Part III — Combinatorics
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

What can one say about a collection of subsets of a finite set satisfying certain conditions in terms of containment, intersection and union? In the past fifty years or so, a good many fundamental results have been proved about such questions: in the course we shall present a selection of these results and their applications, with emphasis on the use of algebraic and probabilistic arguments.

The topics to be covered are likely to include the following:

- The de Bruijn–Erdös theorem and its extensions.
- The Graham–Pollak theorem and its extensions.
- The theorems of Sperner, EKR, LYMB, Katona, Frankl and Füredi.
- Isoperimetric inequalities: Kruskal–Katona, Harper, Bernstein, BTBT, and their applications.
- Correlation inequalities, including those of Harris, van den Berg and Kesten, and the Four Functions Inequality.
- Alon's Combinatorial Nullstellensatz and its applications.
- LLLL and its applications.

Pre-requisites

The main requirement is mathematical maturity, but familiarity with the basic graph theory course in Part II would be helpful.

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1 Hall's theorem

Definition (Bipartite graph)**.** We say $G = (X, Y; E)$ is a *bipartite graph* with bipartition X and Y if $(X \sqcup Y, E)$ is a graph such that every edge is between a vertex in X and a vertex in Y.

We say such a bipartite graph is (k, ℓ)*-regular* if every vertex in X has degree k and every vertex in Y has degree ℓ. A bipartite graph that is (k, ℓ)-regular for some $k, \ell \geq 1$ is said to be *biregular*.

Definition (Complete matching)**.** Let $G = (X, Y; E)$ be a bipartite graph with bipartition X and Y. A *complete matching* from X to Y is an injection $f : X \to Y$ such that $x\,f(x)$ is an edge for every $x \in X$.

Notation. Given a set X, we write $X^{(r)}$ for the set of all subsets of X with r elements, and similarly for $X^{(\geq r)}$ and $X^{(\leq r)}$.

2 Sperner systems

Definition (Chain)**.** A subset $C \subseteq S$ of a poset is a *chain* if any two of its elements are comparable.

Definition (Anti-chain)**.** A subset $A \subseteq S$ is an *anti-chain* if no two of its elements are comparable.

Definition (Graded poest)**.** We say $\mathcal{P} = (S, <)$ is a *graded poset* if we can write S as a disjoint union

$$S = \coprod_{i=0}^{n} S_i$$

such that

- S_i is an anti-chain; and
- $x < y$ iff there exists elements $x = z_i < z_{i+1} < \cdots < z_j = y$ such that $z_h \in S_h$.

Definition (Shadow)**.** Given $A \subseteq S_i$, the *shadow* at level $i - 1$ is

$$\partial A = \{x \in S_{i-1} : x < y \text{ for some } y \in A\}.$$

Definition (Downward-expanding poset)**.** A graded poset $P = (S, <)$ is said to be *downward-expanding* if

$$\frac{|\partial A|}{|S_{i-1}|} \geq \frac{|A|}{|S_i|}$$

for all $A \subseteq A_i$.

We similarly define *upward-expanding*, and say a poset is *expanding* if it is upward or downward expanding.

Definition (Weight)**.** The *weight* of a set $A \subseteq S$ is

$$w(A) = \sum_{i=0}^{n} \frac{|A \cap S_i|}{|S_i|}.$$

Definition (Regular poset)**.** We say a graded poset $(S, <)$ is *regular* if for each i, there exists r_i, s_i such that if $x \in A_i$, then x dominates r_i elements at level $i - 1$, and is dominated by s_i elements at level $i + 1$.

Definition (Symmetric chain)**.** We say a chain $\mathcal{C} = \{C_i, C_{i+1}, \ldots, C_{n-i}\}$ is *symmetric* if $|C_j| = j$ for all j.

3 The Kruskal–Katona theorem

4 Isoperimetric inequalities

Definition (Boundary)**.** Let G be a graph and $A \subseteq V(A)$. Then the *boundary* $b(A)$ is the set of all $x \in G$ such that $x \notin A$ but x is adjacent to A.

Definition (Neighbourhood)**.** Let G be a graph and $A \subseteq V(A)$. Then the *neighbourhood* of A is $N(A) = A \cup b(A)$.

Definition (Discrete cube)**.** Given a set X, we turn $\mathbb{P}(X)$ into a graph as follows: join x to y if $|x \Delta y| = 1$, i.e. if $x = y \cup \{a\}$ for some $a \notin y$, or vice versa.

This is the *discrete cube* Q_n, where $n = |X|$.

Definition (Simplicial ordering)**.** The *simplicial ordering* on Q_n is defined by $x < y$ if either $|x| < |y|$, or $|x| = |y|$ and $x < y$ in lex.

Definition (Binary order)**.** The binary order on $Q_n \cong \mathcal{P}(X)$ is given by $x < y$ if $\max x \Delta y \in y$.

Equivalently, define $\varphi : \mathcal{P}(X) \to \mathbb{N}$ by

$$\varphi(x) = \sum_{i \in x} 2^i.$$

Then $x < y$ if $\varphi(x) < \varphi(y)$.

5 Sum sets

6 Projections

Definition ((Uniform) cover)**.** We say a family $A_1, \ldots, A_r \subseteq [n]$ *covers* $[n]$ if

$$\bigcup_{i=1}^{r} A_r = [n],$$

and is a *uniform k-cover* if each $i \in [n]$ is in exactly k many of the sets.

Definition (Irreducible cover)**.** A uniform k-cover is *reducible* if it is the disjoint union of two uniform covers. Otherwise, it is *irreducible*.

7 Alon's combinatorial Nullstellensatz