Part III — Combinatorics

Theorems with proof

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What can one say about a collection of subsets of a finite set satisfying certain conditions in terms of containment, intersection and union? In the past fifty years or so, a good many fundamental results have been proved about such questions: in the course we shall present a selection of these results and their applications, with emphasis on the use of algebraic and probabilistic arguments.

The topics to be covered are likely to include the following:

- The de Bruijn–Erdös theorem and its extensions.
- The Graham–Pollak theorem and its extensions.
- The theorems of Sperner, EKR, LYMB, Katona, Frankl and Füredi.
- Isoperimetric inequalities: Kruskal–Katona, Harper, Bernstein, BTBT, and their applications.
- Correlation inequalities, including those of Harris, van den Berg and Kesten, and the Four Functions Inequality.
- Alon's Combinatorial Nullstellensatz and its applications.
- LLLL and its applications.

Pre-requisites

The main requirement is mathematical maturity, but familiarity with the basic graph theory course in Part II would be helpful.

Contents

1 Hall's theorem

Theorem (Hall, 1935)**.** A bipartite graph $G = (X, Y; E)$ has a complete matching from X to Y if and only if $|\Gamma(S)| \geq |S|$ for all $S \subseteq X$.

Proof. We may assume G is edge-minimal satisfying Hall's condition. We show that G is a complete matching from X to Y. For G to be a complete matching, we need the following two properties:

(i) Every vertex in X has degree 1

(ii) Every vertex in Y has degree 0 or 1.

We first examine the second condition. Suppose $y \in Y$ is such that there exists edges $x_1y, x_2y \in E$. Then the minimality of G implies there are sets, $X_1, X_2 \subseteq X$ such that $x_i \in X_i$ such that $|\Gamma(X_i)| = |X_i|$ and x_i is the only neighbour of y in X_i.

Now consider the set $X_1 \cap X_2$. We know $\Gamma(X_1 \cap X_2) \subseteq \Gamma(X_1) \cap \Gamma(X_2)$. Moreover, this is strict, as y is in the RHS but not the LHS. So we have

$$\Gamma(X_1 \cap X_2) \leq |\Gamma(X_i) \cap \Gamma(X_2)| - 1.$$

But also

$$\begin{aligned}
|X_1 \cap X_2| &\leq |\Gamma(X_1 \cap X_2)| \\
&\leq |\Gamma(X_1) \cap \Gamma(X_2)| - 1 \\
&= |\Gamma(X_1)| + |\Gamma(X_2)| - |\Gamma(X_1) \cup \Gamma(X_2)| - 1 \\
&= |X_1| + |X_2| - |\Gamma(X_1 \cup X_2)| - 1 \\
&\leq |X_1| + |X_2| - |X_1 \cup X_2| - 1 \\
&= |X_1 \cap X_2| - 1,
\end{aligned}$$

which contradicts Hall's condition.

One then sees that the first condition is also satisfied — if $x \in X$ is a vertex, then the degree of x certainly cannot be 0, or else $|\Gamma(\{x\})| < |\{x\}|$, and we see that $d(x)$ cannot be > 1 or else we can just remove an edge from x without violating Hall's condition. □

Theorem. $\mathcal{A}$ has a set of distinct representatives iff for all $\mathcal{B} \subseteq \mathcal{A}$, we have

$$\left| \bigcup_{B \in \mathcal{B}} B \right| \geq |\mathcal{B}|.$$

Proof. Define a bipartite graph as follows — we let $X = \mathcal{A}$, and $Y = \bigcup_{i \in [m]} A_i$. Then draw an edge from x to A_i if $x \in A_i$. Then there is a complete matching of this graph iff $\mathcal{A}$ has a set of distinct representations, and the condition in the theorem is exactly Hall's condition. So we are done by Hall's theorem. □

Theorem. Let $G = (X, Y; E)$ be a bipartite graph such that $d(x) \geq d(y)$ for all $x \in X$ and $y \in Y$. Then there is a complete matching from X to Y.

Proof. Let d be such that $d(x) \geq d \geq d(y)$ for all $x \in X$ and $y \in Y$. For $S \subseteq X$ and $T \subseteq Y$, we let $e(S, T)$ be the number of edges between S and T. Let $S \subseteq X$, and $T = \Gamma(S)$. Then we have

$$e(S, T) = \sum_{x \in S} d(x) \geq d|S|,$$

but on the other hand, we have

$$e(S, T) \leq \sum_{y \in T} d(y) \leq d|T|.$$

So we find that $|T| \geq |S|$. So Hall's condition is satisfied. □

Corollary. If $G = (X, Y; E)$ is a (k, ℓ)-regular bipartite graph with $1 \leq \ell \leq k$, then there is a complete matching from X to Y.

Theorem. Let $G = (X, Y; E)$ be biregular and $A \subseteq X$. Then

$$\frac{|\Gamma(A)|}{|Y|} \geq \frac{|A|}{|X|}.$$

Proof. Suppose G is (k, ℓ)-regular. Then

$$k|A| = e(A, \Gamma(A)) \leq \ell|\Gamma(A)|.$$

Thus we have

$$\frac{|\Gamma(A)|}{|Y|} \geq \frac{k|A|}{\ell|Y|}.$$

On the other hand, we can count that

$$|E| = |X|k = |Y|\ell,$$

and so

$$\frac{k}{\ell} = \frac{|Y|}{|X|}.$$

So we are done. □

Corollary. Let $G = (X, Y; E)$ be biregular and let $|X| \leq |Y|$. Then there is a complete matching of X into Y.

Corollary. Let $1 \leq r < s \leq |X| = n$. Suppose $|\frac{n}{2} - r| \geq |\frac{n}{2} - s|$. Then there exists an injection $f : X^{(r)} \to X^{(s)}$ such that $A \subseteq f(A)$ for all $A \in X^{(r)}$.

If $|\frac{n}{2} - r| \leq |\frac{n}{2} - s|$, then there exists an injection $g : X^{(s)} \to X^{(r)}$ such that $A \supseteq g(A)$ for all $A \in X^{(s)}$.

Proof. Note that $|\frac{n}{2} - r| \leq |\frac{n}{2} - s|$ iff $\binom{n}{r} \geq \binom{n}{s}$. □

2 Sperner systems

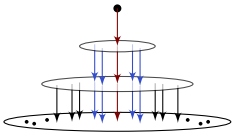
Theorem (Sperner, 1928)**.** For $|X| = n$, the maximal size of an antichain in $\mathcal{P}(X)$ is $\binom{n}{\lfloor n/2 \rfloor}$, witnessed by $X^{\lfloor n/2 \rfloor}$.

Proof. If $\mathcal{C}$ is a chain and $\mathcal{A}$ is an antichain, then $|\mathcal{A} \cap \mathcal{C}| \leq 1$. So it suffices to partition $\mathcal{P}(X)$ into

$$m = \max_k \binom{n}{k} = \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$$

many chains.

We can do so using the injections constructed at the end of the previous section. For $i \geq \lfloor \frac{n}{2} \rfloor$, we can construct injections $f_i : X_{i-1} \to X_i$ such that $A \subseteq f_i(A)$ for all A. By chaining these together, we get m chains ending in $X^{\lfloor \frac{n}{2} \rfloor}$.



Similarly, we can partition $X^{(\leq \lfloor n/2 \rfloor)}$ into m chains with each chain ending in $X^{(\lfloor n/2 \rfloor)}$. Then glue them together. □

Theorem (LYM inequality)**.** Let $\mathcal{A}$ be an antichain in $\mathcal{P}(X)$ with $|X| = n$. Then

$$\sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1.$$

In particular, $|\mathcal{A}| \leq \max_r \binom{n}{r} = \binom{n}{\lfloor n/2 \rfloor}$, as we already know.

Proof. A chain $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_m$ is maximal if it has $n + 1$ elements. Moreover, there are $n!$ maximal chains, since we start with the empty set and then, given C_i, we produce C_{i+1} by picking one unused element and adding it to C_i.

For every maximal chain $\mathcal{C}$, we have $|\mathcal{C} \cap \mathcal{A}| \leq 1$. Moreover, every set of k elements appears in $k!(n-k)!$ maximal chains, by a similar counting argument as above. So

$$\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!.$$

Then the result follows. □

Theorem. If P is downward expanding and A is an anti-chain, then $w(A) \leq 1$. In particular, $|A| \leq \max_i |S_i|$.

Since each S_i is an anti-chain, the largest anti-chain has size $\max_i |S_i|$.

Proof. We define the *span* of A to be

$$\operatorname{span} A = \max_{A_j \neq \emptyset} j - \min_{A_i \neq \emptyset} i.$$

We do induction on $\operatorname{span} A$.

If $\operatorname{span} A = 0$, then we are done. Otherwise, let $h_i = \max_{A_j \neq 0} j$, and set $B_{h-1} = \partial A_h$. Then since A is an anti-chain, we know $A_{h-1} \cap B_{h-1} = \emptyset$.

We set $A' = A \setminus A_h \cup B_{h-1}$. This is then another anti-chain, by the transitivity of $<$. We then have

$$w(A) = w(A') + w(A_h) - w(B_{h-1}) \leq w(A') \leq 1,$$

where the first inequality uses the downward-expanding hypothesis and the second is the induction hypothesis. □

Proposition. An anti-chain in a regular poset has weight ≤ 1.

Proof. Let M be the number of maximal chains of length $(n+1)$, and for each $x \in S_k$, let $m(x)$ be the number of maximal chains through x. Then

$$m(x) = \prod_{i=1}^{k} r_i \prod_{i=k}^{n-1} s_i.$$

So if $x, y \in S_i$, then $m(x) = m(y)$.

Now since every maximal chain passes through a unique element in S_i, for each $x \in S_i$, we have

$$M = \sum_{x \in S_i} m(x) = |S_i| m(x).$$

This gives the formula

$$m(x) = \frac{M}{|S_i|}.$$

now let A be an anti-chain. Then A meets each chain in ≤ 1 elements. So we have

$$M = \sum_{\text{maximal chains}} 1 \geq \sum_{x \in A} m(x) = \sum_{i=0}^{n} |A \cap S_i| \cdot \frac{M}{|S_i|}.$$

So it follows that

$$\sum \frac{|A \cap S_i|}{|S_i|} \leq 1.$$ □

Theorem (Erdös, 1945)**.** Let x_i be all real, $|x_i| \geq 1$. For $A \subseteq [n]$, let

$$x_A = \sum_{i \in A} x_i.$$

Let $\mathcal{A} \subseteq \mathcal{P}(n)$. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. We claim that we may assume $x_i \geq 1$ for all i. To see this, suppose we instead had $x_1 = -2$, say. Then whether or not $i \in A$ determines whether x_A should include 0 or -2 in the sum. If we replace x_i with 2, then whether or not $i \in A$ determines whether x_A should include 0 or 2. So replacing x_i with 2 just essentially shifts all terms by 2, which doesn't affect the difference.

But if we assume that $x_i \geq 1$ for all i, then we are done, since $\mathcal{A}$ must be an anti-chain, for if $A, B \in \mathcal{A}$ and $A \subsetneq B$, then $x_B - x_A = x_{B \setminus A} \geq 1$. □

Theorem. $\mathcal{P}(n)$ has a decomposition into symmetric chain.

Proof. We prove by induction. In the case $n = 1$, we simply have to take $\{\emptyset, \{1\}\}$.

Now suppose $\mathcal{P}(n-1)$ has a symmetric chain decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_t$. Given a symmetric chain

$$\mathcal{C}_j = \{C_i, C_{i+1}, \dots, C_{n-1-i}\},$$

we obtain two chains $\mathcal{C}_j^{(0)}, \mathcal{C}_j^{(1)}$ in $\mathcal{P}(n)$ by

$$\begin{aligned}\mathcal{C}_j^{(0)} &= \{C_i, C_{i+1}, \dots, C_{n-1-i}, C_{n-1-i} \cup \{n\}\}\\ \mathcal{C}_j^{(1)} &= \{C_i \cup \{n\}, C_{i+1} \cup \{n\}, \dots, C_{n-2-i} \cup \{n\}\}.\end{aligned}$$

Note that if $|\mathcal{C}_j| = 1$, then $\mathcal{C}_j^{(1)} = \emptyset$, and we drop this. Under this convention, we note that every $A \in \mathcal{P}(n)$ appears in exactly one $\mathcal{C}_j^{(\varepsilon)}$, and so we are done. □

Theorem (Kleitman, 1970)**.** Let $x_1, x_2, \dots, x_n$ be vectors in a normed space with norm $\|x_I\| \geq 1$ for all i. For $A \in \mathcal{P}(n)$, we set

$$x_A = \sum_{i \in A} x_i.$$

Let $\mathcal{A} \subseteq \mathcal{P}(n)$ be such that $\|x_A - x_B\| < 1$. Then $\|\mathcal{A}\| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. Call $\mathcal{F} \subseteq \mathcal{P}(n)$ *sparse* if $\|x_E - x_F\| \geq 1$ for all $E, F \in \mathcal{F}$, $E \neq F$. Note that if $\mathcal{F}$ is sparse, then $|\mathcal{F} \cap \mathcal{A}| \leq 1$. So if we can find a decomposition of $\mathcal{P}(n)$ into $\binom{n}{\lfloor n/2 \rfloor}$ sparse sets, then we are done.

We call a partition $\mathcal{P}(n) = \mathcal{F}_1 \cup \cdots \cup F_t$ *symmetric* if the number of families with $n + 1 - 2i$ sets is $\ell(n, i)$, i.e. the "profile" is that of a symmetric chain decomposition.

Claim. $\mathcal{P}(n)$ has a symmetric decomposition into sparse families.

We again induct on n. When $n = 1$, we can take $\{\emptyset, \{1\}\}$. Now suppose Δ_{n-1} is a symmetric decomposition of $\mathcal{P}(n-1)$ as $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_t$.

Given $\mathcal{F}_j$, we construct $\mathcal{F}_j^{(0)}$ and $\mathcal{F}_j^{(1)}$ "as before". We pick some $D \in \mathcal{F}_j$, to be decided later, and we take

$$\begin{aligned}\mathcal{F}_j^{(0)} &= \mathcal{F}_j \cup \{D \cup \{n\}\}\\ \mathcal{F}_j^{(1)} &= \{E \cup \{n\} : E \in \mathcal{F}_j \setminus \{D\}\}.\end{aligned}$$

The resulting set is certainly still symmetric. The question is whether it is sparse, and this is where the choice of D comes in. The collection $\mathcal{F}_j^{(1)}$ is certainly still sparse, and we must pick a D such that $\mathcal{F}_j^{(0)}$ is sparse.

To do so, we use Hahn–Banach to obtain a linear functional f such that $\|f\| = 1$ and $f(x_n) = \|x_n\| \geq 1$. We can then pick D to maximize $f(x_D)$. Then we check that if $E \in \mathcal{F}_j$, then

$$f(x_{D \cup \{n\}} - x_2) = f(x_D) - f(x_E) + f(x_n).$$

By assumption, $f(x_n) \geq 1$ and $f(x_D) \geq f(x_E)$. So this is ≥ 1. Since $\|f\| = 1$, it follows that $\|x_{D \cup \{n\}} - x_E\| \geq 1$. □

3 The Kruskal–Katona theorem

Lemma. We have

$$\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial \mathcal{A}).$$

In particular, $|\partial C_{ij}(\mathcal{A})| \leq |\partial \mathcal{A}|$. □

Lemma. Let $\mathcal{A} \subseteq X^{(r)}$ and $U, V \in X^{(s)}$, $U \cap V = \emptyset$. Suppose for all $u \in U$, there exists v such that $\mathcal{A}$ is $(U \setminus \{u\}, V \setminus \{v\})$-compressed. Then

$$\partial C_{UV}(\mathcal{A}) \subseteq C_{UV}(\partial A).$$ □

Lemma. $\mathcal{A} \subseteq X^{(r)}$ is an initial segment of $X^{(r)}$ in colex if and only if it is (U, V)-compressed for all U, V disjoint with $|U| = |V|$ and $\max V > \max U$.

Proof. $\Rightarrow$ is clear. Suppose $\mathcal{A}$ is (U, V) compressed for all such U, V. If $\mathcal{A}$ is not an initial segment, then there exists $B \in \mathcal{A}$ and $C \notin \mathcal{A}$ such that $C < B$. Then $\mathcal{A}$ is not $(C \setminus B, B \setminus C)$-compressed. A contradiction. □

Lemma. Given $\mathcal{A} \in X^{(r)}$, there exists $\mathcal{B} \subseteq X^{(r)}$ such that $\mathcal{B}$ is (U, V)-compressed for all $|U| = |V|$, $U \cap V = \emptyset$, $\max V > \max U$, and moreover

$$|\mathcal{B}| = |\mathcal{A}|, |\partial \mathcal{B}| \leq |\partial \mathcal{A}|. \qquad (*)$$

Proof. Let $\mathcal{B}$ be such that

$$\sum_{B \in \mathcal{B}} \sum_{i \in B} 2^i$$

is minimal among those $\mathcal{B}$'s that satisfy $(*)$. We claim that this $\mathcal{B}$ will do. Indeed, if there exists (U, V) such that $|U| = |V|$, $\max V > \max U$ and $C_{UV}(\mathcal{B}) \neq \mathcal{B}$, then pick such a pair with $|U|$ minimal. Then apply a (U, V)-compression, which is valid since given any $u \in U$ we can pick any $v \in V$ that is not $\max V$ to satisfy the requirements of the previous lemma. This decreases the sum, which is a contradiction. □

Theorem (Kruskal 1963, Katona 1968)**.** Let $\mathcal{A} \subseteq X^{(r)}$, and let $\mathcal{C} \subseteq X^{(r)}$ be the initial segment with $|\mathcal{C}| = |\mathcal{A}|$. Then

$$|\partial \mathcal{A}| \geq |\partial \mathcal{C}|.$$

Theorem (Lovász, 1979)**.** If $\mathcal{A} \subseteq X^{(r)}$ with $|\mathcal{A}| = \binom{x}{r}$ for $x \geq 1, x \in \mathbb{R}$, then

$$|\partial \mathcal{A}| \geq \binom{x}{r-1}.$$

This is best possible if x is an integer.

Proof. Let

$$\mathcal{A}_0 = \{A \in \mathcal{A} : 1 \notin A\}$$
$$\mathcal{A}_1 = \{A \in \mathcal{A} : 1 \in A\}.$$

For convenience, we write

$$\mathcal{A}_1 - 1 = \{A \setminus \{1\} : A \in \mathcal{A}_1\}.$$

We may assume $\mathcal{A}$ is (i,j)-compressed for all $i < j$. We induct on r and then on $|\mathcal{A}|$. We have

$$|\mathcal{A}_0| = |\mathcal{A}| - |\mathcal{A}_1|.$$

We note that $\mathcal{A}_1$ is non-empty, as $\mathcal{A}$ is left-compressed. So $|\mathcal{A}_0| < |\mathcal{A}|$.

If $r = 1$ and $|\mathcal{A}| = 1$ then there is nothing to do.

Now observe that $\partial\mathcal{A} \subseteq \mathcal{A}_1 - 1$, since if $A \in \mathcal{A}$, $1 \notin A$, and $B \subseteq A$ is such that $|A \setminus B| = 1$, then $B \cup \{1\} \in \mathcal{A}_1$ since $\mathcal{A}$ is left-compressed. So it follows that

$$|\partial\mathcal{A}_0| \leq |\mathcal{A}_1|.$$

Suppose $|\mathcal{A}_1| < \binom{x-1}{r-1}$. Then

$$|\mathcal{A}_0| > \binom{x}{r} - \binom{x-1}{r-1} = \binom{x-1}{r}.$$

Therefore by induction, we have

$$|\partial\mathcal{A}_0| > \binom{x-1}{r-1}.$$

This is a contradiction, since $|\partial\mathcal{A}_0| \leq |\mathcal{A}_1|$. Hence $|\mathcal{A}_1| \geq \binom{x-1}{r-1}$. Hence we are done, since

$$|\partial\mathcal{A}| \geq |\partial\mathcal{A}_1| = |\mathcal{A}_1| + |\partial(\mathcal{A}_1 - 1)| \geq \binom{x-1}{r-1} + \binom{x-1}{r-2} = \binom{x}{r-1}. \qquad \square$$

4 Isoperimetric inequalities

Lemma. For $A \subseteq Q_n$, we have $|N(C_i(A))| \leq |N(A)|$.

Proof. We have

$$|N(A)| = |N(A_+) \cup A_-| + |N(A_-) \cup A_+|$$

Take $B = C_i(A)$. Then

$$\begin{aligned}
|N(B)| &= |N(B_+) \cup B_-| + |N(B_-) \cup B_+| \\
&= \max\{|N(B_+)|, |B_-|\} + \max\{|N(B_-)|, |B_+|\} \\
&\leq \max\{|N(A_+)|, |A_-|\} + \max\{|N(A_-)|, |A_+|\} \\
&\leq |N(A_+) \cup A_i| + |N(A_-) \cup A_+| \\
&= |N(A)|
\end{aligned}$$

□

Lemma. For any $A \subseteq Q_n$, there is a compressed set $B \subseteq Q_n$ such that

$$|B| = |A|, \quad |N(B)| \leq |N(A)|.$$

Lemma. For each n, there exists a unique element $z \in Q_n$ such that z^c is the successor of z.

Moreover, if $B \subseteq Q_n$ is compressed but not an initial segment, then $|B| = 2^{n-1}$ and B is obtained from taking the initial segment of size 2^{n-1} and replacing x with x^c.

Proof. For the first part, simply note that complementation is an order-reversing bijection $Q_n \to Q_n$, and $|Q_n|$ is even. So the 2^{n-1}th element is the only such element z.

Now if B is not an initial segment, then we can find some $x < y$ such that $x \notin B$ and $y \in B$. Since B is compressed, it must be the case that for each i, there is exactly one of x and y that contains i. Hence $x = y^c$. Note that this is true for all $x < y$ such that $x \notin B$ and $y \in B$. So if we write out the simplicial order, then B must look like

$$\bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \circ \ \bullet \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \cdots$$

since any $x \notin B$ such that $x < y$ must be given by $x = y^c$, and so there must be a unique such x, and similarly the other way round. So it must be the case that y is the successor of x, and so $x = z$. □

Theorem (Harper, 1967)**.** Let $A \subseteq Q^n$, and let C be the initial segment in the simplicial order with $|C| = |A|$. Then $|N(A)| \geq |N(C)|$. In particular,

$$|A| = \sum_{i=0}^{r} \binom{n}{i} \text{ implies } |N(A)| \geq \sum_{i=0}^{r+1} \binom{n}{i}.$$

Theorem. Let $A \subseteq Q_n$ be a subset, and let $C \subseteq Q_n$ be the initial segment of length $|A|$ in the binary order. Then $|\partial_e C| \leq |\partial_e A|$.

Proof. We induct on n using codimension-1 compressions. Recall that we previously defined the sets $A_\pm^{(i)}$.

The i-compression of A is the set $B \subseteq Q_n$ such that $|B_\pm^{(i)}| = |A_\pm^{(i)}|$, and $B_\pm^{(i)}$ are initial segments in the binary order. We set $D_i(A) = B$.

Observe that performing D_i reduces the edge boundary. Indeed, given any A, we have

$$|\partial_e A| = |\partial_e A_+^{(i)}| + |\partial_e A_-^{(i)}| + |A_+^{(i)} \Delta A_i^{(i)}|.$$

Applying D_i clearly does not increase any of those factors. So we are happy. Now note that if $A \neq D_i A$, then

$$\sum_{x\in A}\sum_{i\in x} 2^i < \sum_{x\in D_i A}\sum_{i\in x} 2^i.$$

So after applying compressions finitely many times, we are left with a compressed set.

We now hope that a compressed subset must be an initial segment, but this is not quite true.

Claim. If A is compressed but not an initial, then

$$A = \tilde{B} = \mathbb{P}(X \setminus \{n\}) \setminus \{123\cdots(n-1)\} \cup \{n\}.$$

By direct computation, we have

$$|\partial_e \tilde{B}| = 2^{n-1} - 2(n-2),$$

and so the initial segment is better. So we are done.

The proof of the claim is the same as last time. Indeed, by definition, we can find some $x < y$ such that $x \notin A$ and $y \in A$. As before, for any i, it cannot be the case that both x and y contain i or neither contain i, since A is compressed. So $x = y^c$, and we are done as before. □

5 Sum sets

Theorem (Cauchy–Davenport theorem)**.** Let A and B be non-empty subsets of $\mathbb{Z}_p$ with p a prime, and $|A| + |B| \leq p + 1$. Then

$$|A + B| \geq |A| + |B| - 1.$$

Proof. We may assume $1 \leq |A| \leq |B|$. Apply induction on $|A|$. If $|A| = 1$, then there is nothing to do. So assume $A \geq 2$.

Since everything is invariant under translation, we may assume $0, a \in A$ with $a \neq 0$. Then $\{a, 2a, \ldots, pa\} = \mathbb{Z}_p$. So there exists $k \geq 0$ such that $ka \in B$ and $(k+1)a \notin B$.

By translating B, we may assume $0 \in B$ and $a \notin B$.

Now $0 \in A \cap B$, while $a \in A \setminus B$. Therefore we have

$$1 \leq |A \cap B| < |A|.$$

Hence

$$|(A \cap B) + (A \cup B)| \geq |A \cap B| + |A \cup B| - 1 = |A| + |B| - 1.$$

Also, clearly

$$(A \cap B) + (A \cup B) \subseteq A + B.$$

So we are done. □

Corollary. Let $A_1, \ldots, A_k$ be non-empty subsets of $\mathbb{Z}_p$ such that

$$\sum_{i=1}^{d} |A_i| \leq p + k - 1.$$

Then

$$|A_1 + \ldots + A_k| \geq \sum_{i=1}^{k} |A_i| - k + 1.$$

Theorem (Erdös–Ginzburg–Ziv)**.** Let $a_1, \ldots, a_{2n-1} \in \mathbb{Z}_n$. Then there exists $I \in [2n-1]^{(n)}$ such that

$$\sum_{i \in I} a_i = 0$$

in $\mathbb{Z}_n$.

Proof. First consider the case $n = p$ is a prime. Write

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_{2p-1} < p.$$

If $a_i = a_{i+p-1}$, then there are p terms that are the same, and so we are done by adding them up. Otherwise, set $A_i = \{a_i, a_{i+p-1}\}$ for $i = 1, \ldots, p-1$, and $A_p = \{a_{2p-1}\}$, then $|A_i| = 2$ for $i = 1, \ldots, p-1$ and $|A_p| = 1$. Hence we know

$$|A_1 + \cdots + A_p| \geq (2(p-1) + 1) - p + 1 = p.$$

Thus, every element in $\mathbb{Z}_p$ is a sum of some p of our terms, and in particular 0 is.

In general, suppose n is not a prime. Write $n = pm$, where p is a prime and $m > 1$. By induction, for every $2m - 1$ terms, we can find m terms whose sum is a multiple of m.

Select *disjoint* $S_1, S_2, \ldots, S_{2p-1} \in [2n-1]^{(m)}$ such that

$$\sum_{j \in S_i} a_j = mb_i.$$

This can be done because after selecting, say, $S_1, \ldots, S_{2p-2}$, we have

$$(2n-1) - (2p-2)m = 2m - 1$$

elements left, and so we can pick the next one.

We are essentially done, because we can pick $j_1, \ldots, j_p$ such that $\sum_{k=1}^{p} b_{i_k}$ is a multiple of p. Then

$$\sum_{k=1}^{p} \sum_{j \in S_{i_k}} a_j$$

is a sum of $mp = n$ terms whose sum is a multiple of mp. □

6 Projections

Proposition. Let K be a body in $\mathbb{R}^3$. Then

$$|K|^2 \leq |K_{12}||K_{13}||K_{23}|.$$

Proof. Suppose first that each section of K is a square, i.e.

$$K(x) = (0, f(x)) \times (0, f(x))\ \mathrm{d}x$$

for all x and some f. Then

$$|K| = \int f(x)^2\ \mathrm{d}x.$$

Moreover,

$$|K_{12}| = \left(\sup_x f(x)\right)^2 \equiv M^2, \quad |K_{13}| = |K_{23}| = \int f(x)\ \mathrm{d}x.$$

So we have to show that

$$\left(\int f(x)^2\ \mathrm{d}x\right)^2 \leq M^2 \left(\int f(x)\ \mathrm{d}x\right)^2,$$

but this is trivial, because $f(x) \leq M$ for all x.

Let's now consider what happens when we compress K. For the general case, define a new body $L \subseteq \mathbb{R}^3$ by setting its sections to be

$$L(x) = (0, \sqrt{|K(x)|}) \times (0, \sqrt{|K(x)|}).$$

Then $|L| = |K|$, and observe that

$$|L_{12}| \leq \sup |K(x)| \leq \left|\bigcup K(x)\right| = |K_{12}|.$$

To understand the other two projections, we introduce

$$g(x) = |K(x)_1|, \quad h(x) = |K(x)_2|.$$

Now observe that

$$|L(x)| = |K(x)| \leq g(x)h(x),$$

Since $L(x)$ is a square, it follows that $L(x)$ has side length $\leq g(x)^{1/2}h(x)^{1/2}$. So

$$|L_{13}| = |L_{23}| \leq \int g(x)^{1/2}h(x)^{1/2}\ \mathrm{d}x.$$

So we want to show that

$$\left(\int g^{1/2}h^{1/2}\ \mathrm{d}x\right)^2 \leq \left(\int g\ \mathrm{d}x\right)\left(\int h\ \mathrm{d}x\right).$$

Observe that this is just the Cauchy–Schwarz inequality applied to $g^{1/2}$ and $h^{1/2}$. So we are done. □

Theorem (Uniform cover inequality)**.** If $A_1, \dots, A_r$ is a uniform k-cover of $[n]$, then

$$|K|^k = \prod_{i=1}^{r} |K_A|.$$

Proof. Let $\mathcal{A}$ be a k-uniform cover of $[k]$. Note that $\mathcal{A}$ is a *multiset*. Write

$$\mathcal{A}_- = \{A \in \mathcal{A} : n \notin A\}$$
$$\mathcal{A}_+ = \{A \setminus \{n\} \in \mathcal{A} : n \in A\}$$

We have $|\mathcal{A}_+| = k$, and $\mathcal{A}_+ \cup \mathcal{A}_-$ forms a k-uniform cover of $[n-1]$.

Now note that if $K = \mathbb{R}^n$ and $n \notin A$, then

$$|K_A| \geq |K(x)_A| \tag{1}$$

for all x. Also, if $n \in A$, then

$$|K_A| = \int |K(x)_{A\setminus\{n\}}| \, \mathrm{d}x. \tag{2}$$

In the previous proof, we used Cauchy–Schwarz. What we need here is Hölder's inequality

$$\int fg \, \mathrm{d}x \leq \left(\int f^p \, \mathrm{d}x\right)^{1/p} \left(\int g^q \, \mathrm{d}x\right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Iterating this, we get

$$\int f_1 \cdots f_k \, \mathrm{d}x \leq \prod_{i=1}^{k} \left(\int f_i^k \, \mathrm{d}x\right)^{1/k}.$$

Now to perform the proof, we induct on n. We are done if $n = 1$. Otherwise, given $K \subseteq \mathbb{R}^n$ and $n \geq 2$, by induction,

$$\begin{aligned}
|K| &= \int |K(x)| \, \mathrm{d}x \\
&\leq \int \prod_{A\in\mathcal{A}_-} |K(x)_A|^{1/k} \prod_{A\in\mathcal{A}_+} |K(x)_A|^{1/k} \, \mathrm{d}x && \text{(by induction)} \\
&\leq \prod_{A\in\mathcal{A}_-} |K_A|^{1/k} \int \prod_{A\in\mathcal{A}_+} |K(x)_A|^{1/k} \, \mathrm{d}x && \text{(by (1))} \\
&\leq \prod_{A\leq|\mathcal{A}_-} |K_A|^{1/k} \prod_{A\in\mathcal{A}_+} \left(\int |K(x)_A|\right)^{1/k} && \text{(by Hölder)} \\
&= \prod_{A\in\mathcal{A}} |K_A|^{1/k} \prod_{A\in\mathcal{A}_+} |K_{A\cup\{n\}}|^{1/k}. && \text{(by (2))}
\end{aligned}$$

□

Theorem (Box Theorem (Bollobás, Thomason))**.** Given a body $K \subseteq \mathbb{R}^n$, i.e. a non-empty bounded open set, there exists a box L such that $|L| = |K|$ and $|L_A| \leq |K_A|$ for all $A \subseteq [n]$.

Lemma. There are only finitely many irreducible covers of $[n]$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be covers. We say $\mathcal{A} < \mathcal{B}$ if $\mathcal{A}$ is a "subset" of $\mathcal{B}$, i.e. for each $A \subseteq [n]$, the multiplicity of A in $\mathcal{A}$ is less than the multiplicity in $\mathcal{B}$.

Then note that the set of irreducible uniform k-covers form an anti-chain, and observe that there cannot be an infinite anti-chain.

□

Proof of box theorem. For $\mathcal{A}$ an irreducible cover, we have

$$|K|^k \leq \prod_{A\in\mathcal{A}} |K_A|.$$

Also,

$$|K_A| \leq \prod_{i\in A} |K_{\{i\}}|.$$

Let $\{x_A : A \subseteq [n]\}$ be a minimal array with $x_A \leq |K_A|$ such that for each irreducible k-cover $\mathcal{A}$, we have

$$|K|^k \leq \prod_{A\in\mathcal{A}} x_A \tag{1}$$

and moreover

$$x_A \leq \prod_{i\in A} x_{\{i\}} \tag{2}$$

for all $A \subseteq [n]$. We know this exists since there are only finitely many inequalities to be satisfied, and we can just decrease the x_A's one by one. Now again by finiteness, for each x_A, there must be at least one inequality involving x_A on the right-hand side that is in fact an equality.

Claim. For each $i \in [n]$, there exists a uniform k_i-cover $\mathcal{C}_i$ containing $\{i\}$ with equality

$$|K|^{k_i} = \prod_{A\in\mathcal{C}_i} x_A.$$

Indeed if x_i occurs on the right of (1), then we are done. Otherwise, it occurs on the right of (2), and then there is some A such that (2) holds with equality. Now there is some cover $\mathcal{A}$ containing A such that (1) holds with equality. Then replace A in $\mathcal{A}$ with $\{\{j\} : j \in A\}$, and we are done.

Now let

$$\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i, \quad \mathcal{C}' = \mathcal{C} \setminus \{\{1\}, \{2\}, \ldots, \{n\}\}, \quad k = \sum_{i=1}^{n} k_i.$$

Then

$$|K|^k = \prod_{A\in\mathcal{C}} x_A = \left(\prod_{A\in\mathcal{C}^1} x_A\right) \geq |K|^{k-1} \prod_{i=1}^{n} x_i.$$

So we have

$$|K| \geq \prod_{i=1}^{n} x_i.$$

But we of course also have the reverse inequality. So it must be the case that they are equal.

Finally, for each A, consider $\mathcal{A} = \{A\} \cup \{\{i\} : i \notin A\}$. Then dividing (1) by $\prod_{i \in A} x_i$ gives us

$$\prod_{i \notin A} x_i \leq x_A.$$

By (2), we have the inverse equality. So we have

$$x_A = \prod_{i \in A} x_i$$

for all i. So we are done by taking L to be the box with side length x_i. □

Corollary. If K is a union of translates of the unit cube, then for any (not necessarily uniform) k-cover $\mathcal{A}$, we have

$$|K|^k \leq \prod_{A \in \mathcal{A}} |K_A|.$$

Proof. Observe that if $B \subseteq A$, then $|K_B| \leq |K_A|$. So we can reduce $\mathcal{A}$ to a uniform k-cover. □

7 Alon's combinatorial Nullstellensatz

Theorem (Alon's combinatorial Nullstellensatz)**.** Let $\mathbb{F}$ be a field, and let $S_1, \dots, S_n$ be non-empty finite subsets of $\mathbb{F}$ with $|S_i| = d_i + 1$. Let $f \in \mathbb{F}[X_1, \dots, X_n]$ have degree $d = \sum_{i=1}^n d_i$, and let the coefficient of $X_1^{d_1} \cdots X_n^{d_n}$ be non-zero. Then f is not identically zero on $S = S_1 \times \cdots \times S_n$.

Proposition (Division algorithm)**.** Let $f, g \in R[X]$ with g monic. Then we can write

$$f = hg + r,$$

where $\deg h \leq \deg f - \deg g$ and $\deg r < \deg g$.

Lemma. Let $f \in R[X]$, and for $i = 1, \dots, n$, let $g_i(X_i) \in R[X_i] \subseteq R[X]$ be monic of degree $\deg g_i = \deg_{X_i} g_i = d_i$. Then there exists polynomials $h_1, \dots, h_n, r \in R[X]$ such that

$$f = \sum f_i g_i + r,$$

where

$$\begin{aligned} \deg h_i &\leq \deg f - \deg d_i & \deg_{X_i} r &\leq d_i - 1 \\ \deg_{X_i} h_i &\leq \deg_{X_i} f - d_i & \deg_{X_i} r &\leq \deg_{X_i} f \\ \deg_{X_j} h_i &\leq \deg_{X_j} f & \deg r &\leq \deg f \end{aligned}$$

for all i, j.

Proof. Consider f as a polynomial with coefficients in $R[X_2, \dots, X_n]$, then divide by g_1 using the division algorithm. So we write

$$f = h_1 g_1 + r_1.$$

Then we have

$$\begin{aligned} \deg_{X_1} h_1 &\leq \deg_{X_1} f - d_1 & \deg_{X_1} r_1 &\leq d_1 - 1 \\ \deg h_1 &\leq \deg f & \deg_{X_j} r_1 &\leq \deg_{X_j} f \\ \deg_{X_j} h_1 &\leq \deg_{X_j} f & \deg r &\leq \deg f. \end{aligned}$$

Then repeat this with f replaced by r_1, g_1 by g_2, and X_1 by X_2. □

Lemma. Let $S_1, \dots, S_n$ be non-empty finite subsets of a field $\mathbb{F}$, and let $h \in \mathbb{F}[X]$ be such that $\deg_{X_i} h < |S_i|$ for $i = 1, \dots, n$. Suppose h is identically 0 on $S = S_1 \times \cdots \times S_n \subseteq \mathbb{F}^n$. Then h is the zero polynomial.

Proof. Let $d_i = |S_i| - 1$. We induct on n. If $n = 1$, then we are done. For $n \geq 2$, consider h as a one-variable polynomial in $F[X_1, \dots, X_{n-1}]$ in X_n. Then we can write

$$h = \sum_{i=0}^{d_n} g_i(X_1, \dots, X_{n-1}) X_m^i.$$

Fix $(x_1, \dots, x_{n-1}) \in S_1 \times \cdots S_{n-1}$, and set $c_i = g_i(x_1, \dots, x_{n-1}) \in \mathbb{F}$. Then $\sum_{i=0}^{d_n} c_i X_n^i$ vanishes on S_n. So $c_i = g_i(x_1, \dots, x_{n-1}) = 0$ for all $(x_1, \dots, x_{n-1}) \in S_1 \times \cdots \times S_{n-1}$. So by induction, $g_i = 0$. So $h = 0$. □

Lemma. For $i = 1, \dots, n$, let S_i be a non-empty finite subset of $\mathbb{F}$, and let

$$g_i(X_i) = \prod_{s \in S_i} (X_i - s) \in \mathbb{F}[X_i] \subseteq F[X].$$

Then if $f \in \mathbb{F}[X]$ is identically zero on $S = S_1 \times \cdots \times S_n$, then there exists $h_i \in \mathbb{F}[X]$, $\deg h_i \leq \deg f - |S_i|$ and

$$f = \sum_{i=1}^{n} h_i g_i.$$

Proof. By the division algorithm, we can write

$$f = \sum_{i=1}^{n} h_i g_i + r,$$

where r satisfies $\deg_{X_i} r < \deg g_i$. But then r vanishes on $S_1 \times \cdots \times S_n$, as both f and g_i do. So $r = 0$. □

Theorem (Alon's combinatorial Nullstellensatz)**.** Let $S_1, \dots, S_n$ be non-empty finite subsets of $\mathbb{F}$ with $|S_i| = d_i + 1$. Let $f \in \mathbb{F}[X]$ have degree $d = \sum_{i=1}^{n} d_i$, and let the coefficient of $X_1^{d_1} \cdots X_n^{d_n}$ be non-zero. Then f is not identically zero on $S = S_1 \times \cdots \times S_n$.

Proof. Suppose for contradiction that f is identically zero on S. Define $g_i(X_i)$ and h_i as before such that

$$f = \sum h_i g_i.$$

Since the coefficient of $X_1^{d_1} \cdots X_n^{d_n}$ is non-zero in f, it is non-zero in some $h_j g_j$. But that's impossible, since

$$\deg h_j \leq \left(\sum_{i=1}^{n} d_i \right) - \deg g_j = \sum_{i \neq j} d_i - 1,$$

and so h_j cannot contain a $X_1^{d_1} \cdots \hat{X_j}^{d_j} \cdots X_n^{d_n}$ term. □

Theorem (Chevalley, 1935)**.** Let $f_1, \dots, f_m \in \mathbb{F}_q[X_1, \dots, X_n]$ be such that

$$\sum_{i=1}^{m} \deg f_i < n.$$

Then the f_i cannot have exactly one common zero.

Proof. Suppose not. We may assume that the common zero is $0 = (0, \dots, 0)$. Define

$$f = \prod_{i=1}^{m} (1 - f_i(X)^{q-1}) - \gamma \prod_{i=1}^{n} \prod_{s \in \mathbb{F}_q^\times} (X_i - s),$$

where γ is chosen so that $F(0) = 0$, namely the inverse of $\left(\prod_{s \in \mathbb{F}_q^\times} (-s) \right)^m$.

Now observe that for any non-zero x, the value of $f_i(x)^{q-1} = 1$, so $f(x) = 0$.

Thus, we can set $S_i = \mathbb{F}_q$, and they satisfy the hypothesis of the theorem. In particular, the coefficient of $X_1^{q-1} \cdots X_n^{q-1}$ is $\gamma \neq 0$. However, f vanishes on $\mathbb{F}_q^n$. This is a contradiction. □

Theorem (Warning)**.** Let $f(X) = f(X_1, \dots, X_n) \in \mathbb{F}_q[X]$ have degree $< n$. Then $N(f)$, the number of zeroes of f is a multiple of p.

Proof. We have

$$1 - f(x)^{q-1} = \begin{cases} 1 & f(x) = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, we know

$$N(f) = \sum_{x \in \mathbb{F}_q^n} (1 - f(x)^{q-1}) = -\sum_{x \in \mathbb{F}_q^n} f(x)^{q-1} \in \mathbb{F}_q.$$

Further, we know that if $k \geq 0$, then

$$\sum_{x \in \mathbb{F}_q^n} x^k = \begin{cases} -1 & k = q-1 \\ 0 & \text{otherwise} \end{cases}.$$

So let's write $f(x)^{q-1}$ as a linear combination of monomials. Each monomial has degree $< n(q-1)$. So there is at least one k such that the power of X_k in that monomial is $< q-1$. Then the sum over X_k vanishes for this monomial. So each monomial contributes 0 to the sum. □

Theorem (Cauchy–Davenport theorem)**.** Let p be a prime and $A, B \subseteq \mathbb{Z}_p$ be non-empty subsets with $|A| + |B| \leq p + 1$. Then $|A + B| \geq |A| + |B| - 1$.

Proof. Suppose for contradiction that $A + B \subseteq C \subseteq \mathbb{Z}_p$, and $|C| = |A| + |B| - 2$. Let's come up with a polynomial that encodes the fact that C contains the sum $A + B$. We let

$$f(X, Y) = \prod_{c \in C} (X + Y - c).$$

Then f vanishes on $A \times B$, and $\deg f = |C|$.

To apply the theorem, we check that the coefficient of $X^{|A|-1}Y^{|B|-1}$ is $\binom{|C|}{|A|-1}$, which is non-zero in $\mathbb{Z}_p$, since $C < p$. This contradicts Alon's combinatorial Nullstellensatz. □

Theorem (Erdös–Ginzburg–Ziv)**.** Let p be a prime and $a_1, \dots, a_{2p+1} \in \mathbb{Z}_p$. Then there exists $I \in [2p-1]^{(p)}$ such that

$$\sum_{i \in I} a_i = 0 \in \mathbb{Z}_p.$$

Proof. Define

$$f_1(X_1, \dots, X_{2p-1}) = \sum_{i=1}^{2p-1} X_i^{p-1}.$$

$$f_2(X_1, \dots, X_{2p-1}) = \sum_{i=1}^{2p-1} a_i X_i^{p-1}.$$

Then by Chevalley's theorem, we know there cannot be exactly one common zero. But 0 is one common zero. So there must be another. Take this solution, and let $I = \{i : x_i \neq 0\}$. Then $f_1(X) = 0$ is the same as saying $|I| = p$, and $f_2(X) = 0$ is the same as saying $\sum_{i \in I} a_i = 0$. □

Theorem. Let $A, B \subseteq \mathbb{Z}_p$ be such that $2 \leq |A| < |B|$ and $|A| + |B| \leq p + 2$. Then $A \dot{+} B \geq |A| + |B| - 2$.

Proof. Suppose not. Define

$$f(X, Y) = (X - Y) \prod_{c \in C} (X + Y - c),$$

where $A \dot{+} B \subseteq C \subseteq \mathbb{Z}_p$ and $|C| = |A| + |B| - 3$.

Then $\deg g = |A| + |B| - 2$, and the coefficient of $X^{|A|-1} Y^{|B|-1}$ is

$$\binom{|A| + |B| - 3}{|A| - 2} - \binom{|A| + |B| - 3}{|A| - 1} \neq 0.$$

Hence by Alon's combinatorial Nullstellensatz, $f(x, y)$ is not identically zero on $A \times B$. A contradiction. □

Corollary (Erdös–Heilbronn conjecture)**.** If $A, B \subseteq \mathbb{Z}_p$, non-empty and $|A| + |B| \leq p + 3$, and p is a prime, then $|A \dot{+} B| \geq |A| + |B| - 3$.

Proof. We may assume $2 \leq |A| \leq |B|$. Pick $a \in A$, and set $A' = A \setminus \{a\}$. Then

$$|A \dot{+} B| \geq |A' \dot{+} B| \geq |A'| + |B| - 2 = |A| + |B| - 3.$$ □

Theorem. If $2n + 1$ is a prime, then this can be done.

Proof. We may wlog assume the host is at 0. We want to partition $\mathbb{Z}_p \setminus \{0\} = \mathbb{Z}_p^\times$ into n pairs $\{x_i, x_i + d_i\}$. Consider the polynomial ring $\mathbb{Z}_p[X_1, \ldots, X_n] = \mathbb{Z}_p[X]$. We define

$$f(x) = \prod_i X_i(X_i + d_i) \prod_{i<j} (X_i - X_j)(X_i + d_i - X_j)(X_i - X_j - d_j)(X_i + d_i - X_j - d_j).$$

We want to show this is not identically zero on $\mathbb{Z}_p^n$

First of all, we have

$$\deg f = 4\binom{n}{2} + 2n = 2n^2.$$

So we are good. The coefficient of $X_1^{2n} \cdots X_n^{2n}$ is the same as that in

$$\prod X_i^2 \prod_{i<j} (X_i - X_j)^4 = \prod X_i^2 \prod_{i \neq j} (X_i - X_j)^2 = \prod X_i^{2n} \prod_{i \neq j} \left(1 - \frac{X_i}{X_j}\right)^2.$$

This, we are looking for the constant term in

$$\prod_{i \neq j} \left(1 - \frac{X_i}{X_j}\right)^2.$$

By a question on the example sheet, this is

$$\binom{2n}{2, 2, \ldots, 2} \neq 0 \text{ in } \mathbb{Z}_p.$$ □

Theorem. If $b_1, \dots, b_p \in \mathbb{Z}_p$ are such that $\sum b_i = 0$, then there exists numerations $a_1, \dots, a_p$ and $b_1, \dots, b_p$ of the elements of $\mathbb{Z}_p$ such that for each i, we have

$$a_i + b_i = c_i.$$

Proof. It suffices to show that for all (b_i), there are distinct $a_1, \cdots, a_{p-1}$ such that $a_i + b_i \neq a_j + b_j$ for all $i \neq j$. Consider the polynomial

$$\prod_{i<j} (X_i - X_j)(X_i + b_i - X_j - b_j).$$

The degree is

$$2\binom{p-1}{2} = (p-1)(p-2).$$

We then inspect the coefficient of $X_1^{p-2} \cdots X_{p-1}^{p-2}$, and checking that this is non-zero is the same as above. □