Part III — Algebraic Topology
Theorems with proof

Based on lectures by O. Randal-Williams
Notes taken by Dexter Chua

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Algebraic Topology assigns algebraic invariants to topological spaces; it permeates modern pure mathematics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. The course will not assume any knowledge of Algebraic Topology, but will go quite fast in order to reach more interesting material, so some previous exposure to simplicial homology or the fundamental group would be helpful. The Part III Differential Geometry course will also contain useful, relevant material.

Hatcher's book is especially recommended for the course, but there are many other suitable texts.

Contents

1 Homotopy

Proposition. If $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

2 Singular (co)homology

2.1 Chain complexes

Lemma. If $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ given by $[x] \mapsto [f_n(x)]$ is a well-defined homomorphism, where $x \in C_n$ is any element representing the homology class $[x] \in H_n(C_\bullet)$.

Proof. Before we check if it is well-defined, we first need to check if it is defined at all! In other words, we need to check if $f_n(x)$ is a cycle. Suppose $x \in C_n$ is a cycle, i.e. $d_n^C(x) = 0$. Then we have

$$d_n^D(f_n(x)) = f_{n-1}(d_n^C(x)) = f_{n-1}(0) = 0.$$

So $f_n(x)$ is a cycle, and it does represent a homology class.

To check this is well-defined, if $[x] = [y] \in H_n(C_\bullet)$, then $x - y = d_{n+1}^C(z)$ for some $z \in C_{n+1}$. So $f_n(x) - f_n(y) = f_n(d_{n+1}^C(z)) = d_{n+1}^D(f_{n+1}(z))$ is a boundary. So we have $[f_n(x)] = [f_n(y)] \in H_n(D_\bullet)$. □

2.2 Singular (co)homology

Lemma. If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} : \Delta^{n-2} \to \Delta^n$.

Proof. Both send $(t_0, \cdots, t_{n-2})$ to

$$(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{j-2}, 0, t_{j-1}, \cdots, t_{n-2}).$$ □

Corollary. The homomorphism $d_{n-1} \circ d_n : C_n(X) \to C_{n-2}(X)$ vanishes.

Proof. It suffices to check this on each basis element $\sigma : \Delta^n \to X$. We have

$$d_{n-1} \circ d_n(\sigma) = \sum_{i=0}^{n-1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \delta_j \circ \delta_i.$$

We use the previous lemma to split the sum up into $i < j$ and $i \geq j$:

$$\begin{aligned}
&= \sum_{i<j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i \\
&= \sum_{i<j} (-1)^{i+j} \sigma \circ \delta_i \circ \delta_{j-1} + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i \\
&= \sum_{i \leq j} (-1)^{i+j+1} \sigma \circ \delta_i \circ \delta_j + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i \\
&= 0.
\end{aligned}$$ □

Proposition. If $f : X \to Y$ is a continuous map of topological spaces, then the maps

$$\begin{aligned}
f_n : C_n(X) &\to C_n(Y) \\
(\sigma : \Delta^n \to X) &\mapsto (f \circ \sigma : \Delta^n \to Y)
\end{aligned}$$

give a chain map. This induces a map on the homology (and cohomology).

Proof. To see that the f_n and d_n commute, we just notice that f_n acts by composing on the left, and d_n acts by composing on the right, and these two operations commute by the associativity of functional composition. $\square$

Proposition. If $f : X \to Y$ is a homeomorphism, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism of abelian groups.

Proof. If $g : Y \to X$ is an inverse to f, then g_* is an inverse to f_*, as $f_* \circ g_* = (f \circ g)_* = (\mathrm{id})_* = \mathrm{id}$, and similarly the other way round. $\square$

Lemma. If X is path-connected and non-empty, then $H_0(X) \cong \mathbb{Z}$.

Proof. Define a homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$ given by

$$\sum n_\sigma \sigma \mapsto \sum n_\sigma.$$

Then this is surjective. We claim that the composition

$$C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

is zero. Indeed, each simplex has two ends, and a $\sigma : \Delta^1 \to X$ is mapped to $\sigma \circ \delta_0 - \sigma \circ \delta_1$, which is mapped by ε to $1 - 1 = 0$.

Thus, we know that $\varepsilon(\sigma) = \varepsilon(\sigma + d\tau)$ for any $\sigma \in C_0(X)$ and $\tau \in C_1(X)$. So we obtain a well-defined map $\varepsilon : H_0(X) \to \mathbb{Z}$ mapping $[x] \mapsto \varepsilon(x)$, and this is surjective as X is non-empty.

So far, this is true for a general space. Now we use the path-connectedness condition to show that this map is indeed injective. Suppose $\sum n_\sigma \sigma \in C_0(X)$ lies in $\ker \varepsilon$. We choose an $x_0 \in X$. As X is path-connected, for each of $\Delta^0 \to X$ we can choose a path $\tau_\sigma : \Delta^1 \to X$ with $\tau_\sigma \circ \delta_0 = \sigma$ and $\tau_\sigma \circ \delta_1 = x_0$.

Given these 1-simplices, we can form a 1-chain $\sum n_\sigma \tau_\sigma \in C_1(X)$, and

$$d_1 \left(\sum n_\sigma \tau_\sigma\right) = \sum n_\sigma(\sigma + x_0) = \sum n_\sigma \cdot \sigma - \left(\sum n_\sigma\right) x_0.$$

Now we use the fact that $\sum n_\sigma = 0$. So $\sum n_\sigma \cdot \sigma$ is a boundary. So it is zero in $H_0(X)$. $\square$

Proposition. For any space X, we have $H_0(X)$ is a free abelian group generated by the path components of X.

3 Four major tools of (co)homology

3.1 Homotopy invariance

Theorem (Homotopy invariance theorem)**.** Let $f \simeq g : X \to Y$ be homotopic maps. Then they induce the same maps on (co)homology, i.e.

$$f_* = g_* : H_\bullet(X) \to H_\bullet(Y)$$

and

$$f^* = g^* : H^\bullet(Y) \to H^\bullet(X).$$

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_\bullet(X) \to H_\bullet(Y)$ and $f^* : H^\bullet(Y) \to H^\bullet(X)$ are isomorphisms.

Proof. If $g : Y \to X$ is a homotopy inverse, then

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_\bullet(X)}.$$

Similarly, we have $f_* \circ g_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_\bullet(Y)}$. So f_* is an isomorphism with an inverse g_*.

The case for cohomology is similar. $\square$

3.2 Mayer-Vietoris

Lemma. In a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0\ ,$$

the map f is injective; g is surjective, and $C \cong B/A$.

Theorem (Mayer-Vietoris theorem)**.** Let $X = A \cup B$ be the union of two open subsets. We have inclusions

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"] & A \arrow[d, hook, "j_A"] \\ B \arrow[r, hook, "j_B"] & X \end{tikzcd}.$$

Then there are homomorphisms $\partial_{MV} : H_n(X) \to H_{n-1}(A \cap B)$ such that the following sequence is exact:

$$\xrightarrow{\partial_{MV}} H_n(A \cap B) \xrightarrow{i_{A*} \oplus i_{B*}} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*} - j_{B*}} H_n(X) \xrightarrow{\partial_{MV}}$$

$$\longrightarrow H_{n-1}(A \cap B) \xrightarrow{i_{A*} \oplus i_{B*}} H_{n-1}(A) \oplus H_{n-1}(B) \xrightarrow{j_{A*} - j_{B*}} H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(A) \oplus H_0(B) \xrightarrow{j_{A*} - j_{B*}} H_0(X) \longrightarrow 0$$

Furthermore, the Mayer-Vietoris sequence is *natural*, i.e. if $f : X = A \cup B \to Y = U \cup V$ satisfies $f(A) \subseteq U$ and $f(B) \subseteq V$, then the diagram

$$\begin{tikzcd}
H_{n+1}(X) \arrow[r, "\partial_{MV}"] \arrow[d, "f_*"] & H_n(A \cap B) \arrow[r, "i_{A*} \oplus i_{B*}"] \arrow[d, "f|_{A \cap B *}"] & H_n(A) \oplus H_n(B) \arrow[r, "j_{A*} - j_{B*}"] \arrow[d, "f|_{A*} \oplus f|_{B*}"] & H_n(X) \arrow[d, "f_*"] \\
H_{n+1}(Y) \arrow[r, "\partial_{MV}"] & H_n(U \cap V) \arrow[r, "i_{U*} \oplus i_{V*}"] & H_n(U) \oplus H_n(V) \arrow[r, "j_{U*} - j_{V*}"] & H_n(Y)
\end{tikzcd}$$

commutes.

3.3 Relative homology

Theorem (Exact sequence for relative homology)**.** There are homomorphisms $\partial : H_n(X, A) \to H_{n-1}(A)$ given by mapping

$$\big[[c]\big] \mapsto [d_n c].$$

This makes sense because if $c \in C_n(X)$, then $[c] \in C_n(X)/C_n(A)$. We know $[d_n c] = 0 \in C_{n-1}(X)/C_{n-1}(A)$. So $d_n c \in C_{n-1}(A)$. So this notation makes sense.

Moreover, there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "q_*"] & H_n(X, A) \arrow[dll, out=0, in=180, "\partial"] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r, "q_*"] & H_{n-1}(X, A) \arrow[r] & \cdots,
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_0(X) \arrow[r, "q_*"] & H_0(X, A) \arrow[r] & 0
\end{tikzcd}$$

where i_* is induced by $i : C_\bullet(A) \to C_\bullet(X)$ and q_* is induced by the quotient $q : C_\bullet(X) \to C_\bullet(X, A)$.

3.4 Excision theorem

Theorem (Excision theorem)**.** Let (X, A) be a pair of spaces, and $Z \subseteq A$ be such that $\overline{Z} \subseteq \mathring{A}$ (the closure is taken in X). Then the map

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

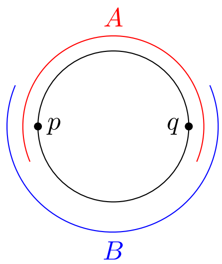
is an isomorphism.

3.5 Applications

Theorem. We have

$$H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We can split S^1 up as



We want to apply Mayer-Vietoris. We have

$$A \cong B \cong \mathbb{R} \simeq *, \quad A \cap B \cong \mathbb{R} \coprod \mathbb{R} \simeq \{p\} \coprod \{q\}.$$

We obtain

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & 0 \arrow[d, equal] & & \\
\cdots \arrow[r] & H_1(A \cap B) \arrow[r] & H_1(A) \oplus H_1(B) \arrow[r] & H_1(S^1) \arrow[dll, "\partial"] & \\
 & H_0(A \cap B) \arrow[r, "i_{A*} \oplus i_{B*}"] \arrow[d, equal] & H_0(A) \oplus H_0(B) \arrow[r] \arrow[d, equal] & H_0(S^1) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \oplus Z & \mathbb{Z} &
\end{tikzcd}$$

Notice that the map into $H_1(S^1)$ is zero. So the kernel of ∂ is trivial, i.e. ∂ is an injection. So $H_1(S^1)$ is isomorphic to the image of ∂, which is, by exactness, the kernel of $i_{A*} \oplus i_{B*}$. So we want to know what this map does.

We know that $H_0(A \cap B) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by p and q, and the inclusion map sends each of p and q to the unique connected components of A and B. So the homology classes are both sent to $(1, 1) \in H_0(A) \oplus H_0(B) \cong \mathbb{Z} \oplus \mathbb{Z}$. We then see that the kernel of $i_{A*} \oplus i_{B*}$ is generated by $(p - q)$, and is thus isomorphic to $\mathbb{Z}$. So $H_1(S^1) \cong \mathbb{Z}$.

By looking higher up the exact sequence, we see that all other homology groups vanish. $\square$

Theorem. For any $n \geq 1$, we have

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We again cut up S^n as

$$\begin{aligned} A &= S^n \setminus \{N\} \cong \mathbb{R}^n \simeq *, \\ B &= S^n \setminus \{S\} \cong \mathbb{R}^n \simeq *, \end{aligned}$$

where N and S are the north and south poles. Moreover, we have

$$A \cap B \cong \mathbb{R} \times S^{n-1} \simeq S^{n-1}$$

So we can "induct up" using the Mayer-Vietoris sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(S^{n-1}) \arrow[r] & H_i(*) \oplus H_i(*) \arrow[r] & H_i(S^n) \arrow[dll, "\partial"] & \\
 & H_{i-1}(S^{n-1}) \arrow[r] & H_{i-1}(*) \oplus H_{i-1}(*) \arrow[r] & H_{i-1}(S^n) \arrow[r] & \cdots
\end{tikzcd}$$

Now suppose $n \geq 2$, as we already did S^1 already. If $i > 1$, then $H_i(*) = 0 = H_{i-1}(*)$. So the Mayer-Vietoris map

$$H_i(S^n) \xrightarrow{\partial} H_{i-1}(S^{n-1})$$

is an isomorphism.

All that remains is to look at $i = 0, 1$. The $i = 0$ case is trivial. For $i = 1$, we look at

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & & \\
 & \cdots \arrow[r] & H_1(*) \oplus H_1(*) \arrow[r] & H_1(S^n) \arrow[dll, "\partial"] & \\
 & H_0(S^{n-1}) \arrow[r, "f"] \arrow[d, equal] & H_0(*) \oplus H_0(*) \arrow[r] \arrow[d, equal] & H_0(S^n) \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

To conclude that $H_1(S^n)$ is trivial, it suffices to show that the map f is injective. By picking the canonical generators, it is given by $1 \mapsto (1, 1)$. So we are done. $\square$

Corollary. If $n \neq m$, then $S^{n-1} \not\simeq S^{m-1}$, since they have different homology groups.

Corollary. If $n \neq m$, then $\mathbb{R}^n \not\cong \mathbb{R}^m$.

Proposition.

(i) $\deg(\mathrm{id}_{S^n}) = 1$.

(ii) If f is not surjective, then $\deg(f) = 0$.

(iii) We have $\deg(f \circ g) = (\deg f)(\deg g)$.

(iv) Homotopic maps have equal degrees.

Proof.

(i) Obvious.

(ii) If f is not surjective, then f can be factored as

$$S^n \xrightarrow{f} S^n \setminus \{p\} \hookrightarrow S^n ,$$

where p is some point not in the image of f. But $S^n \setminus \{p\}$ is contractible. So f_* factors as

$$f_* : H_n(S^n) \longrightarrow H_n(*) = 0 \longrightarrow H_n(S^n) .$$

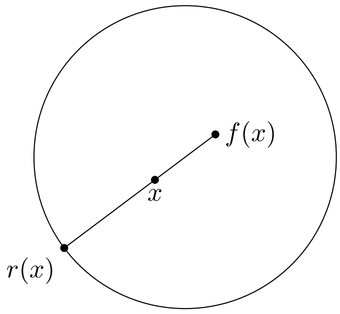
So f_* is the zero homomorphism, and is thus multiplication by 0.

(iii) This follows from the functoriality of H_n.

(iv) Obvious as well. $\square$

Corollary (Brouwer's fixed point theorem)**.** Any map $f : D^n \to D^n$ has a fixed point.

Proof. Suppose f has no fixed point. Define $r : D^n \to S^{n-1} = \partial D^n$ by taking the intersection of the ray from $f(x)$ through x with ∂D^n. This is continuous.



Now if $x \in \partial D^n$, then $r(x) = x$. So we have a map

$$S^{n-1} = \partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n = S^{n-1},$$

and the composition is the identity. This is a contradiction — contracting D^n to a point, this gives a homotopy from the identity map $S^{n-1} \to S^{n-1}$ to the constant map at a point. This is impossible, as the two maps have different degrees. □

Proposition. A reflection $r : S^n \to S^n$ about a hyperplane has degree -1. As before, we cover S^n by

$$A = S^n \setminus \{N\} \cong \mathbb{R}^n \simeq *,$$
$$B = S^n \setminus \{S\} \cong \mathbb{R}^n \simeq *,$$

where we suppose the north and south poles lie in the hyperplane of reflection. Then both A and B are invariant under the reflection. Consider the diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\partial_{MV}", "\sim"'] \arrow[d, "r_*"] & H_{n-1}(A \cap B) \arrow[d, "r_*"] & H_{n-1}(S^{n-1}) \arrow[l, "\sim"'] \arrow[d, "r_*"] \\
H_n(S^n) \arrow[r, "\partial_{MV}", "\sim"'] & H_{n-1}(A \cap B) & H_{n-1}(S^{n-1}) \arrow[l, "\sim"']
\end{tikzcd}$$

where the S^{n-1} on the right most column is given by contracting $A \cap B$ to the equator. Note that r restricts to a reflection on the equator. By tracing through the isomorphisms, we see that $\deg(r) = \deg(r|_{\mathrm{equator}})$. So by induction, we only have to consider the case when $n = 1$. Then we have maps

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^1) \arrow[r, "\partial_{MV}", "\sim"'] \arrow[d, "r_*"] & H_0(A \cap B) \arrow[r] \arrow[d, "r_*"] & H_0(A) \oplus H_0(B) \arrow[d, "r_* \oplus r_*"] \\
0 \arrow[r] & H_1(S^1) \arrow[r, "\partial_{MV}", "\sim"'] & H_0(A \cap B) \arrow[r] & H_0(A) \oplus H_0(B)
\end{tikzcd}$$

Now the middle vertical map sends $p \mapsto q$ and $q \mapsto p$. Since $H_1(S^1)$ is given by the kernel of $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$, and is generated by $p - q$, we see that this sends the generator to its negation. So this is given by multiplication by -1. So the degree is -1.

Corollary. The antipodal map $a : S^n \to S^n$ given by

$$a(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, -x_{n+1})$$

has degree $(-1)^{n+1}$ because it is a composition of $(n+1)$ reflections.

Corollary (Hairy ball theorem)**.** S^n has a nowhere 0 vector field iff n is odd. More precisely, viewing $S^n \subseteq \mathbb{R}^{n+1}$, a vector field on S^n is a map $v : S^n \to \mathbb{R}^{n+1}$ such that $\langle v(x), x\rangle = 0$, i.e. $v(x)$ is perpendicular to x.

Proof. If n is odd, say $n = 2k-1$, then

$$v(x_1, y_1, x_2, y_2, \cdots, x_k, y_k) = (y_1, -x_1, y_2, -x_2, \cdots, y_k, -x_k)$$

works.

Conversely, if $v : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is a vector field, we let $w = \frac{v}{|v|} : S^n \to S^n$. We can construct a homotopy from w to the antipodal map by "linear interpolation", but in a way so that it stays on the sphere. We let

$$\begin{aligned} H : [0, \pi] \times S^n &\to S^n \\ (t, x) &\mapsto \cos(t)x + \sin(t)w(x) \end{aligned}$$

Since $w(x)$ and x are perpendicular, it follows that this always has norm 1, so indeed it stays on the sphere.

Now we have

$$H(0, x) = x, \quad H(\pi, x) = -x.$$

So this gives a homotopy from id to a, which is a contradiction since they have different degrees. $\square$

Lemma. Let M be a d-dimensional manifold (i.e. a Hausdorff, second-countable space locally homeomorphic to $\mathbb{R}^d$). Then

$$H_n(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & n = d \\ 0 & \text{otherwise} \end{cases}.$$

This is known as the *local homology*.

Proof. Let U be an open neighbourhood isomorphic to $\mathbb{R}^d$ that maps $x \mapsto 0$. We let $Z = M \setminus U$. Then

$$\overline{Z} = Z \subseteq M \setminus \{x\} = (M \overset{\circ}{\setminus} \{x\}).$$

So we can apply excision, and get

$$H_n(U, U \setminus \{x\}) = H_n(M \setminus Z, (M \setminus \{x\}) \setminus \mathbb{Z}) \cong H_n(M, M \setminus \{x\}).$$

So it suffices to do this in the case $M \cong \mathbb{R}^d$ and $x = 0$. The long exact sequence for relative homology gives

$$H_n(\mathbb{R}^d) \longrightarrow H_n(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \longrightarrow H_{n-1}(\mathbb{R}^d \setminus \{0\}) \longrightarrow H_{n-1}(\mathbb{R}^d)\ .$$

Since $H_n(\mathbb{R}^d) = H_{n-1}(\mathbb{R}^d) = 0$ for $n \geq 2$ large enough, it follows that

$$H_n(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \cong H_{n-1}(\mathbb{R}^d \setminus \{0\}) \cong H_{n-1}(S^{d-1}),$$

and the result follows from our previous computation of the homology of S^{d-1}. We will have to check the lower-degree terms manually, but we shall not. $\square$

Theorem. Let $f : S^d \to S^d$ be a map. Suppose there is a $y \in S^d$ such that

$$f^{-1}(y) = \{x_1, \cdots, x_k\}$$

is finite. Then

$$\deg(f) = \sum_{i=1}^{k} \deg(f)_{x_i}.$$

Proof. Note that by excision, instead of computing the local degree at x_i via $H_d(S^d, S^d \setminus \{x\})$, we can pick a neighbourhood U_i of x_i and a neighbourhood V of x_i such that $f(U_i) \subseteq V$, and then look at the map

$$f_* : H_d(U_i, U_i \setminus \{x_i\}) \to H_d(V, V \setminus \{y\})$$

instead. Moreover, since S^d is Hausdorff, we can pick the U_i such that they are disjoint. Consider the huge commutative diagram:

$$\begin{tikzcd}
H_d(S^d) \arrow[r, "f_*"] \arrow[d] & H_d(S^d) \arrow[d, "\sim"] \\
H_d(S^d, S^d \setminus \{x_1, \cdots, x_k\}) \arrow[r, "f_*"] & H_d(S^d, S^d \setminus \{y\}) \\
H_d\left(\coprod U_i, \coprod (U_i \setminus x_i)\right) \arrow[u, "\text{excision}"] & \\
\bigoplus_{i=1}^k H_d(U_i, U_i \setminus x_i) \arrow[u, "\sim"] \arrow[r, "\bigoplus f_*"] & H_d(V, V \setminus \{y\}) \arrow[uu, "\sim"]
\end{tikzcd}$$

Consider the generator 1 of $H_d(S^d)$. By definition, its image in $H_d(S^d)$ is $\deg(f)$. Also, its image in $\bigoplus H_d(U_i, U_i \setminus \{x_i\})$ is $(1, \cdots, 1)$. The bottom horizontal map sends this to $\sum \deg(f)_{x_i}$. So by the isomorphisms, it follows that

$$\deg(f) = \sum_{i=1}^{k} \deg(f)_x. \qquad \square$$

3.6 Repaying the technical debt

Theorem (Snake lemma)**.** Suppose we have a short exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{q_\bullet} C_\bullet \longrightarrow 0\ .$$

Then there are maps

$$\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$$

such that there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "q_*"] & H_n(C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[near end, above]{∂_*} ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
 & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "q_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}.$$

Proof. The proof of this is in general not hard. It just involves a lot of checking of the details, such as making sure the homomorphisms are well-defined, are actually homomorphisms, are exact at all the places etc. The only important and non-trivial part is just the construction of the map ∂_*.

First we look at the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "q_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "q_{n-1}"] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

To construct $\partial_* : H_n(C) \to H_{n-1}(A)$, let $[x] \in H_n(C)$ be a class represented by $x \in Z_n(C)$. We need to find a cycle $z \in A_{n-1}$. By exactness, we know the map $q_n : B_n \to C_n$ is surjective. So there is a $y \in B_n$ such that $q_n(y) = x$. Since our target is A_{n-1}, we want to move down to the next level. So consider $d_n(y) \in B_{n-1}$. We would be done if $d_n(y)$ is in the image of i_{n-1}. By exactness, this is equivalent saying $d_n(y)$ is in the kernel of q_{n-1}. Since the diagram is commutative, we know

$$q_{n-1} \circ d_n(y) = d_n \circ q_n(y) = d_n(x) = 0,$$

using the fact that x is a cycle. So $d_n(y) \in \ker q_{n-1} = \operatorname{im} i_{n-1}$. Moreover, by exactness again, i_{n-1} is injective. So there is a unique $z \in A_{n-1}$ such that $i_{n-1}(z) = d_n(y)$. We have now produced our z.

We are not done. We have $\partial_*[x] = [z]$ as our candidate definition, but we need to check many things:

(i) We need to make sure ∂_* is indeed a homomorphism.

(ii) We need $d_{n-1}(z) = 0$ so that $[z] \in H_{n-1}(A)$;

(iii) We need to check $[z]$ is well-defined, i.e. it does not depend on our choice of y and x for the homology class $[x]$.

(iv) We need to check the exactness of the resulting sequence.

We now check them one by one:

(i) Since everything involved in defining ∂_* are homomorphisms, it follows that ∂_* is also a homomorphism.

(ii) We check $d_{n-1}(z) = 0$. To do so, we need to add an additional layer.

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "q_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "d_{n-1}"] & B_{n-1} \arrow[r, "q_{n-1}"] \arrow[d, "d_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & 0 \\ 0 \arrow[r] & A_{n-2} \arrow[r, "i_{n-2}"] & B_{n-2} \arrow[r, "q_{n-2}"] & C_{n-2} \arrow[r] & 0 \end{tikzcd}$$

We want to check that $d_{n-1}(z) = 0$. We will use the commutativity of the diagram. In particular, we know

$$i_{n-2} \circ d_{n-1}(z) = d_{n-1} \circ i_{n-1}(z) = d_{n-1} \circ d_n(y) = 0.$$

By exactness at A_{n-2}, we know i_{n-2} is injective. So we must have $d_{n-1}(z) = 0$.

(iii) (a) First, in the proof, suppose we picked a different y' such that $q_n(y') = q_n(y) = x$. Then $q_n(y' - y) = 0$. So $y' - y \in \ker q_n = \operatorname{im} i_n$. Let $a \in A_n$ be such that $i_n(a) = y' - y$. Then

$$\begin{aligned} d_n(y') &= d_n(y' - y) + d_n(y) \\ &= d_n \circ i_n(a) + d_n(y) \\ &= i_{n-1} \circ d_n(a) + d_n(y). \end{aligned}$$

Hence when we pull back $d_n(y')$ and $d_n(y)$ to A_{n-1}, the results differ by the boundary $d_n(a)$, and hence produce the same homology class.

(b) Suppose $[x'] = [x]$. We want to show that $\partial_*[x] = \partial_*[x']$. This time, we add a layer above.

$$\begin{tikzcd} 0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d_{n+1}"] & B_{n+1} \arrow[r, "q_{n+1}"] \arrow[d, "d_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d_{n+1}"] & 0 \\ 0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n"] & B_n \arrow[r, "q_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "q_{n-1}"] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

By definition, since $[x'] = [x]$, there is some $c \in C_{n+1}$ such that

$$x' = x + d_{n+1}(c).$$

By surjectivity of q_{n+1}, we can write $c = q_{n+1}(b)$ for some $b \in B_{n+1}$. By commutativity of the squares, we know

$$x' = x + q_n \circ d_{n+1}(b).$$

The next step of the proof is to find some y such that $q_n(y) = x$. Then

$$q_n(y + d_{n+1}(b)) = x'.$$

So the corresponding y' is $y' = y + d_{n+1}(b)$. So $d_n(y) = d_n(y')$, and hence $\partial_*[x] = \partial_*[x']$.

(iv) This is yet another standard diagram chasing argument. When reading this, it is helpful to look at a diagram and see how the elements are chased along. It is even more beneficial to attempt to prove this yourself.

(a) $\operatorname{im} i_* \subseteq \ker q_*$: This follows from the assumption that $i_n \circ q_n = 0$.

(b) $\ker q_* \subseteq \operatorname{im} i_*$: Let $[b] \in H_n(B)$. Suppose $q_*([b]) = 0$. Then there is some $c \in C_{n+1}$ such that $q_n(b) = d_{n+1}(c)$. By surjectivity of q_{n+1}, there is some $b' \in B_{n+1}$ such that $q_{n+1}(b') = c$. By commutativity, we know $q_n(b) = q_n \circ d_{n+1}(b')$, i.e.

$$q_n(b - d_{n+1}(b')) = 0.$$

By exactness of the sequence, we know there is some $a \in A_n$ such that

$$i_n(a) = b - d_{n+1}(b').$$

Moreover,

$$i_{n-1} \circ d_n(a) = d_n \circ i_n(a) = d_n(b - d_{n+1}(b')) = 0,$$

using the fact that b is a cycle. Since i_{n-1} is injective, it follows that $d_n(a) = 0$. So $[a] \in H_n(A)$. Then

$$i_*([a]) = [b] - [d_{n+1}(b')] = [b].$$

So $[b] \in \operatorname{im} i_*$.

(c) $\operatorname{im} q_* \subseteq \ker \partial_*$: Let $[b] \in H_n(B)$. To compute $\partial_*(q_*([b]))$, we first pull back $q_n(b)$ to $b \in B_n$. Then we compute $d_n(b)$ and then pull it back to A_{n+1}. However, we know $d_n(b) = 0$ since b is a cycle. So $\partial_*(q_*([b])) = 0$, i.e. $\partial_* \circ q_* = 0$.

(d) $\ker \partial_* \subseteq \operatorname{im} q_*$: Let $[c] \in H_n(C)$ and suppose $\partial_*([c]) = 0$. Let $b \in B_n$ be such that $q_n(b) = c$, and $a \in A_{n-1}$ such that $i_{n-1}(a) = d_n(b)$. By assumption, $\partial_*([c]) = [a] = 0$. So we know a is a boundary, say $a = d_n(a')$ for some $a' \in A_n$. Then by commutativity we know $d_n(b) = d_n \circ i_n(a')$. In other words,

$$d_n(b - i_n(a')) = 0.$$

So $[b - i_n(a')] \in H_n(B)$. Moreover,

$$q_*([b - i_n(a')]) = [q_n(b) - q_n \circ i_n(a')] = [c].$$

So $[c] \in \operatorname{im} q_*$.

(e) $\operatorname{im} \partial_* \subseteq \ker i_*$: Let $[c] \in H_n(C)$. Let $b \in B_n$ be such that $q_n(b) = c$, and $a \in A_{n-1}$ be such that $i_n(a) = d_n(b)$. Then $\partial_*([c]) = [a]$. Then

$$i_*([a]) = [i_n(a)] = [d_n(b)] = 0.$$

So $i_* \circ \partial_* = 0$.

(f) $\ker i_* \subseteq \operatorname{im} \partial_*$: Let $[a] \in H_n(A)$ and suppose $i_*([a]) = 0$. So we can find some $b \in B_{n+1}$ such that $i_n(a) = d_{n+1}(b)$. Let $c = q_{n+1}(b)$. Then

$$d_{n+1}(c) = d_{n+1} \circ q_{n+1}(b) = q_n \circ d_{n+1}(b) = q_n \circ i_n(a) = 0.$$

So $[c] \in H_n(C)$. Then $[a] = \partial_*([c])$ by definition of ∂_*. So $[a] \in \operatorname{im} \partial_*$. □

Lemma (Five lemma)**.** Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\
\downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E'
\end{array}$$

If the two rows are exact, m and p are isomorphisms, q is injective and ℓ is surjective, then n is also an isomorphism.

Proof. The philosophy is exactly the same as last time.

We first show that n is surjective. Let $c' \in C'$. Then we obtain $d' = t(c') \in D'$. Since p is an isomorphism, we can find $d \in D$ such that $p(d) = d'$. Then we have

$$q(j(d)) = u(p(d)) = u(f(c')) = 0.$$

Since q is injective, we know $j(d) = 0$. Since the sequence is exact, there is some $c \in C$ such that $h(c) = d$.

We are not yet done. We do not know that $n(c) = c'$. All we know is that $d(n(c)) = d(c')$. So $d(c' - n(c)) = 0$. By exactness at C', we can find some b' such that $s(b') = n(c) - c'$. Since m was surjective, we can find $b \in B$ such that $m(b) = b'$. Then we have

$$n(g(b)) = n(c) - c'.$$

So we have

$$n(c - g(b)) = c'.$$

So n is surjective.

Showing that n is injective is similar. □

Corollary. Let $f : (X, A) \to (Y, B)$ be a map of pairs, and that any two of $f_* : H_*(X, A) \to H_*(Y, B)$, $H_*(X) \to H_*(Y)$ and $H_*(A) \to H_*(B)$ are isomorphisms. Then the third is also an isomorphism.

Proof. Follows from the long exact sequence and the five lemma. □

Lemma. If $f_\bullet$ and $g_\bullet$ are chain homotopic, then $f_* = g_* : H_*(C_\bullet) \to H_*(D_\bullet)$.

Proof. Let $[c] \in H_n(C_\bullet)$. Then we have

$$g_n(c) - f_n(c) = d_{n+1}^D F_n(c) + F_{n-1}(d_n^C(c)) = d_{n+1}^D F_n(c),$$

where the second term dies because c is a cycle. So we have $[g_n(c)] = [f_n(c)]$. □

Theorem (Small simplices theorem)**.** The natural map $H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.

Proof of Mayer-Vietoris. Let $X = A \cup B$, with A, B open in X. We let $\mathcal{U} = \{A, B\}$, and write $C_\bullet(A + B) = C_\bullet^{\mathcal{U}}(X)$. Then we have a natural chain map

$$C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{j_A - j_B} C_\bullet(A + B)$$

that is surjective. The kernel consists of (x, y) such that $j_A(x) - j_B(y) = 0$, i.e. $j_A(x) = j_B(y)$. But j doesn't really do anything. It just forgets that the simplices lie in A or B. So this means $y = x$ is a chain in $A \cap B$. We thus deduce that we have a short exact sequence of chain complexes

$$C_\bullet(A \cap B) \xrightarrow{(i_A, i_B)} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{j_A - j_B} C_\bullet(A + B).$$

Then the snake lemma shows that we obtain a long exact sequence of homology groups. So we get a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{(i_A, i_B)} H_n(A) \oplus H_n(B) \xrightarrow{j_A - j_B} H_n^{\mathcal{U}}(X) \longrightarrow \cdots.$$

By the small simplices theorem, we can replace $H_n^{\mathcal{U}}(X)$ with $H_n(X)$. So we obtain Mayer-Vietoris. □

Proof of excision. Let $X \supseteq A \supseteq Z$ be such that $\overline{Z} \supseteq \mathring{A}$. Let $B = X \setminus Z$. Then again take

$$\mathcal{U} = \{A, B\}.$$

By assumption, their interiors cover X. We consider the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d, equal] & C_\bullet(A+B) \arrow[r] \arrow[d] & C_\bullet(A+B)/C_\bullet(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_\bullet(A) \arrow[r] & C_\bullet(X) \arrow[r] & C_\bullet(X, A) \arrow[r] & 0 \end{tikzcd}$$

Looking at the induced map between long exact sequences on homology, the middle and left terms induce isomorphisms, so the right term does too by the 5-lemma.

On the other hand, the map

$$C_\bullet(B)/C_\bullet(A \cap B) \longrightarrow C_\bullet(A+B)/C_\bullet(A)$$

is an isomorphism of chain complexes. Since their homologies are $H_\bullet(B, A \cap B)$ and $H_\bullet(X, A)$, we infer they the two are isomorphic. Recalling that $B = X \setminus \bar{Z}$, we have shown that

$$H_*(X \setminus Z, A \setminus Z) \cong H_*(X, A). \qquad \square$$

Lemma. $\rho_\bullet^X$ is a natural chain map.

Lemma. $\rho_\bullet^X$ is chain homotopic to the identity.

Proof. No one cares. $\square$

Lemma. The diameter of each subdivided simplex in $(\rho_n^{\Delta^n})^k(\iota_n)$ is bounded by $\left(\frac{n}{n+1}\right)^k \operatorname{diam}(\Delta^n)$.

Proof. Basic geometry. $\square$

Proposition. If $c \in C_n^{\mathcal{U}}(X)$, then $p^X(c) \in C_n^{\mathcal{U}}(X)$.

Moreover, if $c \in C_n(X)$, then there is some k such that $(\rho_n^X)^k(c) \in C_n^{\mathcal{U}}(X)$.

Proof. The first part is clear. For the second part, note that every chain is a finite sum of simplices. So we only have to check it for single simplices. We let σ be a simplex, and let

$$\mathcal{V} = \{\sigma^{-1}\mathring{U}_\alpha\}$$

be an open cover of Δ^n. By the Lebesgue number lemma, there is some $\varepsilon > 0$ such that any set of diameter $< \varepsilon$ is contained in some $\sigma^{-1}\mathring{U}_\alpha$. So we can choose $k > 0$ such that $(\rho_n^{\Delta^n})(\iota_n)$ is a sum of simplices which each has diameter $< \varepsilon$. So each lies in some $\sigma^{-1}\mathring{U}_\alpha$. So

$$(\rho_n^{\Delta^n})^k(\iota_n) = C_n^{\mathcal{V}}(\Delta^n).$$

So applying σ tells us

$$(\rho_n^{\Delta^n})^k(\sigma) \in C_n^{\mathcal{U}}(X). \qquad \square$$

Theorem (Small simplices theorem)**.** The natural map $U : H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.

Proof. Let $[c] \in H_n(X)$. By the proposition, there is some $k > 0$ such that $(\rho_n^X)^k(c) \in C_n^{\mathcal{U}}(x)$. We know that ρ_n^X is chain homotopic to the identity. Thus so is $(\rho_n^X)^k$. So $[(\rho_n^X)^k(c)] = [c]$. So the map $H_n^{\mathcal{U}}(X) \to H_n(X)$ is surjective.

To show that it is injective, we suppose $U([c]) = 0$. Then we can find some $z \in H_{n+1}(X)$ such that $dz = c$. We can then similarly subdivide z enough such that it lies in $C_{n+1}^{\mathcal{U}}(X)$. So this shows that $[c] = 0 \in H_n^{\mathcal{U}}(X)$. □

4 Reduced homology

Theorem. If (X, A) is good, then the natural map

$$H_*(X, A) \longrightarrow H_*(X/A, A/A) = \tilde{H}_*(X/A)$$

is an isomorphism.

Proof. As $i : A \hookrightarrow U$ is in particular a homotopy equivalence, the map

$$H_*(A) \longrightarrow H_*(U)$$

is an isomorphism. So by the five lemma, the map on relative homology

$$H_*(X, A) \longrightarrow H_*(X, U)$$

is an isomorphism as well.

As $i : A \hookrightarrow U$ is a deformation retraction with homotopy H, the inclusion

$$\{*\} = A/A \hookrightarrow U/A$$

is also a deformation retraction. So again by the five lemma, the map

$$H_*(X/A, A/A) \longrightarrow H_*(X/A, U/A)$$

is also an isomorphism. Now we have

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\sim"] \arrow[d] & H_n(X, U) \arrow[r, "\text{excise } A"] \arrow[d] & H_n(X \setminus A, U \setminus A) \arrow[d] \\
H_n(X/A, A/A) \arrow[r, "\sim"] & H_n(X/A, U/A) \arrow[r, "\text{excise } A/A"] & H_n\left(\frac{X}{A} \setminus \frac{A}{A}, \frac{U}{A} \setminus \frac{A}{A}\right)
\end{tikzcd}$$

We now notice that $X \setminus A = \frac{X}{A} \setminus \frac{A}{A}$ and $U \setminus A = \frac{U}{A} \setminus \frac{A}{A}$. So the right-hand vertical map is actually an isomorphism. So the result follows. ☐

5 Cell complexes

Lemma. If $A \subseteq X$ is a subcomplex, then the pair (X, A) is *good*.

Proof. See Hatcher 0.16. $\square$

Corollary. If $A \subseteq X$ is a subcomplex, then

$$H_n(X, A) \xrightarrow{\sim} \tilde{H}_n(X/A)$$

is an isomorphism.

Lemma. Let X be a cell complex. Then

(i)
$$H_i(X^n, X^{n-1}) = \begin{cases} 0 & i \neq n \\ \bigoplus_{i \in I_n} \mathbb{Z} & i = n \end{cases}.$$

(ii) $H_i(X^n) = 0$ for all $i > n$.

(iii) $H_i(X^n) \to H_i(X)$ is an isomorphism for $i < n$.

Proof.

(i) As (X^n, X^{n-1}) is good, we have an isomorphism

$$H_i(X^n, X^{n-1}) \xrightarrow{\sim} \tilde{H}_i(X^n/X^{n-1}) .$$

But we have
$$X^n/X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n_\alpha,$$
the space obtained from $Y = \coprod_{\alpha \in I_n} S^n_\alpha$ by collapsing down the subspace $Z = \{x_\alpha : \alpha \in I_n\}$, where each x_α is the south pole of the sphere. To compute the homology of the wedge X^n/X^{n-1}, we then note that (Y, Z) is good, and so we have a long exact sequence

$$H_i(Z) \longrightarrow H_i(Y) \longrightarrow \tilde{H}_i(Y/Z) \longrightarrow H_{i-1}(Z) \longrightarrow H_{i-1}(Y) .$$

Since $H_i(Z)$ vanishes for $i \geq 1$, the result then follows from the homology of the spheres plus the fact that $H_i(\coprod X_\alpha) = \bigoplus H_i(X_\alpha)$.

(ii) This follows by induction on n. We have (part of) a long exact sequence

$$H_i(X^{n-1}) \longrightarrow H_i(X^n) \longrightarrow H_i(X^n, X^{n-1})$$

We know the first term vanishes by induction, and the third term vanishes for $i > n$. So it follows that $H_i(X^n)$ vanishes.

(iii) To avoid doing too much point-set topology, we suppose X is finite-dimensional, so $X = X^m$ for some m. Then we have a long exact sequence

$$H_{i+1}(X^{n+1}, X^n) \to H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}, X^n)$$

Now if $i < n$, we know the first and last groups vanish. So we have $H_i(X^n) \cong H_i(X^{n+1})$. By continuing, we know that

$$H_i(X^n) \cong H_i(X^{n+1}) \cong H_i(X^{n+2}) \cong \cdots \cong H_i(X^m) = H_i(X).$$

To prove this for the general case, we need to use the fact that any map from a compact space to a cell complex hits only finitely many cells, and then the result would follow from this special case. $\square$

Theorem.

$$H_n^{\mathrm{cell}}(X) \cong H_n(X).$$

Proof. We have

$$\begin{aligned} H_n(X) &\cong H_n(X^{n+1}) \\ &= H_n(X^n)/\operatorname{im}(\partial : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n)) \end{aligned}$$

Since q_n is injective, we apply it to and bottom to get

$$= q_n(H_n(X^n))/\operatorname{im}(d_{n+1}^{\mathrm{cell}} : H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}))$$

By exactness, the image of q_n is the kernel of ∂. So we have

$$\begin{aligned} &= \ker(\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}))/\operatorname{im}(d_{n+1}^{\mathrm{cell}}) \\ &= \ker(d_n^{\mathrm{cell}})/\operatorname{im}(d_{n+1}^{\mathrm{cell}}) \\ &= H_n^{\mathrm{cell}}(X). \end{aligned}$$ $\square$

Corollary. If X is a finite cell complex, then $H_n(X)$ is a finitely-generated abelian group for all n, generated by at most $|I_n|$ elements. In particular, if there are no n-cells, then $H_n(X)$ vanishes.

If X has a cell-structure with cells in even-dimensional cells only, then $H_*(X)$ are all free.

Lemma. The coefficients $d_{\alpha\beta}$ are given by the degree of the map

$$\begin{tikzcd} S^{n-1}_\alpha = \partial D^n_\alpha \arrow[r, "\varphi_\alpha"] \arrow[rrr, bend right, "f_{\alpha\beta}"'] & X^{n-1} \arrow[r] & X^{n-1}/X^{n-2} = \bigvee_{\gamma \in I_{n-1}} S^{n-1}_\gamma \arrow[r] & S^{n-1}_\beta \end{tikzcd},$$

where the final map is obtained by collapsing the other spheres in the wedge.

In the case of cohomology, the maps are given by the transposes of these.

Proof. Consider the diagram

$$\begin{tikzcd} H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\sim"'] \arrow[d, "(\Phi_\alpha)_*"] & H_{n-1}(\partial D^n_\alpha) \arrow[r, dashed] \arrow[d, "(\varphi_\alpha)_*"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\ H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "d_n^{\mathrm{cell}}"] & H_{n-1}(X^{n-1}) \arrow[d, "q"] & \tilde{H}_{n-1}\left(\bigvee S^{n-1}_\gamma\right) \arrow[u, "\text{collapse}"'] \arrow[d, equal] \\ & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\text{excision}", "\sim"'] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \end{tikzcd}$$

By the long exact sequence, the top left horizontal map is an isomorphism.

Now let's try to trace through the diagram. We can find

$$\begin{tikzcd}
1 \arrow[r, "\text{isomorphism}"] \arrow[d, mapsto] & 1 \arrow[r, mapsto, "f_{\alpha\beta}"] & d_{\alpha\beta} \\
e_\alpha \arrow[dr, mapsto] & & \\
 & \sum d_{\alpha\gamma} e_\gamma \arrow[r, mapsto] & \sum d_{\alpha\gamma} e_\gamma \arrow[uu, mapsto]
\end{tikzcd}$$

So the degree of $f_{\alpha\beta}$ is indeed $d_{\alpha\beta}$. $\square$

6 (Co)homology with coefficients

7 Euler characteristic

Theorem. We have
$$\chi = \chi_{\mathbb{Z}} = \chi_{\mathbb{F}}.$$

Proof. First note that the number of n cells of X is the rank of $C_n^{\text{cell}}(X)$, which we will just write as C_n. Let

$$\begin{aligned} Z_n &= \ker(d_n : C_n \to C_{n-1}) \\ B_n &= \operatorname{im}(d_{n+1} : C_{n+1} \to C_n). \end{aligned}$$

We are now going to write down two short exact sequences. By definition of homology, we have

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow 0 \ .$$

Also, the definition of Z_n and B_n give us

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0 \ .$$

We will now use the first isomorphism theorem to know that the rank of the middle term is the sum of ranks of the outer terms. So we have

$$\chi_{\mathbb{Z}}(X) = \sum (-1)^n \operatorname{rank} H_n(X) = \sum (-1)^n (\operatorname{rank} Z_n - \operatorname{rank} B_n).$$

We also have
$$\operatorname{rank} B_n = \operatorname{rank} C_{n+1} - \operatorname{rank} Z_{n+1}.$$

So we have

$$\begin{aligned} \chi_{\mathbb{Z}}(X) &= \sum_n (-1)^n (\operatorname{rank} Z_n - \operatorname{rank} C_{n+1} + \operatorname{rank} Z_{n+1}) \\ &= \sum_n (-1)^{n+1} \operatorname{rank} C_{n+1} \\ &= \chi(X). \end{aligned}$$

For $\chi_{\mathbb{F}}$, we use the fact that

$$\operatorname{rank} C_n = \dim_{\mathbb{F}} C_n \otimes \mathbb{F}. \qquad \square$$

8 Cup product

Lemma. If $\phi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$, then

$$d(\phi \smile \psi) = (d\phi) \smile \psi + (-1)^k \phi \smile (d\psi).$$

Proof. This is a straightforward computation.

Let $\sigma : \Delta^{k+\ell+1} \to X$ be a simplex. Then we have

$$\begin{aligned}
((d\phi) \smile \psi)(\sigma) &= (d\phi)(\sigma|_{[v_0, \ldots, v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1}, \ldots, v_{k+\ell+1}]})\\
&= \phi\left(\sum_{i=0}^{k+1} (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}]}\right) \cdot \psi(\sigma|_{[v_{k+1}, \ldots, v_{k+\ell+1}]})\\
(\phi \smile (d\psi))(\sigma) &= \phi(\sigma|_{[v_0, \ldots, v_k]}) \cdot (d\psi)(\sigma|_{v_k, \ldots, v_{k+\ell+1}]})\\
&= \phi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \psi\left(\sum_{i=k}^{k+\ell+1} (-1)^{i-k} \sigma|_{[v_k, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}]}\right)\\
&= (-1)^k \phi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \psi\left(\sum_{i=k}^{k+\ell+1} (-1)^{i} \sigma|_{[v_k, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}]}\right).
\end{aligned}$$

We notice that the last term of the first expression, and the first term of the second expression are exactly the same, except the signs differ by -1. Then the remaining terms overlap in exactly 1 vertex, so we have

$$((d\phi) \smile \psi)(\sigma) + (-1)^k \phi \smile (d\psi)(\sigma) = (\phi \smile \psi)(d\sigma) = (d(\phi \smile \psi))(\sigma)$$

as required. □

Corollary. The cup product induces a well-defined map

$$\begin{aligned}
\smile : H^k(X; R) \times H^\ell(X; R) &\longrightarrow H^{k+\ell}(X; R)\\
([\phi], [\psi]) &\longmapsto [\phi \smile \psi]
\end{aligned}$$

Proof. To see this is defined at all, as $d\phi = 0 = d\psi$, we have

$$d(\phi \smile \psi) = (d\phi) \smile \psi \pm \phi \smile (d\psi) = 0.$$

So $\phi \smile \psi$ is a cocycle, and represents the cohomology class. To see this is well-defined, if $\phi' = \phi + d\tau$, then

$$\phi' \smile \psi = \phi \smile \psi + d\tau \smile \psi = \phi \smile \psi + d(\tau \smile \psi) \pm \tau \smile (d\psi).$$

Using the fact that $d\psi = 0$, we know that $\phi' \smile \psi$ and $\phi \smile \psi$ differ by a boundary, so $[\phi' \smile \psi] = [\phi \smile \psi]$. The case where we change ψ is similar. □

Proposition. $(H^*(X; R), \smile, [1])$ is a unital ring.

Proposition. Let R be a commutative ring. If $\alpha \in H^k(X; R)$ and $\beta \in H^\ell(X; R)$, then we have

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

Proposition. The cup product is natural, i.e. if $f : X \to Y$ is a map, and $\alpha, \beta \in H^*(Y; R)$, then

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

So f^* is a homomorphism of unital rings.

Proof of previous proposition. Let $\rho_n : C_n(X) \to C_n(x)$ be given by

$$\sigma \mapsto (-1)^{n(n+1)/2} \sigma|_{[v_n, v_{n-1}, \ldots, v_0]}$$

The $\sigma|_{[v_n, v_{n-1}, \ldots, v_0]}$ tells us that we reverse the order of the vertices, and the factor of $(-1)^{n(n+1)/2}$ is the sign of the permutation that reverses $0, \cdots, n$. For convenience, we write

$$\varepsilon_n = (-1)^{n(n+1)/2}.$$

Claim. We claim that $\rho_\bullet$ is a chain map, and is chain homotopic to the identity.

We will prove this later.

Suppose the claim holds. We let $\phi \in C^k(X; R)$ represent α and $\psi \in C^\ell(X; R)$ represent β. Then we have

$$\begin{aligned}
(\rho^*\phi \smile \rho^*\psi)(\sigma) &= (\rho^*\phi)(\sigma|_{[v_0, \ldots, v_k]}(\rho^*\psi)(\sigma|_{[v_k, \ldots, v_{k+\ell}]}) \\
&= \phi(\varepsilon_k \cdot \sigma|_{[v_k, \ldots, v_0]})\psi(\varepsilon_\ell \sigma|_{[v_{k+\ell}, \ldots, v_k]}).
\end{aligned}$$

Thus, we can compute

$$\begin{aligned}
\rho^*(\psi \smile \phi)(\sigma) &= (\psi \smile \phi)(\varepsilon_{k+\ell}\sigma|_{[v_{k+\ell}, \ldots, v_0]}) \\
&= \varepsilon_{k+\ell}\psi(\sigma|_{[v_{k+\ell}, \ldots, v_k}])\phi(\sigma|_{[v_k, \ldots, v_0]}) \\
&= \varepsilon_{k+\ell}\varepsilon_k\varepsilon_\ell(\rho^*\phi \smile \rho^*\psi)(\sigma).
\end{aligned}$$

By checking it directly, we can see that $\varepsilon_{n+\ell}\varepsilon_k\varepsilon_\ell = (-1)^{k\ell}$. So we have

$$\begin{aligned}
\alpha \smile \beta &= [\phi \smile \psi] \\
&= [\rho^*\phi \smile \rho^*\psi] \\
&= (-1)^{k\ell}[\rho^*(\psi \smile \phi)] \\
&= (-1)^{k\ell}[\psi \smile \phi] \\
&= (-1)^{kl}\beta \smile \alpha.
\end{aligned}$$

Now it remains to prove the claim. We have

$$\begin{aligned}
\mathrm{d}\rho(\sigma) &= \varepsilon_n \sum_{i=0}^n (-1)^j \sigma|_{[v_n, \ldots, \hat{v}_{n-i}, \ldots, v_0]} \\
\rho(\mathrm{d}\sigma) &= \rho\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}\right) \\
&= \varepsilon_{n-1} \sum_{j=0}^n (-1)^j \sigma|_{[v_n, \ldots, \hat{v}_j, v_0]}.
\end{aligned}$$

We now notice that $\varepsilon_{n-1}(-1)^{n-i} = \varepsilon_n(-1)^i$. So this is a chain map!

We now define a chain homotopy. This time, we need a "twisted prism". We let

$$P_n = \sum_i (-1)^i \varepsilon_{n-i}[v_0, \cdots, v_i, w_n, \cdots, w_i] \in C_{n+1}([0,1] \times \Delta^n),$$

where $v_0, \cdots, v_n$ are the vertices of $\{0\} \times \Delta^n$ and $w_0, \cdots, w_n$ are the vertices of $\{1\} \times \Delta^n$.

We let $\pi : [0,1] \times \Delta^n \to \Delta^n$ be the projection, and let $F_n^X : C_n(X) \to C_{n+1}(X)$ be given by

$$\sigma \mapsto (\sigma \circ \pi)_\#(P_n).$$

We calculate

$$\begin{aligned} \mathrm{d}F_n^X(\sigma) &= (\sigma \circ \pi)_\#(\mathrm{d}P_n) \\ &= (\sigma \circ \pi_\#) \left(\sum_i \left(\sum_{j \leq i} (-1)^j (-1)^i \varepsilon_{n-i}[v_0, \cdots, \hat{v}_j, \cdots, v_i, w_0, \cdots, w_i] \right) \right. \\ &\quad \left. + \left(\sum_{j \geq i} (-1)^{n+i+1-j} (-1)^i \varepsilon_{n-i}[v_0, \cdots, v_i, w_n, \cdots, \hat{w}_j, \cdots, v_i] \right) \right). \end{aligned}$$

The terms with $j = i$ give

$$\begin{aligned} &(\sigma \circ \pi)_\# \left(\sum_i \varepsilon_{n-i}[v_0, \cdots, v_{i-1}, w_n, \cdots, w_i] \right. \\ &\qquad \left. + \sum_i (-1)^{n+1}(-1)^i \varepsilon_{n-i}[v_0, \cdots, v_i, w_n, \cdots, w_{i+1}] \right) \\ &= (\sigma \circ \pi)_\#(\varepsilon_n[w_n, \cdots, w_0] - [v_0, \cdots, v_n]) \\ &= \rho(\sigma) - \sigma \end{aligned}$$

The terms with $j \neq i$ are precisely $-F_{n-1}^X(\mathrm{d}\sigma)$ as required. It is easy to see that the terms are indeed the right terms, and we just have to check that the signs are right. I'm not doing that. $\square$

9 Künneth theorem and universal coefficients theorem

Theorem (Künneth's theorem)**.** Let R be a commutative ring, and suppose that $H^n(Y; R)$ is a free R-module for each n. Then the cross product map

$$\bigoplus_{k+\ell=n} H^k(X; R) \otimes H^\ell(Y; R) \xrightarrow{\times} H^n(X \times Y; R)$$

is an isomorphism for every n, for every finite cell complex X.

It follows from the five lemma that the same holds if we have a relative complex (Y, A) instead of just Y.

Proof. Let

$$F^n(-) = \bigoplus_{k+\ell=n} H^k(-; R) \otimes H^\ell(Y; R).$$

We similarly define

$$G^n(-) = H^n(- \times Y; R).$$

We observe that for each X, the cross product gives a map $\times : F^n(X) \to G^n(X)$, and, crucially, we know $\times_* : F^n(*) \to G^n(*)$ is an isomorphism, since $F^n(*) \cong G^n(*) \cong H^n(Y; R)$.

The strategy is to show that $F^n(-)$ and $G^n(-)$ have the same formal structure as cohomology and agree on a point, and so must agree on all finite cell complexes.

It is clear that both F^n and G^n are homotopy invariant, because they are built out of homotopy invariant things.

We now want to define the cohomology of pairs. This is easy. We define

$$\begin{aligned} F^n(X, A) &= \bigoplus_{i+j=n} H^i(X, A; R) \otimes H^j(Y; R) \\ G^n(X, A) &= H^n(X \times Y, A \times Y; R). \end{aligned}$$

Again, the relative cup product gives us a relative cross product, which gives us a map $F^n(X, A) \to G^n(X, A)$.

It is immediate G^n has a long exact sequence associated to (X, A) given by the usual long exact sequence of $(X \times Y, A \times Y)$. We would like to say F has a long exact sequence as well, and this is where our hypothesis comes in.

If $H^*(Y; R)$ is a free R-module, then we can take the long exact sequence of (X, A)

$$\cdots \longrightarrow H^n(A; R) \xrightarrow{\partial} H^n(X, A; R) \longrightarrow H^n(X; R) \longrightarrow H^n(A; R) \longrightarrow \cdots,$$

and then tensor with $H^j(Y; R)$. This preserves exactness, since $H^j(Y; R) \cong R^k$ for some k, so tensoring with $H^j(Y; R)$ just takes k copies of this long exact sequence. By adding the different long exact sequences for different j (with appropriate translations), we get a long exact sequence for F.

We now want to prove Künneth by induction on the number of cells and the dimension at the same time. We are going to prove that if $X = X' \cup_f D^n$ for some $S^{n-1} \to X'$, and $\times : F(X') \to G(X')$ is an isomorphism, then $\times : F(X) \to G(X)$ is also an isomorphism. In doing so, we will assume that the result is true for attaching *any* cells of dimension less than n.

Suppose $X = X' \cup_f D^n$ for some $f : S^{n-1} \to X'$. We get long exact sequences

$$\begin{tikzcd} F^{*-1}(X') \arrow[r] \arrow[d, "\times", "\sim"'] & F^*(X, X') \arrow[r] \arrow[d, "\times"] & F^*(X) \arrow[r] \arrow[d, "\times"] & F^*(X') \arrow[r] \arrow[d, "\times", "\sim"'] & F^{*+1}(X, X') \arrow[d, "\times"] \\ G^{*-1}(X') \arrow[r] & G^*(X, X') \arrow[r] & G^*(X) \arrow[r] & G^*(X') \arrow[r] & G^{*+1}(X, X') \end{tikzcd}$$

Note that we need to manually check that the boundary maps ∂ commute with the cross product, since this is not induced by maps of spaces, but we will not do it here.

Now by the five lemma, it suffices to show that the maps on the relative cohomology $\times : F^n(X, X') \to G^n(X, X')$ is an isomorphism.

We now notice that $F^*(-)$ and $G^*(-)$ have excision. Since (X, X') is a good pair, we have a commutative square

$$\begin{tikzcd} F^*(D^n, \partial D^n) \arrow[d, "\times"] & F^*(X, X') \arrow[l, "\sim"'] \arrow[d, "\times"] \\ G^*(D^n, \partial D^n) & G^*(X, X') \arrow[l, "\sim"'] \end{tikzcd}$$

So we now only need the left-hand map to be an isomorphism. We look at the long exact sequence for $(D^n, \partial D^n)$!

$$\begin{tikzcd} F^{*-1}(\partial D^n) \arrow[r] \arrow[d, "\times", "\sim"'] & F^*(D^n, \partial D^n) \arrow[r] \arrow[d, "\times", "\sim"'] & F^*(D^n) \arrow[r] \arrow[d, "\times"] & F^*(\partial D^n) \arrow[r] \arrow[d, "\times", "\sim"'] & F^{*+1}(D^n, \partial D^n) \arrow[d, "\times", "\times"'] \\ G^{*-1}(\partial D^n) \arrow[r] & G^*(D^n, \partial D^n) \arrow[r] & G^*(D^n) \arrow[r] & G^*(\partial D^n) \arrow[r] & G^{*+1}(D^n, \partial D^n) \end{tikzcd}$$

But now we know the vertical maps for D^n and ∂D^n are isomorphisms — the ones for D^n are because they are contractible, and we have seen the result of $*$ already; whereas the result for ∂D^n follows by induction.

So we are done. $\square$

Theorem (Universal coefficients theorem for (co)homology)**.** Let R be a PID and M an R-module. Then there is a natural map

$$H_*(X; R) \otimes M \to H_*(X; M).$$

If $H_*(X; R)$ is a free module for each n, then this is an isomorphism. Similarly, there is a natural map

$$H^*(X; M) \to \mathrm{Hom}_R(H_*(X; R), M), ,$$

which is an isomorphism again if $H^*(X; R)$ is free.

Proof. Let C_n be $C_n(X;R)$ and $Z_n \subseteq C_n$ be the cycles and $B_n \subseteq Z_n$ the boundaries. Then there is a short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{\ i\ } C_n \xrightarrow{\ g\ } B_{n-1} \longrightarrow 0\ ,$$

and $B_{n-1} \leq C_{n-1}$ is a submodule of a free R-module, and is free, since R is a PID. So by picking a basis, we can find a map $s : B_{n-1} \to C_n$ such that $g \circ s = \mathrm{id}_{B_{n-1}}$. This induces an isomorphism

$$i \oplus s : Z_n \oplus B_{n-1} \xrightarrow{\ \sim\ } C_n.$$

Now tensoring with M, we obtain

$$0 \longrightarrow Z_n \otimes M \longrightarrow C_n \otimes M \longrightarrow B_{n-1} \otimes M \longrightarrow 0\ ,$$

which is exact because we have

$$C_n \otimes M \cong (Z_n \oplus B_{n-1}) \otimes M \cong (Z_n \otimes M) \oplus (B_{n-1} \otimes M).$$

So we obtain a short exact sequence of chain complexes

$$0 \longrightarrow (Z_n \otimes M, 0) \longrightarrow (C_n \otimes M, d \otimes \mathrm{id}) \longrightarrow (B_{n-1} \otimes M, 0) \longrightarrow 0\ ,$$

which gives a long exact sequence in homology:

$$\cdots \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X;M) \longrightarrow B_{n-1} \otimes M \longrightarrow \cdots$$

We'll leave this for a while, and look at another short exact sequence. By definition of homology, we have a long exact sequence

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(X;R) \longrightarrow 0\ .$$

As $H_n(X;R)$ is free, we have a splitting $t : H_n(X;R) \to Z_n$, so as above, tensoring with M preserves exactness, so we have

$$0 \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X;R) \otimes M \longrightarrow 0\ .$$

Hence we know that $B_n \otimes M \to Z_n \otimes M$ is injective. So our previous long exact sequence breaks up to

$$0 \longrightarrow B_n \otimes M \longrightarrow Z_n \otimes M \longrightarrow H_n(X;M) \longrightarrow 0.$$

Since we have two short exact sequence with first two terms equal, the last terms have to be equal as well.

The cohomology version is similar. $\square$

10 Vector bundles

10.1 Vector bundles

Theorem (Tubular neighbourhood theorem)**.** Let $M \subseteq N$ be a smooth submanifold. Then there is an open neighbourhood U of M and a homeomorphism $\nu_{M \subseteq N} \to U$, and moreover, this homeomorphism is the identity on M (where we view M as a submanifold of $\nu_{M \subseteq N}$ by the image of the zero section).

Proposition. Partitions of unity exist for any open cover.

Lemma. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is a continuous family of inner products on E. In other words, there is a map $E \otimes E \to \mathbb{R}$ which restricts to an inner product on each E_x.

Proof. We notice that every trivial bundle has an inner product, and since every bundle is locally trivial, we can patch these up using partitions of unity.

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X with local trivializations

$$\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^d.$$

The inner product on $\mathbb{R}^d$ then gives us an inner product on $E|_{U_\alpha}$, say $\langle \cdot\,, \cdot \rangle_\alpha$. We let λ_α be a partition of unity associated to $\{U_\alpha\}$. Then for $u \otimes v \in E \otimes E$, we define

$$\langle u, v \rangle = \sum_{\alpha \in I} \lambda_\alpha(\pi(u)) \langle u, v \rangle_\alpha.$$

Now if $\pi(u) = \pi(v)$ is not in U_α, then we don't know what we mean by $\langle u, v \rangle_\alpha$, but it doesn't matter, because $\lambda_\alpha(\pi(u)) = 0$. Also, since the partition of unity is locally finite, we know this is a finite sum.

It is then straightforward to see that this is indeed an inner product, since a positive linear combination of inner products is an inner product. □

Lemma. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some N such that E is a vector subbundle of $X \times \mathbb{R}^N$.

Proof. Let $\{U_\alpha\}$ be a trivializing cover of X. Since X is compact, we may wlog assume the cover is finite. Call them $U_1, \cdots, U_n$. We let

$$\varphi_i : E|_{U_i} \to U_i \times \mathbb{R}^d.$$

We note that on each patch, $E|_{U_i}$ embeds into a trivial bundle, because it *is* a trivial bundle. So we can add all of these together. The trick is to use a partition of unity, again.

We define f_i to be the composition

$$E|_{U_i} \xrightarrow{\varphi_i} U_i \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d\ .$$

Then given a partition of unity λ_i, we define

$$\begin{aligned} f : E &\to X \times (\mathbb{R}^d)^n \\ v &\mapsto (\pi(v), \lambda_1(\pi(v)) f_1(v), \lambda_2(\pi(v)) f_2(v), \cdots, \lambda_n(\pi(v)) f_n(v)). \end{aligned}$$

We see that this is injective. If v, w belong to different fibers, then the first coordinate distinguishes them. If they are in the same fiber, then there is some U_i with $\lambda_i(\pi(u)) \neq 0$. Then looking at the ith coordinate gives us distinguishes them. This then exhibits E as a subbundle of $X \times \mathbb{R}^n$. □

Corollary. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some $p : F \to X$ such that $E \oplus F \cong X \times \mathbb{R}^n$. In particular, E embeds as a subbundle of a trivial bundle.

Proof. By above, we can assume E is a subbundle of a trivial bundle. We can then take the orthogonal complement of E. $\square$

Theorem. There is a correspondence

$$\left\{\begin{array}{c}\text{homotopy classess}\\ \text{of maps}\\ f : X \to \mathrm{Gr}_d(\mathbb{R}^\infty)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}d\text{-dimensional}\\ \text{vector bundles}\\ \pi : E \to X\end{array}\right\}$$

$$[f] \longmapsto f^*\gamma_d^{\mathbb{R}}$$

$$[f_\pi] \longleftarrow\!\shortmid \pi$$

10.2 Vector bundle orientations

Lemma. Every vector bundle is $\mathbb{F}_2$-orientable.

Proof. There is only one possible choice of generator. $\square$

Lemma. If $\{U_\alpha\}_{\alpha \in I}$ is a family of covers such that for each $\alpha, \beta \in I$, the homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^d \xleftarrow[\cong]{\varphi_\alpha} E|_{U_\alpha \cap U_\beta} \xrightarrow[\varphi_\beta]{\cong} (U_\alpha \cap U_\beta) \times \mathbb{R}^d$$

gives an orientation preserving map from $(U_\alpha \cap U_\beta) \times \mathbb{R}^d$ to itself, i.e. has a positive determinant on each fiber, then E is orientable for any R.

Proof. Choose a generator $u \in H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$. Then for $x \in U_\alpha$, we define ε_x by pulling back u along

$$E_x \hookrightarrow E|_{U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times \mathbb{R}^d \xrightarrow{\pi_2} \mathbb{R}^d \ . \qquad (\dagger_\alpha)$$

If $x \in U_\beta$ as well, then the analogous linear isomorphism $\dagger_\alpha$ differs from $\dagger_\beta$ by post-composition with a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ of *positive* determinant. We now use the fact that any linear map of positive determinant is homotopic to the identity. Indeed, both L and id lies in $\mathrm{GL}_d^+(\mathbb{R})$, a connected group, and a path between them will give a homotopy between the maps they represent. So we know $(\dagger_\alpha)$ is homotopic to $(\dagger_\beta)$. So they induce the same maps on cohomology classes. $\square$

10.3 The Thom isomorphism theorem

Theorem (Thom isomorphism theorem)**.** Let $\pi : E \to X$ be a d-dimensional vector bundle, and $\{\varepsilon_x\}_{x \in X}$ be an R-orientation of E. Then

(i) $H^i(E, E^\#; R) = 0$ for $i < d$.

(ii) There is a unique class $u_E \in H^d(E, E^\#; R)$ which restricts to ε_x on each fiber. This is known as the *Thom class*.

(iii) The map Φ given by the composition

$$H^i(X;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{-\smile u_E} H^{i+d}(E,E^\#;R)$$

is an isomorphism.

Note that (i) follows from (iii), since $H^i(X;R) = 0$ for $i < 0$.

Theorem. If there is a section $s : X \to E$ which is nowhere zero, then $e(E) = 0 \in H^d(X;R)$.

Proof. Notice that any two sections of $E \to X$ are homotopic. So we have $e \equiv s_0^* u_E = s^* u_E$. But since $u_E \in H^d(E,E^\#;R)$, and s maps into $E^\#$, we have $s^* u_E$.

Perhaps more precisely, we look at the long exact sequence for the pair $(E,E^\#)$, giving the diagram

$$\begin{tikzcd}
H^d(E,E^\#;R) \arrow[r] & H^d(E;R) \arrow[r] \arrow[d, "s_0^*"] & H^d(E^\#;R) \arrow[dl, "s^*"] \\
& H^d(X;R)
\end{tikzcd}$$

Since s and s_0 are homotopic, the diagram commutes. Also, the top row is exact. So $u_E \in H^d(E,E^\#;R)$ gets sent along the top row to $0 \in H^d(E^\#;R)$, and thus s^* sends it to $0 \in H^d(X;R)$. But the image in $H^d(X;R)$ is exactly the Euler class. So the Euler class vanishes. □

Theorem. We have

$$u_E \smile u_E = \Phi(e(E)) = \pi^*(e(E)) \smile u_E \in H^*(E,E^\#;R).$$

Proof. By construction, we know the following maps commute:

$$\begin{tikzcd}
H^d(E,E^\#;R) \otimes H^d(E,E^\#;R) \arrow[r, "\smile"] \arrow[d, "q^* \otimes \mathrm{id}"] & H^{2d}(E,E^\#;R) \\
H^d(E;R) \otimes H^d(E,E^\#;R) \arrow[ur, "\smile"']
\end{tikzcd}$$

We claim that the Thom class $u_E \otimes u_E \in H^d(E,E^\#;R) \otimes H^d(E,E^\#;R)$ is sent to $\pi^*(e(E)) \otimes u_E \in H^d(E;R) \otimes H^d(E,E^\#;R)$.

By definition, this means we need

$$q^* u_E = \pi^*(e(E)),$$

and this is true because π^* is homotopy inverse to s_0^* and $e(E) = s_0^* q^* u_E$. □

Lemma. If $\pi : E \to X$ is a d-dimensional R-module vector bundle with d odd, then $2e(E) = 0 \in H^d(X;R)$.

Proof. Consider the map $\alpha : E \to E$ given by negation on each fiber. This then gives an isomorphism

$$a^* : H^d(E,E^\#;R) \xrightarrow{\cong} H^d(E,E^\#;R).$$

This acts by negation on the Thom class, i.e.

$$a^*(u_E) = -u_E,$$

as on the fiber E_x, we know a is given by an odd number of reflections, each of which acts on $H^d(E_x, E_x^\#; R)$ by -1 (by the analogous result on S^n). So we change ε_x by a sign. We then lift this to a statement about u_E by the fact that u_E is the unique thing that restricts to ε_x for each x.

But we also know

$$a \circ s_0 = s_0,$$

which implies

$$s_0^*(a^*(u_E)) = s_0^*(u_E).$$

Combining this with the result that $a^*(u_E) = -u_E$, we get that

$$2e(E) = 2s_0^*(u_E) = 0. \qquad \square$$

Proof of Thom isomorphism theorem. We will drop the "R" in all our diagrams for readability (and also so that it fits in the page).

We first consider the case where the bundle is trivial, so $E = X \times \mathbb{R}^d$. Then we note that

$$H^*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) = \begin{cases} R & * = d \\ 0 & * \neq d \end{cases}.$$

In particular, the modules are free, and (a relative version of) Künneth's theorem tells us the map

$$\times : H^*(X) \otimes H^*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \xrightarrow{\cong} H^*(X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\}))$$

is an isomorphism. Then the claims of the Thom isomorphism theorem follow immediately.

(i) For $i < d$, all the summands corresponding to $H^i(X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\}))$ vanish since the $H^*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\})$ term vanishes.

(ii) The only non-vanishing summand for $H^d(X \times \mathbb{R}^d, X \times (\mathbb{R}^d \setminus \{0\})$ is

$$H^0(X) \otimes H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}).$$

Then the Thom class must be $1 \otimes u$, where u is the object corresponding to $\varepsilon_x \in H^d(E_x, E_x^\#) = H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\})$, and this is unique.

(iii) We notice that Φ is just given by

$$\Phi(x) = \pi^*(x) \smile u_E = x \times u_E,$$

which is an isomorphism by Künneth.

We now patch the result up for a general bundle. Suppose $\pi : E \to X$ is a bundle. Then it has an open cover of trivializations, and moreover if we assume our X is compact, there are finitely many of them. So it suffices to show that if $U, V \subseteq X$ are open sets such that the Thom isomorphism the holds for E restricted to $U, V, U \cap V$, then it also holds on $U \cup V$.

The relative Mayer-Vietoris sequence gives us

$$H^{d-1}(E|_{U\cap V}, E^{\#}|_{U\cap V}) \xrightarrow{\partial^{MV}} H^d(E|_{U\cup V}, E^{\#}|_{U\cup V}) \longrightarrow H^d(E|_U, E^{\#}|_U) \oplus H^d(E|_V, E^{\#}|_V) \longrightarrow H^d(E|_{U\cap V}, E^{\#}|_{U\cap V}).$$

We first construct the Thom class. We have

$$u_{E|_V} \in H^d(E|_V, E^{\#}), \quad u_{E|_U} \in H^d(E|_U, E^{\#}).$$

We claim that $(u_{E|_U}, u_{E|_V}) \in H^d(E|_U, E^{\#}|_U) \oplus H^d(E|_V, E^{\#}|_V)$ gets sent to 0 by $i_U^* - i_V^*$. Indeed, both the restriction of $u_{E|_U}$ and $u_{E|_V}$ to $U \cap V$ are Thom classes, so they are equal by uniqueness, so the difference vanishes.

Then by exactness, there must be some $u_{E|_{U\cup V}} \in H^d(E|_{U\cup V}, E^{\#}|_{U\cup V})$ that restricts to $u_{E|_U}$ and $u_{E|_V}$ in U and V respectively. Then this must be a Thom class, since the property of being a Thom class is checked on each fiber. Moreover, we get uniqueness because $H^{d-1}(E|_{U\cap V}, E^{\#}|_{U\cap V}) = 0$, so $u_{E|_U}$ and $u_{E|_V}$ must be the restriction of a unique thing.

The last part in the Thom isomorphism theorem come from a routine application of the five lemma, and the first part follows from the last as previously mentioned. $\square$

10.4 Gysin sequence

11 Manifolds and Poincaré duality

11.1 Compactly supported cohomology

Theorem. For any space X, we let

$$\mathcal{K}(X) = \{K \subseteq X : K \text{ is compact}\}.$$

This is a directed set under inclusion, and the map

$$K \mapsto H^n(X, X \setminus K)$$

gives a direct system of abelian groups indexed by $\mathcal{K}(X)$, where the maps ρ are given by restriction.

Then we have

$$H_c^*(X) \cong \varinjlim_{\mathbf{K}(X)} H^n(X, X \setminus K).$$

Proof. We have

$$C_c^n(X) \cong \varinjlim_{\mathcal{K}(X)} C^n(X, X \setminus K),$$

where we have a map

$$\varinjlim_{K(\alpha)} C^n(X, X \setminus K) \to C_c^n(X)$$

given in each component of the direct limit by inclusion, and it is easy to see that this is well-defined and bijective.

It is then a general algebraic fact that H^* commutes with inverse limits, and we will not prove it. $\square$

Lemma. We have

$$H_c^i(\mathbb{R}^d; R) \cong \begin{cases} R & i = d \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $\mathcal{B} \in \mathcal{K}(\mathbb{R}^d)$ be the balls, namely

$$\mathcal{B} = \{nD^d, n = 0, 1, 2, \cdots\}.$$

Then since every compact set is contained in one of them, we have

$$H_c^n(X) \cong \varinjlim_{K \in \mathcal{K}(\mathbb{R}^d)} H^n(\mathbb{R}^d, \mathbb{R}^d \setminus K; R) \cong \varinjlim_{nD^d \in \mathcal{B}} H^n(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d; R)$$

We can compute that directly. Since $\mathbb{R}^d$ is contractible, the connecting map

$$H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d; R) \to H^{i-1}(\mathbb{R}^d \setminus nD^d; R)$$

in the long exact sequence is an isomorphism. Moreover, the following diagram commutes:

$$\begin{tikzcd}
H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^n; R) \arrow[r, "\rho_{n,n+1}"] \arrow[d, "\partial"] & H^i(\mathbb{R}^d, \mathbb{R}^d \setminus (n+1)D^d; R) \arrow[d, "\partial"] \\
H^{i-1}(\mathbb{R}^d \setminus nD^d; R) \arrow[r] & H^{i-1}(\mathbb{R}^d \setminus (n+1)D^d; R)
\end{tikzcd}$$

But all maps here are isomorphisms because the horizontal maps are homotopy equivalences. So we know

$$\varinjlim H^i(\mathbb{R}^d, \mathbb{R}^d \setminus nD^d; R) \cong H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \cong H^{i-1}(\mathbb{R}^d \setminus \{0\}; R).$$

So it follows that

$$H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) = \begin{cases} \mathbb{R} & i = d \\ 0 & \text{otherwise} \end{cases}. \qquad \square$$

Proposition. Let $K, L \subseteq X$ be compact. Then there is a long exact sequence

$$\begin{tikzcd} H^n(X \mid K \cap L) \arrow[r] & H^n(X \mid K) \oplus H^n(X \mid L) \arrow[r] & H^n(X \mid K \cup L) \arrow[dll, "\partial"] \\ H^{n+1}(X \mid K \cap L) \arrow[r] & H^{n+1}(X \mid K) \oplus H^{n+1}(X \mid L) \arrow[r] & \cdots \end{tikzcd},$$

where the unlabelled maps are those induced by inclusion.

Proof. We cover $X \setminus K \cap L$ by

$$\mathcal{U} = \{X \setminus K, X \setminus L\}.$$

We then draw a huge diagram (here * denotes the dual, i.e. $X^* = \mathrm{Hom}(X; R)$, and $C^\bullet(X \mid K) = C^\bullet(X, X \setminus K)$):

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \left(\frac{C_\bullet(X)}{C_\bullet^{\mathcal{U}}(X \setminus K \cap L)}\right)^* \arrow[r] \arrow[d] & C^\bullet(X \mid K) \oplus C^\bullet(X \mid L) \arrow[r] \arrow[d] & C^\bullet(X \mid K \cup L) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^\bullet(X) \arrow[r, "(\mathrm{id}, -\mathrm{id})"] \arrow[d] & C^\bullet(X) \oplus C^\bullet(X) \arrow[r, "\mathrm{id} + \mathrm{id}"] \arrow[d] & C^\bullet(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^\bullet_{\mathcal{U}}(X \setminus K \cap L) \arrow[r, "(j_1^*, -j_2^*)"] \arrow[d] & C^\bullet(X \setminus K) \oplus C^\bullet(X \setminus L) \arrow[r, "i_1^* + i_2^*"] \arrow[d] & C^\bullet(X \setminus K \cup L) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 \end{tikzcd}$$

This is a diagram. Certainly.

The bottom two rows and all columns are exact. By a diagram chase (the *nine lemma*), we know the top row is exact. Taking the long exact sequence almost gives what we want, except the first term is a funny thing.

We now analyze that object. We look at the left vertical column:

$$0 \longrightarrow \mathrm{Hom}\left(\frac{C_\bullet(x)}{C_\bullet^{\mathcal{U}}(X \setminus K \cap L)}, R\right) \longrightarrow C^\bullet(X) \longrightarrow \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X \setminus K \cap L), R) \longrightarrow 0$$

Now by the small simplices theorem, the right hand object gives the same (co)homology as $C_\bullet(X \setminus K \cap L; R)$. So we can produce another short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}\left(\frac{C_\bullet(x)}{C_\bullet^{\mathcal{U}}(X\setminus(K\cap L))}, R\right) \arrow[r] & C^\bullet(X) \arrow[r] & \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X \setminus K \cap L), R) \arrow[r] & 0 \\
0 \arrow[r] & C^\bullet(X, X \setminus K \cap L) \arrow[r] \arrow[u] & C^\bullet(X) \arrow[r] \arrow[u, equal] & \mathrm{Hom}(C_\bullet(X \setminus K \cap L), R) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Now the two right vertical arrows induce isomorphisms when we pass on to homology. So by taking the long exact sequence, the five lemma tells us the left hand map is an isomorphism on homology. So we know

$$H_*\left(\mathrm{Hom}\left(\frac{C_\bullet(x)}{C_\bullet^{\mathcal{U}}(X \setminus (K \cap L))}, R\right)\right) \cong H^*(X \mid K \cap L).$$

So the long exact of the top row gives what we want. $\square$

Corollary. Let X be a manifold, and $X = A \cup B$, where A, B are open sets. Then there is a long exact sequence

$$\begin{tikzcd}
H_c^n(A \cap B) \arrow[r] & H_c^n(A) \oplus H_c^n(B) \arrow[r] & H_c^n(X) \arrow[dll, "\partial"] \\
H_c^{n+1}(A \cap B) \arrow[r] & H_c^{n+1}(A) \oplus H_c^{n+1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Let $K \subseteq A$ and $L \subseteq B$ be compact sets. Then by excision, we have isomorphisms

$$\begin{aligned}
H^n(X \mid K) &\cong H^n(A \mid K) \\
H^n(X \mid L) &\cong H^n(B \mid L) \\
H^n(X \mid K \cap L) &\cong H^n(A \cap B \mid K \cap L).
\end{aligned}$$

So the long exact sequence from the previous proposition gives us

$$\begin{tikzcd}
H^n(A \cap B \mid K \cap L) \arrow[r] & H^n(A \mid K) \oplus H^n(B \mid L) \arrow[r] & H^n(X \mid K \cup L) \arrow[dll, "\partial"] \\
H^{n+1}(A \cap B \mid K \cap L) \arrow[r] & H^{n+1}(A \mid K) \oplus H^{n+1}(B \mid L) \arrow[r] & \cdots
\end{tikzcd}$$

The next step is to take the direct limit over $K \in \mathcal{K}(A)$ and $L \in \mathcal{K}(B)$. We need to make sure that these do indeed give the right compactly supported cohomology. The terms $H^n(A \mid K) \oplus H^n(B \mid L)$ are exactly right, and the one for $H^n(A \cap B \mid K \cap L)$ also works because every compact set in $A \cap B$ is a compact set in A intersect a compact set in B (take those to be both the original compact set).

So we get a long exact sequence

$$H_c^n(A \cap B) \to H_c^n(A) \oplus H_c^n(B) \to \varinjlim_{\substack{K \in \mathcal{K}(A) \\ L \in \mathcal{K}(B)}} H^n(X \mid K \cup L) \xrightarrow{\partial} H_c^{n+1}(A \cap B)$$

To show that that funny direct limit is really what we want, we have to show that every compact set $C \in X$ lies inside some $K \cup L$, where $K \subseteq A$ and $L \subseteq B$ are compact.

Indeed, as X is a manifold, and C is compact, we can find a finite set of closed balls in X, each in A or in B, such that their interiors cover C. So done. (In general, this will work for any locally compact space) $\square$

11.2 Orientation of manifolds

Lemma.

(i) If $R = \mathbb{F}_2$, then every manifold is R-orientable.

(ii) If $\{\varphi_\alpha : \mathbb{R}^d \to U_\alpha \subseteq M\}$ is an open cover of M by Euclidean space such that each homeomorphism

$$\mathbb{R}^d \supseteq \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \xleftarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^d$$

is orientation-preserving, then M is R-orientable.

Proof.

(i) $\mathbb{F}_2$ has a unique $\mathbb{F}_2$-module generator.

(ii) For $x \in U_\alpha$, we define μ_x to be the image of the standard orientation of $\mathbb{R}^d$ via

$$H_d(M \mid x) \xleftarrow[\cong]{} H_\alpha(U_d \mid x) \xleftarrow[(\varphi_\alpha)_*]{} H_d(\mathbb{R}^d \mid \varphi_\alpha^{-1}(x)) \xleftarrow[\text{trans.}]{} \mathbb{R}_d(\mathbb{R}^d \mid 0)$$

If this is well-defined, then it is obvious that this is compatible. However, we have to check it is well-defined, because to define this, we need to pick a chart.

If $x \in U_\beta$ as well, we need to look at the corresponding μ'_x defined using U_β instead. But they have to agree by definition of orientation-preserving. $\square$

Theorem. Let M be an R-oriented manifold and $A \subseteq M$ be compact. Then

(i) There is a unique class $\mu_A \in H_d(M \mid A; R)$ which restricts to $\mu_x \in H_d(M \mid x; R)$ for all $x \in A$.

(ii) $H_i(M \mid A; R) = 0$ for $i > d$.

Proof. Call a compact set A "good" if it satisfies the conclusion of the theorem.

Claim. We first show that if K, L and $K \cap L$ is good, then $K \cup L$ is good.

This is analogous to the proof of the Thom isomorphism theorem, and we will omit this.

Now our strategy is to prove the following in order:

(i) If $A \subseteq \mathbb{R}^d$ is convex, then A is good.

(ii) If $A \subseteq \mathbb{R}^d$, then A is good.

(iii) If $A \subseteq M$, then A is good.

Claim. If $A \subseteq \mathbb{R}^d$ is convex, then A is good.

Let $x \in A$. Then we have an inclusion

$$\mathbb{R}^d \setminus A \hookrightarrow \mathbb{R}^d \setminus \{x\}.$$

This is in fact a homotopy equivalence by scaling away from x. Thus the map

$$H_i(\mathbb{R}^d \mid A) \to H_i(\mathbb{R}^d \mid x)$$

is an isomorphism by the five lemma for all i. Then in degree d, there is some μ_A corresponding to μ_x. This μ_A is then has the required property by definition of orientability. The second part of the theorem also follows by what we know about $H_i(\mathbb{R}^d \mid x)$.

Claim. If $A \subseteq \mathbb{R}^d$, then A is good.

For $A \subseteq \mathbb{R}^d$ compact, we can find a finite collection of closed balls B_i such that

$$A \subseteq \bigcup_{i=1}^n \mathring{B}_i = B.$$

Moreover, if $U \supseteq A$ for any open U, then we can in fact take $B_i \subseteq U$. By induction on the number of balls n, the first claim tells us that any B of this form is good.

We now let

$$\mathcal{G} = \{B \subseteq \mathbb{R}^d : A \subseteq \mathring{B}, B \text{ compact and good}\}.$$

We claim that this is a directed set under inverse inclusion. To see this, for $B, B' \in \mathcal{G}$, we need to find a $B'' \in \mathcal{G}$ such that $B'' \subseteq B, B'$ and B'' is good and compact. But the above argument tells us we can find one contained in $\mathring{B}' \cup \mathring{B}''$. So we are safe.

Now consider the directed system of groups given by

$$B \mapsto H_i(\mathbb{R}^d \mid B),$$

and there is an induced map

$$\varinjlim_{B \in \mathcal{G}} H_i(\mathbb{R}^d \mid B) \to H_i(\mathbb{R}^d \mid A),$$

since each $H_i(\mathbb{R}^d \mid B)$ maps to $H_i(\mathbb{R}^d \mid A)$ by inclusion, and these maps are compatible. We claim that this is an isomorphism. We first show that this is surjective. Let $[c] \in H_i(\mathbb{R}^d \mid A)$. Then the boundary of $c \in C_i(\mathbb{R}^d)$ is a finite sum of simplices in $\mathbb{R}^d \setminus A$. So it is a sum of simplices in some compact $C \subseteq \mathbb{R}^d \setminus A$. But then $A \subseteq \mathbb{R}^d \setminus C$, and $\mathbb{R}^d \setminus C$ is an open neighbourhood of A. So we can find a good B such that

$$A \subseteq B \subseteq \mathbb{R}^d \setminus C.$$

Then $c \in C_i(\mathbb{R}^d \mid B)$ is a cycle. So we know $[c] \in H_i(\mathbb{R}^d \mid B)$. So the map is surjective. Injectivity is obvious.

An immediate consequence of this is that for $i > d$, we have $H_i(\mathbb{R}^d \mid A) = 0$. Also, if $i = d$, we know that μ_A is given uniquely by the collection $\{\mu_B\}_{B \in \mathcal{G}}$ (uniqueness follows from injectivity).

Claim. If $A \subseteq M$, then A is good.

This follows from the fact that any compact $A \subseteq M$ can be written as a finite union of compact A_α with $A_\alpha \subseteq U_\alpha \cong \mathbb{R}^d$. So A_α and their intersections are good. So done. □

Corollary. If M is compact, then we get a unique class $[M] = \mu_M \in H_n(M; R)$ such that it restricts to μ_x for each $x \in M$. Moreover, $H^i(M; R) = 0$ for $i > d$.

11.3 Poincaré duality

Theorem (Poincaré duality)**.** Let M be a d-dimensional R-oriented manifold. Then there is a map

$$D_M : H_c^k(M; R) \to H_{d-k}(M; R)$$

that is an isomorphism.

Lemma. We have

$$d(\sigma \frown \varphi) = (-1)^d((d\sigma) \frown \varphi - \sigma \frown (d\varphi)).$$

Proof. Write both sides out. □

Lemma. If $f : X \to Y$ is a map, and $x \in H_k(X; R)$ And $y \in H^\ell(Y; R)$, then we have

$$f_*(x) \frown y = f_*(x \frown f^*(y)) \in H_{k-\ell}(Y; R).$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
 & H_k(Y; R) \times H^\ell(Y; R) \arrow[r, "\frown"] & H_{k-\ell}(Y; R) \\
H_k(X; R) \times H^\ell(Y; R) \arrow[ur, "f_* \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times f^*"'] & & \\
 & H_k(X; R) \times H^\ell(X; R) \arrow[r, "\frown"] & H_{k-\ell}(X; R) \arrow[uu, "f_*"']
\end{tikzcd}$$

Proof. We check this on the cochain level. We let $x = \sigma : \Delta^k \to X$. Then we have

$$\begin{aligned}
f_\#(\sigma \frown f^\# y) &= f_\# \left((f^\# y)(\sigma|_{[v_0, \ldots, v_\ell]})\sigma|_{[v_\ell, \ldots, v_k]}\right) \\
&= y(f_\#(\sigma|_{[v_0, \ldots, v_\ell]}))f_\#(\sigma|_{[v_\ell, \ldots, v_k]}) \\
&= y((f_\#\sigma)|_{[v_0, \ldots, v_\ell]})(f_\#\sigma)|_{[v_\ell, \ldots, v_k]} \\
&= (f_\#\sigma) \frown y.
\end{aligned}$$

So done. □

Proof. We say M is "good" if the Poincaré duality theorem holds for M. We now do the most important step in the proof:

Claim 0. $\mathbb{R}^d$ is good.

The only non-trivial degrees to check are $\ell = 0, d$, and $\ell = 0$ is straightforward. For $\ell = d$, we have shown that the maps

$$H_c^d(\mathbb{R}^d; R) \xleftarrow[\sim]{} H^d(\mathbb{R}^d \mid 0; R) \xrightarrow{\text{UCT}} \operatorname{Hom}_R(H_d(\mathbb{R}^d \mid 0; R), R)$$

are isomorphisms, where the last map is given by the universal coefficients theorem.

Under these isomorphisms, the map

$$H_c^d(\mathbb{R}^d; R) \xrightarrow{D_{\mathbb{R}^d}} H_0(\mathbb{R}^d; R) \xrightarrow{\varepsilon} R$$

corresponds to the map $\operatorname{Hom}_K(H_d(\mathbb{R}^d \mid 0; R), R) \to R$ is given by evaluating a function at the fundamental class μ_0. But as $\mu_0 \in H_d(\mathbb{R}^d \mid 0; R)$ is an R-module generator, this map is an isomorphism.

Claim 1. If $M = U \cup V$ and $U, V, U \cap V$ are good, then M is good.

Again, this is an application of the five lemma with the Mayer-Vietoris sequence. We have

$$\begin{tikzcd}
H_c^\ell(U \frown V) \arrow[r] \arrow[d, "D_{U \frown V}"] & H_c^\ell(U) \oplus H_c^d(V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H_c^\ell(M) \arrow[r] \arrow[d, "D_M"] & H_c^{\ell+1}(U \frown V) \arrow[d, "D_{U \frown V}"] \\
H_{d-\ell}(U \frown V) \arrow[r] & H_{d-\ell}(U) \oplus H_{d-\ell}(V) \arrow[r] & H_{d-\ell}(M) \arrow[r] & H_{d-\ell-1}(U \frown V)
\end{tikzcd}$$

We are done by the five lemma if this commutes. But unfortunately, it doesn't. It only commutes up to a sign, but it is sufficient for the five lemma to apply if we trace through the proof of the five lemma.

Claim 2. If $U_1 \subseteq U_2 \subseteq \cdots$ with $M = \bigcup_n U_n$, and U_i are all good, then M is good.

Any compact set in M lies in some U_n, so the map

$$\varinjlim H_c^\ell(U_n) \to H_c^\ell(U_n)$$

is an isomorphism. Similarly, since simplices are compact, we also have

$$H_{d-k}(M) = \varinjlim H_{d-k}(U_n).$$

Since the direct limit of open sets is open, we are done.

Claim 3. Any open subset of $\mathbb{R}^d$ is good.

Any U is a countable union of open balls (something something rational points something something). For finite unions, we can use Claims 0 and 1 and induction. For countable unions, we use Claim 2.

Claim 4. If M has a countable cover by $\mathbb{R}^d$'s it is good.

Same argument as above, where we instead use Claim 3 instead of Claim 0 for the base case.

Claim 5. Any manifold M is good.

Any manifold is second-countable by definition, so has a countable open cover by $\mathbb{R}^d$. ☐

Corollary. For any compact d-dimensional R-oriented manifold M, the map

$$[M] \frown \cdot : H^\ell(M;R) \to H_{d-\ell}(M;R)$$

is an isomorphism.

Corollary. Let M be an odd-dimensional compact manifold. Then the Euler characteristic $\chi(M) = 0$.

Proof. Pick $R = \mathbb{F}_2$. Then M is $\mathbb{F}_2$-oriented. Since we can compute Euler characteristics using coefficients in $\mathbb{F}_2$. We then have

$$\chi(M) = \sum_{r=0}^{2n+1} (-1)^i \dim_{\mathbb{F}_2} H_i(M, \mathbb{F}_2).$$

But we know

$$H_i(M, \mathbb{F}_2) \cong H^{2n+1-i}(M, \mathbb{F}_2) \cong (H_{2n+1-i}(M, \mathbb{F}_2))^* \cong H_{2n+1-i}(M, \mathbb{F}_2)$$

by Poincaré duality and the universal coefficients theorem.

But the dimensions of these show up in the sum above with opposite signs. So they cancel, and $\chi(M) = 0$. ☐

Theorem. Let M be a d-dimensional compact R-oriented manifold, and consider the following pairing:

$$\begin{aligned} \langle \cdot\,, \cdot \rangle : H^k(M;R) \otimes H^{d-k}(M,R) &\longrightarrow R \\ [\varphi] \otimes [\psi] &\longmapsto (\varphi \smile \psi)[M] \end{aligned}.$$

If $H_*(M;R)$ is free, then $\langle \cdot\,, \cdot \rangle$ is non-singular, i.e. both adjoints are isomorphisms, i.e. both

$$\begin{aligned} H^k(M;R) &\longrightarrow \mathrm{Hom}(H^{d-k}(M;R), R) \\ [\varphi] &\longmapsto ([\psi] \mapsto \langle \varphi, \psi \rangle) \end{aligned}$$

and the other way round are isomorphisms.

Proof. We have

$$\langle \varphi, \psi \rangle = (-1)^{|\varphi||\psi|} \langle \psi, \varphi \rangle,$$

as we know

$$\varphi \smile \psi = (-1)^{|\varphi||\psi|} \psi \smile \varphi.$$

So if one adjoint is an isomorphism, then so is the other.

To see that they are isomorphsims, we notice that we have an isomorphism

$$\begin{aligned} H^k(M;R) \xrightarrow{\text{UCT}} \mathrm{Hom}_R(H_k(M;R), R) &\xrightarrow{D_m^*} \mathrm{Hom}_R(H^{d-k}(M;R), R) \\ [\varphi] \longmapsto ([\sigma] \mapsto \varphi(\sigma)) &\longmapsto ([\psi] \mapsto \varphi([M] \frown \psi)) \end{aligned}.$$

But we know

$$\varphi([M] \frown \psi) = (\psi \smile \varphi)([M]) = \langle \psi, \varphi \rangle.$$

So this is just the adjoint. So the adjoint is an isomorphism. ☐

11.4 Applications

Corollary. Let $f : M \to N$ be a map between manifolds. If $\mathbb{F}$ is a field and $\deg(f) \neq 0 \in \mathbb{F}$, then then the induced map

$$f^* : H^*(N, \mathbb{F}) \to H^*(M, \mathbb{F})$$

is injective.

Proof. Suppose not. Let $\alpha \in H^k(N, \mathbb{F})$ be non-zero but $f^*(\alpha) = 0$. As

$$\langle \cdot\,, \cdot \rangle : H^k(N, \mathbb{F}) \otimes H^{d-k}(N, \mathbb{F}) \to \mathbb{F}$$

is non-singular, we know there is some $\beta \in H^{d-k}(N, \mathbb{F})$ such that

$$\langle \alpha, \beta \rangle = (\alpha \smile \beta)[N] = 1.$$

Then we have

$$\begin{aligned} \deg(f) &= \deg(f) \cdot 1 \\ &= (\alpha \smile \beta)(\deg(f)[N]) \\ &= (\alpha \smile \beta)(f_*[M]) \\ &= (f^*(\alpha) \smile f^*(\beta))([M]) \\ &= 0. \end{aligned}$$

This is a contradiction. □

11.5 Intersection product

Lemma. Let X be a space and V a vector bundle over X. If $V = U \oplus W$, then orientations for any two of U, W, V give an orientation for the third.

Proof. Say $\dim V = d$, $\dim U = n$, $\dim W = m$. Then at each point $x \in X$, by Künneth's theorem, we have an isomorphism

$$H^d(V_x, V_x^\#; R) \cong H^n(U_x, U_x^\#; R) \otimes H^m(W_x, W_x^\#; R) \cong R.$$

So any local R-orientation on any two induces one on the third, and it is straightforward to check the local compatibility condition. □

Theorem. The Poincaré dual of a submanifold is (the extension by zero of) the normal Thom class.

11.6 The diagonal

Theorem. We have

$$\delta = \sum_i (-1)^{|a_i|} a_i \otimes b_i.$$

Proof. We can certainly write

$$\delta = \sum_{i,} C_{ij} a_i \otimes b_j$$

for some C_{ij}. So we need to figure out what the coefficients C_{ij} are. We try to compute

$$\begin{aligned}((b_k \otimes a_\ell) \smile \delta)[M \times M] &= \sum C_{ij}(b_k \otimes a_\ell) \smile (a_i \otimes b_j)[M \times M]\\ &= \sum C_{ij}(-1)^{|a_\ell||a_i|}(b_k \smile a_i) \otimes (a_\ell \smile b_i)[M] \otimes [M]\\ &= \sum C_{ij}(-1)^{|a_\ell||a_i|}(\delta_{ik}(-1)^{|a_i||b_k|})\delta_{j\ell}\\ &= (-1)^{|a_k||a_\ell| + |a_k||b_k|} C_{k\ell}.\end{aligned}$$

But we can also compute this a different way, using the definition of δ:

$$(b_k \otimes a_\ell \smile \delta)[M \times M] = (b_k \otimes a_\ell)(\Delta_*[M]) = (b_k \smile a_\ell)[M] = (-1)^{|a_\ell||b_k|}\delta_{k\ell}.$$

So we see that

$$C_{k\ell} = \delta_{k\ell}(-1)^{|a_\ell|}.$$

□

Corollary. We have

$$\Delta^*(\delta)[M] = \chi(M),$$

the Euler characteristic.

Proof. We note that for $a \otimes b \in H^n(M \times M)$, we have

$$\Delta^*(a \otimes b) = \Delta^*(\pi_1^* a \smile \pi_2^* b) = a \smile b$$

because $\pi_i \circ \Delta = \mathrm{id}$. So we have

$$\Delta^*(\delta) = \sum (-1)^{|a_i|} a_i \smile b_i.$$

Thus

$$\Delta^*(\delta)[M] = \sum_i (-1)^{|a_i|} = \sum_k (-1)^k \dim_{\mathbb{Q}} H^k(M; \mathbb{Q}) = \chi(M).$$

□

Corollary. We have

$$e(TM)[M] = \chi(M).$$

Corollary. If M has a nowhere-zero vector field, then $\chi(M) = 0$.

Lemma. Suppose we have R-oriented vector bundles $E \to X$ and $F \to X$ with Thom classes u_E, u_F. Then the Thom class for $E \oplus F \to X$ is $u_E \smile u_F$. Thus

$$e(E \oplus F) = e(E) \smile e(F).$$

Proof. More precisely, we have projection maps

$$\begin{tikzcd} & E \oplus F \arrow[dl, "\pi_E"'] \arrow[dr, "\pi_F"] & \\ E & & F \end{tikzcd}.$$

We let $U = \pi_E^{-1}(E^\#)$ and $V = \pi_F^{-1}(F^\#)$. Now observe that

$$U \cup V = (E \oplus F)^\#.$$

So if $\dim E = e, \dim F = f$, then we have a map

$$\begin{tikzcd} H^e(E, E^\#) \otimes H^f(F, F^\#) \arrow[r, "\pi_E^* \otimes \pi_F^*"] & H^e(E \oplus F, U) \otimes H^f(E \oplus F, V) \arrow[d, "\smile"] \\ & H^{e+f}(E \oplus F, (E \oplus F)^\#) \end{tikzcd},$$

and it is easy to see that the image of $u_E \otimes u_F$ is the Thom class of $E \oplus F$ by checking the fibers. $\square$

Corollary. TS^{2n} has no proper subbundles.

Proof. We know $e(TS^{2n}) \neq 0$ as $e(TS^{2n})[S^2] = \chi(S^{2n}) = 2$. But it cannot be a proper cup product of two classes, since there is nothing in lower cohomology groups. So TS^{2n} is not the sum of two subbundles. Hence TS^{2n} cannot have a proper subbundle E or else $TS^{2n} = E \oplus E^\perp$ (for any choice of inner product). $\square$

11.7 Lefschetz fixed point theorem

Theorem (Lefschetz fixed point theorem)**.** Let M be a compact d-dimensional $\mathbb{Z}$-oriented manifold, and let $f : M \to M$ be a map such that the graph Γ_f and diagonal Δ intersect transversely. Then Then we have

$$\sum_{x \in \mathrm{fix}(f)} \operatorname{sgn} \det(I - \mathrm{D}_x f) = \sum_k (-1)^k \operatorname{tr}(f^* : H^i(M; \mathbb{Q}) \to H^k(M; \mathbb{Q})).$$

Proof. We have

$$[\Gamma_f] \cdot [\Delta(M)] \in H_0(M \times M; \mathbb{Q}).$$

We now want to calculate ε of this. By Poincaré duality, this is equal to

$$(D_{M \times M}^{-1}[\Gamma_f] \smile D_{M \times M}^{-1}[\Delta(M)])[M \times M] \in \mathbb{Q}.$$

This is the same as

$$(D_{M \times M}^{-1}[\Delta(M)])([\Gamma_f]) = \delta(F_*[M]) = (F^*\delta)[M],$$

where $F : M \to M \times M$ is given by

$$F(x) = (x, f(x)).$$

We now use the fact that

$$\delta = \sum (-1)^{|a_i|} a_i \otimes b_i.$$

So we have

$$F^*\delta = \sum (-1)^{|a_i|} a_i \otimes f^* b_i.$$

We write

$$f^* b_i = \sum C_{ij} b_j.$$

Then we have

$$(F^*\delta)[M] = \sum_{i,j} (-1)^{|a_i|} C_{ij} (a_i \otimes b_j)[M] = \sum_i (-1)^{|a_i|} C_{ii},$$

and C_{ii} is just the trace of f^*.

We now compute this product in a different way. As Γ_f and $\Delta(M)$ are transverse, we know $\Gamma_f \cap \Delta(M)$ is a 0-manifold, and the orientation of Γ_f and $\Delta(M)$ induces an orientation of it. So we have

$$[\Gamma_f] \cdot [\Delta(m)] = [\Gamma_f \cap \Delta(M)] \in H_0(M \times M; \mathbb{Q}).$$

We know this $\Gamma_f \cap \Delta(M)$ has $|\operatorname{fix}(f)|$ many points, so $[\Gamma_f \cap \Delta(M)]$ is the sum of $|\operatorname{fix}(f)|$ many things, which is what we've got on the left above. We have to figure out the sign of each term is actually $\operatorname{sgn}\det(I - \mathrm{D}_x f)$, and this is left as an exercise on the example sheet. $\square$