Part III — Analysis of Partial Differential Equations

Theorems

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This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on four prototype linear equations: Laplace's equation, the heat equation, the wave equation and Schrödinger's equation. Emphasis will be given to modern functional analytic techniques, relying on a priori estimates, rather than explicit solutions, although the interaction with classical methods (such as the fundamental solution and Fourier representation) will be discussed. The following basic unifying concepts will be studied: well-posedness, energy estimates, elliptic regularity, characteristics, propagation of singularities, group velocity, and the maximum principle. Some non-linear equations may also be discussed. The course will end with a discussion of major open problems in PDEs.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

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1 Basics of PDEs

2 The Cauchy–Kovalevskaya theorem

2.1 The Cauchy–Kovalevskaya theorem

Theorem (Picard–Lindelöf theorem)**.** Suppose that there exists $r, K > 0$ such that $B_r(u_0) \subseteq U$, and

$$\|f(x) - f(y)\| \leq K\|x - u_0\|$$

for all $x, y \in B_r(u_0)$. Then there exists an $\varepsilon > 0$ depending on K, r and a unique C^1 function $u : (-\varepsilon, \varepsilon) \to U$ solving the Cauchy problem.

Theorem (Cauchy–Kovalevskaya for ODEs)**.** The series

$$u(t) = \sum_{k=0}^{\infty} u_k \frac{t^k}{k!}.$$

converges to the Picard–Lindelöf solution of the Cauchy problem if f is real analytic in a neighbourhood of u_0.

Lemma.

(i) If $g \gg f$ and g converges for $|x| < r$, then f converges for $|x| < r$.

(ii) If $f(x) = \sum_\alpha f_\alpha x^\alpha$ converges for $x < r$ and $0 < s\sqrt{n} < r$, then f has a majorant which converges on $|x| < s$.

Theorem (Cauchy–Kovalevskaya theorem)**.** Given the above assumptions, there exists a real analytic function $\mathbf{u} = \sum_\alpha \mathbf{u}_\alpha x^\alpha$ solving the PDE in a neighbourhood of the origin. Moreover, it is unique among real analytic functions.

Lemma. For $k = 1, \ldots, m$ and α a multi-index in $\mathbb{N}^n$, there exists a polynomial q_α^k in the power series coefficients of B and $\mathbf{c}$ such that any analytic solution to the PDE must be given by

$$\mathbf{u} = \sum_\alpha \mathbf{q}_\alpha(B, \mathbf{c}) x^\alpha,$$

where $\mathbf{q}_\alpha$ is the vector with entries q_α^k.

Moreover, all coefficients of q_α are non-negative.

Lemma. If $\tilde{B}_j \gg B_j$ and $\tilde{\mathbf{c}} \gg \mathbf{c}$, then

$$q_\alpha^k(\tilde{B}, \tilde{\mathbf{c}}) > q_\alpha^k(B, \mathbf{c}).$$

for all α. In particular, $\tilde{\mathbf{u}} \gg \mathbf{u}$.

Lemma. For any C and r, define

$$h(z, x') = \frac{Cr}{r - (x_1 + \cdots + x_{n-1}) - (z_1 + \cdots + z_m)}$$

If B and $\mathbf{c}$ are given by

$$B_j^*(z, x') = h(z, x') \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad \mathbf{c}^*(z, x') = h(z, x') \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

then the power series
$$\mathbf{u} = \sum_\alpha \mathbf{q}_\alpha(B, \mathbf{c}) x^\alpha$$
converges in a neighbourhood of the origin.

2.2 Reduction to first-order systems

3 Function spaces

3.1 The Hölder spaces

3.2 Sobolev spaces

Theorem. $L^P(U)$ is a Banach space with the L^p norm. □

Lemma. Suppose $v, \tilde{v} \in L^1_{loc}(U)$ are both αth weak derivatives of $u \in L^1_{loc}(U)$, then $v = \tilde{v}$ almost everywhere.

Theorem. For each $k = 0, 1, \ldots$ and $1 \leq p \leq \infty$, the space $W^{k,p}(U)$ is a Banach space.

3.3 Approximation of functions in Sobolev spaces

Theorem. Let $f \in L^1_{loc}(U)$. Then

(i) $f_\varepsilon \in C^\infty(U_\varepsilon)$.

(ii) $f_\varepsilon \to f$ almost everywhere as $\varepsilon \to 0$.

(iii) If in fact $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$, then $f_\varepsilon \to f$ in $L^p_{loc}(U)$, i.e. we have convergence in L^p on any $V \Subset U$. □

Lemma. Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set

$$u_\varepsilon = \eta_\varepsilon * u \text{ on } U_\varepsilon.$$

Then

(i) $u_\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$

(ii) If $V \Subset U$, then $u_\varepsilon \to u$ in $W^{k,p}(V)$.

Theorem (Global approximation)**.** Let $1 \leq p < \infty$, and $U \subseteq \mathbb{R}^n$ be open and bounded. Then $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

Theorem (Smooth approximation up to boundary)**.** Let $1 \leq p < \infty$, and $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose ∂U is $C^{0,1}$. Then $C^\infty(\bar{U}) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

3.4 Extensions and traces

Theorem (Extension of $W^{1,p}$ functions)**.** Suppose U is open, bounded and ∂U is C^1. Pick a bounded V such that $U \Subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

for $1 \leq p < \infty$ such that for any $u \in W^{1,p}(U)$,

(i) $Eu = u$ almost everywhere in U

(ii) Eu has support in V

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$, where the constant C depends on U, V, p but not u.

Theorem (Trace theorem)**.** Assume U is bounded and has C^1 boundary. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ for $1 \leq p < \infty$ such that $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$.

3.5 Sobolev inequalities

Lemma. Let $n \geq 2$ and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. For $1 \leq i \leq n$, denote

$$\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

and set

$$f(x) = f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n).$$

Then $f \in L^1(\mathbb{R}^n)$ with

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Theorem (Gagliardo–Nirenberg–Sobolev inequality)**.** Assume $n > p$. Then we have

$$W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*}(\mathbb{R}^n),$$

where

$$p^* = \frac{np}{n-p} > p,$$

and there exists $c > 0$ depending on n, p such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

In other words, $W^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^{p^*}(\mathbb{R}^n)$.

Corollary. Suppose $U \subseteq \mathbb{R}^n$ is open and bounded with C^1-boundary, and $1 \leq p < n$. Then if $p^* = \frac{np}{n-p}$, we have

$$W^{1,p}(U) \subseteq L^{p^*}(U),$$

and there exists $C = C(U, p, n)$ such that

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Corollary. Suppose U is open and bounded, and suppose $u \in W_0^{1,p}(U)$. For some $1 \leq p < n$, then we have the estimates

$$\|u\|_{L^q(U)} \leq C\|\mathrm{D}u\|_{L^p(U)}$$

for any $q \in [1, p^*]$. In particular,

$$\|u\|_{L^p(U)} \leq C\|\mathrm{D}u\|_{L^p(U)}.$$

Theorem (Morrey's inequality)**.** Suppose $n < p < \infty$. Then there exists a constant C depending only on p and n such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$ where $C = C(p, n)$ and $\gamma = 1 - \frac{n}{p} < 1$.

Corollary. Suppose $u \in W^{1,p}(U)$ for U open, bounded with C^1 boundary. Then there exists $u^* \in C^{0,\gamma}(U)$ such that $u = u^*$ almost everywhere and $\|u^*\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}$.

4 Elliptic boundary value problems

4.1 Existence of weak solutions

Theorem (Lax–Milgram theorem)**.** Let H be a real Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$. Suppose $B : H \times H \to \mathbb{R}$ is a bilinear mapping such that there exists constants $\alpha, \beta > 0$ so that

– $|B[u, v]| \leq \alpha \|u\|\|v\|$ for all $u, v \in H$ (boundedness)

– $\beta \|u\|^2 \leq B[u, u]$ (coercivity)

Then if $f : H \to \mathbb{R}$ is a bounded linear map, then there exists a unique $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Theorem (Energy estimates for B)**.** Suppose $a^{ij} = a^{ji}, b^i, c \in L^\infty(U)$, and there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for almost every $x \in U$ and $\xi \in \mathbb{R}^n$. Then if B is defined by

$$B[u, v] = \int_U \left(\sum_{ij} v_{x_i} a^{ij} u_{x_j} + \sum_i b^i u_{x_i} v + cuv \right) \,\mathrm{d}x,$$

then there exists $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

(i) $|B[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)}$ for all $u, v \in H^1_0(U)$

(ii) $\beta \|u\|^2_{H^1(U)} \leq B[u, u] + \gamma \|u\|^2_{L^2(U)}$.

Moreover, if $b^i \equiv 0$ and $c \geq 0$, then we can take γ.

Theorem. Let U, L be as above. There is a $\gamma \geq 0$ such that for any $\mu \geq \gamma$ and any $f \in L^2(U)$, there exists a unique weak solution to

$$\begin{aligned} Lu + \mu u &= f \text{ on } U \\ u &= 0 \text{ on } \partial U. \end{aligned}$$

Moreover, we have

$$\|u\|_{H^1(U)} \leq C \|f\|_{L^2(U)}$$

for some $C = C(L, U) \geq 0$.

Again, if $b^i \equiv 0$ and $c \geq 0$, then we may take $\gamma = 0$.

4.2 The Fredholm alternative

Theorem (Fredholm alternative)**.** Consider the problem

$$Lu = f, \quad u|_{\partial U} = 0. \tag{$*$}$$

For L a uniformly elliptic operator on an open bounded set U with C^1 boundary, either

(i) For each $f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ to $(*)$; or

(ii) There exists a non-zero weak solution $u \in H_0^1(U)$ to the *homogeneous problem*, i.e. $(*)$ with $f = 0$.

Theorem (Fredholm alternative)**.** Let H be a Hilbert space and $K : H \to H$ be a compact operator. Then

(i) $\ker(I - K)$ is finite-dimensional.

(ii) $\mathrm{im}(I - K)$ is finite-dimensional.

(iii) $\mathrm{im}(I - K) = \ker(I - K^\dagger)^\perp$.

(iv) $\ker(I - K) = \{0\}$ iff $\mathrm{im}(I - K) = H$.

(v) $\dim \ker(I - K) = \dim \ker(I - K^\dagger) = \dim \mathrm{coker}(I - K)$.

Lemma. Weak limits are unique. □

Lemma. Strong convergence implies weak convergence. □

Theorem (Weak compactness)**.** Let H be a separable Hilbert space, and suppose $(u_m)_{m=1}^\infty$ is a bounded sequence in H with $\|u_m\| \leq K$ for all m. Then u_m admits a subsequence $(u_{m_j})_{j=1}^\infty$ such that $u_{m_j} \rightharpoonup u$ for some $u \in H$ with $\|u\| \leq K$.

Lemma (Poincaré revisited)**.** Suppose $u \in H^1(\mathbb{R}^n)$. Let $Q = [\xi_1, \xi_1 + L] \times \cdots \times [\xi_n, \xi_n + L]$ be a cube of length L. Then we have

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|} \left(\int_Q u(x) \;\mathrm{d}x\right)^2 + \frac{nL^2}{2} \|\mathrm{D}u\|^2_{L^2(Q)}.$$

Theorem (Rellich–Kondrachov)**.** Let $U \subseteq \mathbb{R}^n$ be open, bounded with C^1 boundary. Then if $(u_m)_{m=1}^\infty$ is a sequence in $H^1(U)$ with $u_m \rightharpoonup u$, then $u_m \to u$ in L^2.

In particular, by weak compactness any sequence in $H^1(U)$ has a subsequence that is convergent in $L^2(U)$.

Corollary. Suppose $K : L^2(U) \to H^1(U)$ is a bounded linear operator. Then the composition

$$L^2(U) \xrightarrow{K} H^1(U) \hookrightarrow L^2(U)$$

is compact.

Theorem (Fredholm alternative for elliptic BVP)**.** Let L be a uniformly elliptic operator on an open bounded set U with C^1 boundary. Consider the problem

$$Lu = f, \quad u|_{\partial U} = 0. \tag{$*$}$$

Then exactly one of the following are true:

(i) For each $f \in L^2(U)$, there is a unique weak solution $u \in H_0^1(U)$ to $(*)$

(ii) There exists a non-zero weak solution $u \in H_0^1(U)$ to the *homogeneous problem*, i.e. $(*)$ with $f = 0$.

If this holds, then the dimension of $N = \ker L \subseteq H_0^1(U)$ is equal to the dimension of $N^* = \ker L^\dagger \subseteq H_0^1(U)$.

Finally, $(*)$ has a solution if and only if $(f, v)_{L^2(U)} = 0$ for all $v \in N^*$

4.3 The spectrum of elliptic operators

Theorem (Spectral theorem of compact operators)**.** Let $\dim H = \infty$, and $K : H \to H$ a compact operator. Then

- $\sigma(K) = \sigma_p(K) \cup \{0\}$. Note that 0 may or may not be in $\sigma_p(K)$.
- $\sigma(K) \setminus \{0\}$ is either finite or is a sequence tending to 0.
- If $\lambda \in \sigma_p(K)$, then $\ker(K - \lambda I)$ is finite-dimensional.
- If K is self-adjoint, i.e. $K = K^\dagger$ and H is separable, then there exists a countable orthonormal basis of eigenvectors.

Theorem (Spectrum of L)**.**

(i) There exists a countable set $\Sigma \subseteq \mathbb{R}$ such that there is a non-trivial solution to $Lu = \lambda u$ iff $\lambda \in \Sigma$.

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$, the values of an increasing sequence with $\lambda_k \to \infty$.

(iii) To each $\lambda \in \Sigma$ there is an associated finite-dimensional space

$$\mathcal{E}(\lambda) = \{u \in H_0^1(U) \mid u \text{ is a weak solution of } (*) \text{ with } f = 0\}.$$

We say $\lambda \in \Sigma$ is an *eigenvalue* and $u \in \mathcal{E}(\lambda)$ is the associated *eigenfunction*.

Theorem. Suppose L is a formally self-adjoint, positive, uniformly elliptic operator on U, an open bounded set with C^1 boundary. Then we can represent the eigenvalues of L as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

where each eigenvalue appears according to its multiplicity ($\dim \mathcal{E}(\lambda)$), and there exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(U)$ with $w_k \in H_0^1(U)$ an eigenfunction of L with eigenvalue λ_k.

4.4 Elliptic regularity

Lemma. If $u \in L^2(U)$, then $u \in H^1(V)$ iff

$$\|\Delta^h u\|_{L^2(V)} \leq C$$

for some C and all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$. In this case, we have

$$\frac{1}{\tilde{C}}\|\mathrm{D}u\|_{L^2(V)} \leq \|\Delta^h u\|_{L^2(V)} \leq \tilde{C}\|\mathrm{D}u\|_{L^2(V)}.$$

Lemma. If w, v and compactly supported in U, then

$$\int_U w\Delta_k^{-h} v \;\mathrm{d}x = \int_U (\Delta_k^h w) v \;\mathrm{d}x$$
$$\Delta_k^h(wv) = (\tau_k^h w)\Delta_k^h v + (\Delta_k^h w)v,$$

where $\tau_k^h w(x) = w(x + he_k)$.

Theorem (Interior regularity)**.** Suppose L is uniformly elliptic on an open set $U \subseteq \mathbb{R}^n$, and assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is such that

$$B[u, v] = (f, v)_{L^2(U)} \tag{†}$$

for all $v \in H_0^1(U)$. Then $u \in H_{loc}^2(U)$, and for each $V \Subset U$, we have

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

with C depending on L, V, U, but not f or u.

Theorem (Elliptic regularity)**.** If a^{ij}, b^i and c are $C^{m+1}(U)$ for some $m \in \mathbb{N}$, and $f \in H^m(U)$, then $u \in H_{loc}^{m+2}(U)$ and for $V \Subset W \Subset U$, we can estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(W)} + \|u\|_{L^2(W)}).$$

In particular, if m is large enough, then $u \in C_{loc}^2(U)$, and if all a^{ij}, b^i, c, f are smooth, then u is also smooth.

Theorem (Boundary H^2 regularity)**.** Assume $a^{ij} \in C^1(\bar{U})$, $b^1, c \in L^\infty(U)$, and $f \in L^2(U)$. Suppose $u \in H_0^1(U)$ is a weak solution of $Lu = f, u|_{\partial U} = 0$. Finally, we assume that ∂U is C^2. Then

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

If u is the *unique* weak solution, we can drop the $\|u\|_{L^2(U)}$ from the right hand side.

5 Hyperbolic equations

Theorem (Uniqueness of weak solution)**.** A weak solution, if exists, is unique.

Theorem (Existence of weak solution)**.** Given $\psi \in H_0^1(U)$ and $\psi' \in L^2(U)$, $f \in L^2(U_T)$, there exists a (unique) weak solution with

$$\|u\|_{H^1(U_T)} \leq C(\|\psi\|_{H^1(U)} + \|\psi'\|_{L^2(U)} + \|f\|_{L^2(U_T)}). \tag{$\dagger$}$$

Theorem. If $a^{ij}, b^i, c \in C^2(U_T)$ and $\partial U \in C^2$, then for $\psi \in H^2(U)$ and $\psi' \in H_0^1(U)$, and $f, f_t \in L^2(U_T)$, we have

$$\begin{aligned} u &\in H^2(U_T) \cap L^\infty((0,T); H^2(U)) \\ u_t &\in L^\infty((0,T), H_0^1(U)) \\ u_{tt} &\in L^\infty((0,T); L^2(U)) \end{aligned}$$

Theorem. If $a^{ij}, b^i, c \in C^{k+1}(\bar{U}_T)$ and ∂U is C^{k+1}, and

$$\begin{aligned} \partial_t^i u|_{\Sigma_0} &\in H_0^1(U) & i &= 0, \ldots, k \\ \partial_t^{k+1} u|_{\Sigma_0} &\in L^2(U) \\ \partial_t^i f &\in L^2((0,T); H^{k-i}(U)) & i &= 0, \ldots, k \end{aligned}$$

then $u \in H^{k+1}(U)$ and

$$\partial_t^i u \in L^\infty((0,T); H^{k+1-i}(U))$$

for $i = 0, \ldots, k+1$.

In particular, if everything is smooth, then we get a smooth solution.

Theorem. If u is a weak solution of the usual thing, and S' is spacelike, then $u|_D$ depends only on $\psi|_{S_0}, \psi'|_{S_0}$ and $f|_D$.