $g, h \in G$.

We write $\mathrm{Vect}^L(G)$ for the collection of all left invariant vector fields.

Definition (Lie algebra of a Lie group)**.** Let G be a Lie group. The *Lie algebra* $\mathfrak{g}$ of G is the Lie algebra T_eG whose Lie bracket is induced by that of the isomorphism with $\mathrm{Vect}^L(G)$. So

$$[\xi, \eta] = [X_\xi, X_\eta]|_e.$$

We also write $\mathrm{Lie}(G)$ for $\mathfrak{g}$.

Definition (Lie group homomorphisms)**.** Let G, H be Lie groups. A *Lie group homomorphism* is a smooth map that is also a homomorphism.

Definition (Lie algebra homomorphism)**.** Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. Then a *Lie algebra homomorphism* is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$ such that

$$\beta[\xi, \eta] = [\beta(\xi), \beta(\eta)]$$

for all $\xi, \eta \in \mathfrak{g}$.

Definition (Exponential map)**.** The *exponential map* of a Lie group G is $\exp : \mathfrak{g} \to G$ given by

$$\exp(\xi) = \gamma_\xi(1),$$

where γ_ξ is the integral curve of X_ξ through $e \in G$.

Definition (Lie subgroup)**.** A *Lie subgroup* of G is a subgroup H with a smooth structure on H making H an *immersed* submanifold.

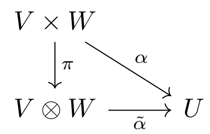
4 Vector bundles

4.1 Tensors

Definition (Bilinear map)**.** Let U, V, W be vector spaces. We define $\mathrm{Bilin}(V \times W, U)$ to be the functions $V \times W \to U$ that are bilinear, i.e.

$$\begin{aligned}\alpha(\lambda_1 v_1 + \lambda_2 v_2, w) &= \lambda_1 \alpha(v_1, w) + \lambda_2 \alpha(v_2, w)\\ \alpha(v, \lambda_1 w_1 + \lambda_2 w_2) &= \lambda_1 \alpha(v, w_1) + \lambda_2 \alpha(v, w_2).\end{aligned}$$

Definition (Tensor product)**.** A *tensor product* of two vector spaces V, W is a vector space $V \otimes W$ and a bilinear map $\pi : V \times W \to V \otimes W$ such that a bilinear map from $V \times W$ is "the same as" a linear map from $V \otimes W$. More precisely, given any bilinear map $\alpha : V \times W \to U$, we can find a unique linear map $\tilde{\alpha} : V \otimes W \to U$ such that the following diagram commutes:



So we have

$$\mathrm{Bilin}(V \times W, U) \cong \mathrm{Hom}(V \otimes W, U).$$

Given $v \in V$ and $w \in W$, we obtain $\pi(v, w) \in V \otimes W$, called the *tensor product* of v and w, written $v \otimes w$.

Definition (Covariant tensor)**.** A *covariant tensor* of rank k on V is an element of

$$\alpha \in \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}},$$

i.e. α is a multilinear map $V \times \cdots \times V \to \mathbb{R}$.

Definition (Tensor)**.** A *tensor* of type (k, ℓ) is an element in

$$T^k_\ell(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{\ell \text{ times}}.$$

Definition (Exterior product)**.** Consider

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

as an algebra (with multiplication given by the tensor product) (with $V^{\otimes 0} = \mathbb{R}$). We let $I(V)$ be the ideal (as algebras!) generated by $\{v \otimes v : v \in V\} \subseteq T(V)$. We define

$$\Lambda(V) = T(V)/I(V),$$

with a projection map $\pi : T(V) \to \Lambda(V)$. This is known as the *exterior algebra*. We let

$$\Lambda^k(V) = \pi(V^{\otimes k}),$$

the *k-th exterior product* of V.

We write $a \wedge b$ for $\pi(\alpha \otimes \beta)$.

4.2 Vector bundles

Definition (Vector bundle)**.** A *vector bundle* of rank r on M is a smooth manifold E with a smooth $\pi : E \to M$ such that

(i) For each $p \in M$, the fiber $\pi^{-1}(p) = E_p$ is an r-dimensional vector space,

(ii) For all $p \in M$, there is an open $U \subseteq M$ containing p and a diffeomorphism

$$t : E|_U = \pi^{-1}(U) \to U \times \mathbb{R}^r$$

such that

$$\begin{array}{ccc} E|_U & \xrightarrow{t} & U \times \mathbb{R}^r \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle p_1} & \\ U & & \end{array}$$

commutes, and the induced map $E_q \to \{q\} \times \mathbb{R}^r$ is a linear isomorphism for all $q \in U$.

We call t a *trivialization* of E over U; call E the *total space*; call M the *base space*; and call π the *projection*. Also, for each $q \in M$, the vector space $E_q = \pi^{-1}(\{q\})$ is called the *fiber* over q.

Note that the vector space structure on E_p is part of the data of a vector bundle.

Definition (Section)**.** A *(smooth) section* of a vector bundle $E \to M$ over some open $U \subseteq M$ is a smooth $s : U \to E$ such that $s(p) \in E_p$ for all $p \in U$, that is $\pi \circ s = \mathrm{id}$. We write $C^\infty(U, E)$ for the set of smooth sections of E over U.

Definition (Transition function)**.** Suppose that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ and $t_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{R}^r$ are trivializations of E. Then

$$t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is fiberwise linear, i.e.

$$t_\alpha \circ t_\beta^{-1}(q, v) = (q, \varphi_{\alpha\beta}(q)v),$$

where $\varphi_{\alpha\beta}(q)$ is in $\mathrm{GL}_r(\mathbb{R})$.

In fact, $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{R})$ is smooth. Then $\varphi_{\alpha\beta}$ is known as the *transition function* from β to α.

Definition (Direct sum of vector bundles)**.** Let $E, \tilde{E}$ be vector bundles on M. Suppose $t_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r$ is a trivialization for E over U_α, and $\tilde{t}_\alpha : \tilde{E}|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^{\tilde{r}}$ is a trivialization for $\tilde{E}$ over U_α.

We let $\varphi_{\alpha\beta}$ be transition functions for $\{t_\alpha\}$ and $\tilde{\varphi}_{\alpha\beta}$ be transition functions for $\{\tilde{t}_\alpha\}$.

Define

$$E \oplus \tilde{E} = \bigcup_p E_p \oplus \tilde{E}_p,$$

and define

$$T_\alpha : (E \oplus \tilde{E})|_{U_\alpha} = E|_{U_\alpha} \oplus \tilde{E}|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^r \oplus \mathbb{R}^{\tilde{r}}) = U_\alpha \times \mathbb{R}^{r+\tilde{r}}$$

be the fiberwise direct sum of the two trivializations. Then T_α clearly gives a linear isomorphism $(E \oplus \tilde{E})_p \cong \mathbb{R}^{r+\tilde{r}}$, and the transition function for T_α is

$$T_\alpha \circ T_\beta^{-1} = \varphi_{\alpha\beta} \oplus \tilde{\varphi}_{\alpha\beta},$$

which is clearly smooth. So this makes $E \oplus \tilde{E}$ into a vector bundle.

Definition (Tensor product of vector bundles)**.** Given two vector bundles $E, \tilde{E}$ over M, we can construct $E \otimes \tilde{E}$ similarly with fibers $(E \otimes \tilde{E})|_p = E|_p \otimes \tilde{E}|_p$.

Definition (Dual vector bundle)**.** Given a vector bundle $E \to M$, we define the *dual vector bundle* by

$$E^* = \bigcup_{p \in M} (E_p)^*.$$

Suppose again that $t_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r$ is a local trivialization. Taking the dual of this map gives

$$t_\alpha^* : U_\alpha \times (\mathbb{R}^r)^* \to E|_{U_\alpha}^*.$$

since taking the dual reverses the direction of the map. We pick an isomorphism $(\mathbb{R}^r)^* \to \mathbb{R}^r$ once and for all, and then reverse the above isomorphism to get a map

$$E|_{U_\alpha}^* \to U_\alpha \times \mathbb{R}^r.$$

This gives a local trivialization.

Definition (Cotangent bundle)**.** The *cotangent bundle* of a manifold M is

$$T^*M = (TM)^*.$$

In local coordinate charts, we have a frame $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ of TM over U. The dual frame is written as $\mathrm{d}x_1, \cdots, \mathrm{d}x_n$. In other words, we have

$$\mathrm{d}x_i|_p \in (T_pM)^*$$

and

$$\mathrm{d}x_i|_p \left(\left.\frac{\partial}{\partial x_j}\right|_p \right) = \delta_{ij}.$$

Definition (p-form)**.** A *p-form* on a manifold M over U is a smooth section of $\Lambda^p T^*M$, i.e. an element in $C^\infty(U, \Lambda^p T^*M)$.

Definition (Tensors on manifolds)**.** Let M be a manifold. We define

$$T_\ell^k M = \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ times}} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{\ell \text{ times}}.$$

A *tensor of type* (k, ℓ) is an element of

$$C^\infty(M, T_\ell^k M).$$

The convention when $k = \ell = 0$ is to set $T_0^0 M = M \times \mathbb{R}$.

Definition (Riemannian metric)**.** A *Riemannian metric* on M is a $(2,0)$-tensor g such that for all p, the bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric and positive definite, i.e. an inner product.

Given such a g and $v_p \in T_pM$, we write $\|v_p\|$ for $\sqrt{g_p(v_p, v_p)}$.

Definition (Length of curve)**.** Let $\gamma : I \to M$ be a curve. The *length* of γ is

$$\ell(\gamma) = \int_I \|\dot{\gamma}(t)\| \;\mathrm{d}t.$$

Definition (Vector bundle morphisms)**.** Let $E \to M$ and $E' \to M'$ be vector bundles. A *bundle morphism* from E to E' is a pair of smooth maps $(F : E \to E', f : M \to M')$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array}.$$

i.e. such that $F_p : E_p \to E'_{f(p)}$ is linear for each p.

Definition (Bundle morphism over M)**.** Given two bundles E, E' over the same base M, a *bundle morphism over M* is a bundle morphism $E \to E'$ of the form (F, id_M).

5 Differential forms and de Rham cohomology

5.1 Differential forms

Definition (Differential form)**.** We write

$$\Omega^p(M) = C^\infty(M, \Lambda^p T^*M) = \{p\text{-forms on } M\}.$$

An element of $\Omega^p(M)$ is known as a *differential p-form.*

In particular, we have

$$\Omega^0(M) = C^\infty(M, \mathbb{R}).$$

Definition (Non-degenerate form)**.** A 2-form $\omega \in \Omega^2(M)$ is *non-degenerate* if $\omega(X_p, X_p) = 0$ implies $X_p = 0$.

Definition (Symplectic form)**.** A symplectic form is a non-degenerate 2-form ω such that $\mathrm{d}\omega = 0$.

Definition (Pullback of differential form)**.** Let $\omega \in \Omega^p(N)$ and $F \in C^\infty(M, N)$. We define the *pullback* of ω along F to be

$$F^*\omega|_x = \Lambda^p(\mathrm{D}F|_x)^*(\omega|_{F(x)}).$$

In other words, for $v_1, \cdots, v_p \in T_xM$, we have

$$(F^*\omega|_x)(v_1, \cdots, v_p) = \omega|_{F(x)}(\mathrm{D}F|_x(v_1), \cdots, \mathrm{D}F|_x(v_p)).$$

5.2 De Rham cohomology

Definition (Closed form)**.** A p-form $\omega \in \Omega^p(M)$ is *closed* if $\mathrm{d}\omega = 0$.

Definition (Exact form)**.** A p-form $\omega \in \Omega^p(M)$ is *exact* if there is some $\sigma \in \Omega^{p-1}(M)$ such that $\omega = \mathrm{d}\sigma$.

Definition (de Rham cohomology)**.** The *pth de Rham cohomology* is given by the $\mathbb{R}$-vector space

$$H^p_{\mathrm{dR}}(M) = \frac{\ker \mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)}{\operatorname{im} \mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M)} = \frac{\text{closed forms}}{\text{exact forms}}.$$

In particular, we have

$$H^0_{\mathrm{dR}}(M) = \ker \mathrm{d} : \Omega^0(M) \to \Omega^1(M).$$

Definition (Smooth homotopy)**.** Let $F_0, F_1 : M \to N$ be smooth maps. A *smooth homotopy* from F_0 to F_1 is a smooth map $F : [0, 1] \times M \to N$ such that

$$F_0(x) = F(0, x), \quad F_1(x) = F(1, x).$$

If such a map exists, we say F_0 and F_1 are *homotopic.*

Definition (Smooth homotopy equivalence)**.** We say two manifolds M, N are *smoothly homotopy equivalent* if there are smooth maps $F : M \to N$ and $G : N \to M$ such that both $F \circ G$ and $G \circ F$ are homotopic to the identity.

5.3 Homological algebra and Mayer-Vietoris theorem

Definition (Cochain complex and exact sequence)**.** A sequence of vector spaces and linear maps

$$\cdots \longrightarrow V^{p-1} \xrightarrow{\mathrm{d}_{p-1}} V^p \xrightarrow{\mathrm{d}_p} V^{p+1} \longrightarrow \cdots$$

is a *cochain complex* if $\mathrm{d}_p \circ \mathrm{d}_{p-1} = 0$ for all $p \in \mathbb{Z}$. Usually we have $V^p = 0$ for $p < 0$ and we do not write them out. Keeping these negative degree V^p rather than throwing them away completely helps us state our theorems more nicely, so that we don't have to view V^0 as a special case when we state our theorems.

It is *exact at* p if $\ker \mathrm{d}_p = \operatorname{im} \mathrm{d}_{p-1}$, and *exact* if it is exact at every p.

Definition (Cohomology)**.** Let

$$V^\bullet = \cdots \longrightarrow V^{p-1} \xrightarrow{\mathrm{d}_{p-1}} V^p \xrightarrow{\mathrm{d}_p} V^{p+1} \longrightarrow \cdots$$

be a cochain complex. The *cohomology* of $V^\bullet$ at p is given by

$$H^p(V^\bullet) = \frac{\ker \mathrm{d}_p}{\operatorname{im} \mathrm{d}_{p-1}}.$$

Definition (Cochain map)**.** Let $V^\bullet$ and $W^\bullet$ be cochain complexes. A *cochain map* $V^\bullet \to W^\bullet$ is a collection of maps $f^p : V^p \to W^p$ such that the following diagram commutes for all p:

$$\begin{array}{ccc} V^p & \xrightarrow{f^p} & W^p \\ \downarrow{\scriptstyle \mathrm{d}_p} & & \downarrow{\scriptstyle \mathrm{d}_p} \\ V^{p+1} & \xrightarrow{f^{p+1}} & W^{p+1} \end{array}$$

Definition (Short exact sequence)**.** A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow V^1 \xrightarrow{\alpha} V^2 \xrightarrow{\beta} V^3 \longrightarrow 0\ .$$

This implies that α is injective, β is surjective, and $\operatorname{im}(\alpha) = \ker(\beta)$. By the rank-nullity theorem, we know

$$\dim V^2 = \operatorname{rank}(\beta) + \operatorname{null}(\beta) = \dim V^3 + \dim V^1.$$

6 Integration

6.1 Orientation

Definition (Orientation of vector space)**.** Let V be a vector space with $\dim V = n$. An *orientation* is an equivalence class of elements $\omega \in \Lambda^n(V^*)$, where we say $\omega \sim \omega'$ iff $\omega = \lambda\omega'$ for some $\lambda > 0$. A basis $(e_1, \cdots, e_n)$ is *oriented* if

$$\omega(e_1, \cdots, e_n) > 0.$$

By convention, if $V = \{0\}$, an orientation is just a choice of number in $\{\pm 1\}$.

Definition (Orientation of a manifold)**.** An *orientation* of a manifold M is defined to be an equivalence class of elements $\omega \in \Omega^n(M)$ that are nowhere vanishing, under the equivalence relation $\omega \sim \omega'$ if there is some smooth $f : M \to \mathbb{R}_{>0}$ such that $\omega = f\omega'$.

Definition (Orientable manifold)**.** A manifold is *orientable* if it has some orientation.

Definition (Oriented manifold)**.** An *oriented manifold* is a manifold with a choice of orientation.

Definition (Oriented coordinates)**.** Let M be an oriented manifold. We say coordinates $x_1, \cdots, x_n$ on a chart U are *oriented coordinates* if

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p$$

is an oriented basis for T_pM for all $p \in U$.

Definition (Orientation-preserving diffeomorphism)**.** Let M, N be oriented manifolds, and $F \in C^\infty(M, N)$ be a diffeomorphism. We say F *preserves orientation* if $\mathrm{D}F|_p : T_pM \to T_{F(p)}N$ takes an oriented basis to an oriented basis.

Alternatively, this says the pullback of the orientation on N is the orientation on M (up to equivalence).

6.2 Integration

Definition (Domain of integration)**.** Let $D \subseteq \mathbb{R}^n$. We say D is a *domain of integration* if D is bounded and ∂D has measure zero.

Definition (Integration on $\mathbb{R}^n$)**.** Let D be a compact domain of integration, and

$$\omega = f \;\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

be an n-form on D. Then we define

$$\int_D \omega = \int_D f(x_1, \cdots, x_n)\;\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

In general, let $U \subseteq \mathbb{R}^n$ and let $\omega \in \Omega^n(\mathbb{R}^n)$ have compact support. We define

$$\int_U \omega = \int_D \omega$$

for some $D \subseteq U$ containing $\operatorname{supp} \omega$.

Definition (Smooth function). Let $D \subseteq \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$. We say f is *smooth* if it is a restriction of some smooth function $\tilde{f} : U \to \mathbb{R}^m$ where $U \supseteq D$.

Definition (Integration on manifolds). Let M be an oriented manifold. Let $\omega \in \Omega^n(M)$. Suppose that $\operatorname{supp}(\omega)$ is a compact subset of some oriented chart (U, φ). We set

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

By the previous lemma, this does not depend on the oriented chart (U, φ).

If $\omega \in \Omega^n(M)$ is a general form with compact support, we do the following: cover the support by finitely many oriented charts $\{U_\alpha\}_{\alpha=1,\ldots,m}$. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. We then set

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \chi_\alpha \omega.$$

Definition (Parametrization). Let M be either an oriented manifold of dimension n, or a domain of integration in $\mathbb{R}^n$. By a *parametrization* of M we mean a decomposition

$$M = S_1 \cup \cdots \cup S_n,$$

with smooth maps $F_i : D_i \to S_i$, where D_i is a compact domain of integration, such that

(i) $F_i|_{\mathring{D}_i} : \mathring{D}_i \to \mathring{S}_i$ is an orientation-preserving diffeomorphism

(ii) ∂S_i has measure zero (if M is a manifold, this means $\varphi(\partial S_i \cap U)$ for all charts (U, φ)).

(iii) For $i \neq j$, S_i intersects S_j only in their common boundary.

6.3 Stokes Theorem

Definition (Manifold with boundary). Let

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

A *chart-with-boundary* on a set M is a bijection $\varphi : U \to \varphi(U)$ for some $U \subseteq M$ such that $\varphi(U) \subseteq \mathbb{H}^n$ is open. Note that this image may or may not hit the boundary of $\mathbb{H}^n$. So a "normal" chart is also a chart with boundary.

An *atlas-with-boundary* on M is a cover by charts-with-boundary $(U_\alpha, \varphi_\alpha)$ such that the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth (in the usual sense) for all α, β.

A *manifold-with-boundary* is a set M with an (equivalence class of) atlas with boundary whose induced topology is Hausdorff and second-countable.

Definition (Boundary point). If M is a manifold with boundary and $p \in M$, then we say p is a *boundary point* if $\varphi(p) \in \partial\mathbb{H}^n$ for some (hence any) chart-with-boundary (U, φ) containing p. We let ∂M be the set of boundary points and $\operatorname{Int}(M) = M \setminus \partial M$.

Definition (Outward/Inward pointing)**.** Let $p \in \partial M$. We then have an inclusion $T_p\partial M \subseteq T_pM$. If $X_p \in T_pM$, then in a chart, we can write

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where $a_i \in \mathbb{R}$ and $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}$ are a basis for $T_p\partial M$. We say X_p is *outward pointing* if $a_n < 0$, and *inward pointing* if $a_n > 0$.

Definition (Induced orientation)**.** Let M be an oriented manifold with boundary. We say a basis $e_1, \cdots, e_{n-1}$ is an oriented basis for $T_p\partial M$ if $(X_p, e_1, \cdots, e_{n-1})$ is an oriented basis for T_pM, where X_p is any outward pointing element in T_pM. This orientation is known as the *induced orientation.*

7 De Rham's theorem*

Definition (Singular p-complex)**.** Let M be a manifold. Then a *singular p-simplex* is a continuous map

$$\sigma : \Delta_p \to M,$$

where

$$\Delta_p = \left\{\sum_{i=0}^{p} t_i e_i : \sum t_I = 1\right\} \subseteq \mathbb{R}^{n+1}.$$

We define

$$C_p(M) = \{\text{formal sums } \sum a_i\sigma_i : a_i \in \mathbb{R}, \sigma_i \text{ a singular } p \text{ simplex}\}.$$

We define

$$C_p^\infty(m) = \{\text{formal sums } \sum a_i\sigma_i : a_i \in \mathbb{R}, \sigma_i \text{ a smooth singular } p \text{ simplex}\}.$$

Definition (Boundary map)**.** The *boundary map*

$$\partial : C_p(M) \to C_{p-1}(M)$$

is the linear map such that if $\sigma : \Delta_p \to M$ is a p simplex, then

$$\partial\sigma = \sum(-1)^i \sigma \circ F_{i,p},$$

where $F_{i,p}$ maps Δ_{p-1} affine linearly to the face of Δ_p opposite the ith vertex. We similarly have

$$\partial : C_p^\infty(M) \to C_{p-1}^\infty(M).$$

Definition (Singular homology)**.** The *singular homology* of M is

$$H_p(M, \mathbb{R}) = \frac{\ker \partial : C_p(M) \to C_{p-1}(M)}{\operatorname{im} \partial : C_{p+1}(M) \to C_p(M)}.$$

The *smooth singular homology* is the same thing with $C_p(M)$ replaced with $C_p^\infty(M)$.

Definition (Singular cohomology)**.** The *singular cohomology* of M is defined as

$$H^p(M, \mathbb{R}) = \operatorname{Hom}(H_p(M, \mathbb{R}), \mathbb{R}).$$

Similarly, the smooth singular cohomology is

$$H^p_\infty(M, \mathbb{R}) = \operatorname{Hom}(H_p^\infty(M, \mathbb{R}), \mathbb{R}).$$

Definition (de Rham)**.**

(i) We say a manifold M is *de Rham* if I is an isomorphism.

(ii) We say an open cover $\{U_\alpha\}$ of M is *de Rham* if $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ is de Rham for all $\alpha_1, \cdots, \alpha_p$.

(iii) A *de Rham basis* is a de Rham cover that is a basis for the topology on M.

8 Connections

8.1 Basic properties of connections

Notation. Let E be a vector bundle on M. Then we write

$$\Omega^p(E) = \Omega^0(E \otimes \Lambda^p(T^*M)).$$

So an element in $\Omega^p(E)$ takes in p tangent vectors and outputs a vector in E.

Definition (Connection)**.** Let E be a vector bundle on M. A *connection* on E is a linear map

$$\mathrm{d}_E : \Omega^0(E) \to \Omega^1(E)$$

such that

$$\mathrm{d}_E(fs) = \mathrm{d}f \otimes s + f\mathrm{d}_E s$$

for all $f \in C^\infty(M)$ and $s \in \Omega^0(E)$.

A connection on TM is called a *linear* or *Koszul connection*.

Definition (Vector field along curve)**.** Let $\gamma : I \to M$ be a curve. A *vector field* along γ is a smooth $V : I \to TM$ such that

$$V(t) \in T_{\gamma(t)}M$$

for all $t \in I$. We write

$$J(\gamma) = \{\text{vector fields along } \gamma\}.$$

Definition (Induced connection on tensor product)**.** Let E, F be vector bundles with connections $\mathrm{d}_E, \mathrm{d}_F$ respectively. The *induced connection* is the connection $\mathrm{d}_{E\otimes F}$ on $E \otimes F$ given by

$$\mathrm{d}_{E\otimes F}(s \otimes t) = \mathrm{d}_E s \otimes t + s \otimes \mathrm{d}_F t$$

for $s \in \Omega^0(E)$ and $t \in \Omega^0(F)$, and then extending linearly.

Definition (Induced connection on dual bundle)**.** Let E be a vector bundle with connection d_E. Then there is an induced connection d_{E^*} on E^* given by requiring

$$\mathrm{d}\langle s, \xi\rangle = \langle \mathrm{d}_E s, \xi\rangle + \langle s, \mathrm{d}_{E^*}\xi\rangle,$$

for $s \in \Omega^0(E)$ and $\xi \in \Omega^0(E^*)$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the natural pairing $\Omega^0(E) \times \Omega^0(E^*) \to C^\infty(M, \mathbb{R})$.

8.2 Geodesics and parallel transport

Definition (Geodesic)**.** Let M be a manifold with a linear connection ∇. We say that $\gamma : I \to M$ is a *geodesic* if

$$\mathrm{D}_t\dot{\gamma}(t) = 0.$$

Definition (Parallel vector field)**.** Let ∇ be a linear connection on M, and $\gamma : I \to M$ be a path. We say a vector field $V \in J(\gamma)$ along γ is *parallel* if $\mathrm{D}_t V(t) \equiv 0$ for all $t \in I$.

Definition (Parallel transport)**.** Let $\gamma : I \to M$ be a curve. For t_0, t_1, we define the *parallel transport map*

$$P_{t_0t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$

given by $\xi \mapsto V_\xi(t_1)$.

8.3 Riemannian connections

Definition (Metric connection)**.** A linear connection ∇ is *compatible* with g (or is a *metric connection*) if for all $X, Y, Z \in \mathrm{Vect}(M)$,

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Note that the first term is just $X(g(Y, Z))$.

Definition (Torsion of linear connection)**.** Let ∇ be a linear connection on M. The *torsion* of ∇ is defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for $X, Y \in \mathrm{Vect}(M)$.

Definition (Symmetric/torsion free connection)**.** A linear connection is *symmetric* or *torsion-free* if $\tau(X, Y) = 0$ for all X, Y.

Definition (Riemannian/Levi-Civita connection)**.** The unique torsion-free metric connection on a Riemannian manifold is called the *Riemannian connection* or *Levi-Civita connection*.

8.4 Curvature

Definition (Parallel vector field)**.** We say a vector field $V \in \mathrm{Vect}(M)$ is *parallel* if V is parallel along any curve in M.

Definition (Curvature)**.** The *curvature* of a connection $\mathrm{d}_E : \Omega^0(E) \to \Omega^1(E)$ is the map

$$F_E = \mathrm{d}_E \circ \mathrm{d}_E : \Omega^0(E) \to \Omega^2(E).$$

Definition (Flat connection)**.** A connection d_E is *flat* if $F_E = 0$.

Definition (Curvature of metric)**.** Let (M, g) be a Riemannian manifold with metric g. The *curvature* of g is the curvature of the Levi-Civita connection, denoted by

$$F_g \in \Omega^2(\mathrm{End}(TM)) = \Omega^0(\Lambda^2 T^*M \otimes TM \otimes T^*M).$$

Definition (Flat metric)**.** A Riemannian manifold (M, g) is *flat* if $F_g = 0$.

Definition (Isometry)**.** Let (M, g) and (N, g') be Riemannian manifolds. We say $G \in C^\infty(M, N)$ is an *isometry* if G is a diffeomorphism and $G^*g' = g$, i.e.

$$\mathrm{D}G|_p : T_pM \to T_{G(p)}N$$

is an isometry for all $p \in M$.

Definition (Locally isometric)**.** A manifold M is *locally isometric* to N if for all $p \in M$, there is a neighbourhood U of p and a $V \subseteq N$ and an isometry $G : U \to V$.

Definition (Holonomy)**.** Consider a piecewise smooth curve $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p$. Say we have a linear connection ∇. Then we have a notion of parallel transport along γ.

The *holonomy* of ∇ around γ: is the map

$$H : T_p M \to T_p M$$

given by

$$H(\xi) = V(1),$$

where V is the parallel transport of ξ along γ.