Part III — Algebraic Topology
Theorems

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Algebraic Topology assigns algebraic invariants to topological spaces; it permeates modern pure mathematics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. The course will not assume any knowledge of Algebraic Topology, but will go quite fast in order to reach more interesting material, so some previous exposure to simplicial homology or the fundamental group would be helpful. The Part III Differential Geometry course will also contain useful, relevant material.

Hatcher's book is especially recommended for the course, but there are many other suitable texts.

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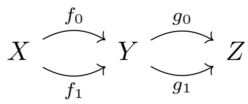
1 Homotopy

Proposition. If $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 : X \to Z$.



2 Singular (co)homology

2.1 Chain complexes

Lemma. If $f_\bullet : C_\bullet \to D_\bullet$ is a chain map, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ given by $[x] \mapsto [f_n(x)]$ is a well-defined homomorphism, where $x \in C_n$ is any element representing the homology class $[x] \in H_n(C_\bullet)$.

2.2 Singular (co)homology

Lemma. If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1} : \Delta^{n-2} \to \Delta^n$.

Corollary. The homomorphism $d_{n-1} \circ d_n : C_n(X) \to C_{n-2}(X)$ vanishes.

Proposition. If $f : X \to Y$ is a continuous map of topological spaces, then the maps

$$\begin{aligned} f_n : C_n(X) &\to C_n(Y) \\ (\sigma : \Delta^n \to X) &\mapsto (f \circ \sigma : \Delta^n \to Y) \end{aligned}$$

give a chain map. This induces a map on the homology (and cohomology).

Proposition. If $f : X \to Y$ is a homeomorphism, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism of abelian groups.

Lemma. If X is path-connected and non-empty, then $H_0(X) \cong \mathbb{Z}$.

Proposition. For any space X, we have $H_0(X)$ is a free abelian group generated by the path components of X.

3 Four major tools of (co)homology

3.1 Homotopy invariance

Theorem (Homotopy invariance theorem)**.** Let $f \simeq g : X \to Y$ be homotopic maps. Then they induce the same maps on (co)homology, i.e.

$$f_* = g_* : H_\bullet(X) \to H_\bullet(Y)$$

and

$$f^* = g^* : H^\bullet(Y) \to H^\bullet(X).$$

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_\bullet(X) \to H_\bullet(Y)$ and $f^* : H^\bullet(Y) \to H^\bullet(X)$ are isomorphisms.

3.2 Mayer-Vietoris

Lemma. In a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0\ ,$$

the map f is injective; g is surjective, and $C \cong B/A$.

Theorem (Mayer-Vietoris theorem)**.** Let $X = A \cup B$ be the union of two open subsets. We have inclusions

$$\begin{array}{ccc} A \cap B & \xhookrightarrow{i_A} & A \\ \big\downarrow{\scriptstyle i_B} & & \big\downarrow{\scriptstyle j_A} \\ B & \xhookrightarrow{j_B} & X \end{array}.$$

Then there are homomorphisms $\partial_{MV} : H_n(X) \to H_{n-1}(A \cap B)$ such that the following sequence is exact:

$$\xrightarrow{\partial_{MV}} H_n(A\cap B) \xrightarrow{i_{A*}\oplus i_{B*}} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*}-j_{B*}} H_n(X) \xrightarrow{\partial_{MV}}$$

$$\longrightarrow H_{n-1}(A\cap B) \xrightarrow{i_{A*}\oplus i_{B*}} H_{n-1}(A) \oplus H_{n-1}(B) \xrightarrow{j_{A*}-j_{B*}} H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(A) \oplus H_0(B) \xrightarrow{j_{A*}-j_{B*}} H_0(X) \longrightarrow 0$$

Furthermore, the Mayer-Vietoris sequence is *natural*, i.e. if $f : X = A \cup B \to Y = U \cup V$ satisfies $f(A) \subseteq U$ and $f(B) \subseteq V$, then the diagram

$$\begin{array}{ccccccc} H_{n+1}(X) & \xrightarrow{\partial_{MV}} & H_n(A\cap B) & \xrightarrow{i_{A*}\oplus i_{B*}} & H_n(A)\oplus H_n(B) & \xrightarrow{j_{A*}-j_{B*}} & H_n(X) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f|_{A\cap B *}} & & \big\downarrow{\scriptstyle f|_{A*}\oplus f|_{B*}} & & \big\downarrow{\scriptstyle f_*} \\ H_{n+1}(Y) & \xrightarrow{\partial_{MV}} & H_n(U\cap V) & \xrightarrow{i_{U*}\oplus i_{V*}} & H_n(U)\oplus H_n(V) & \xrightarrow{j_{U*}-j_{V*}} & H_n(Y) \end{array}$$

commutes.

3.3 Relative homology

Theorem (Exact sequence for relative homology)**.** There are homomorphisms $\partial : H_n(X, A) \to H_{n-1}(A)$ given by mapping

$$\big[[c]\big] \mapsto [d_n c].$$

This makes sense because if $c \in C_n(X)$, then $[c] \in C_n(X)/C_n(A)$. We know $[d_n c] = 0 \in C_{n-1}(X)/C_{n-1}(A)$. So $d_n c \in C_{n-1}(A)$. So this notation makes sense.

Moreover, there is a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{q_*} H_{n-1}(X, A) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_0(X) \xrightarrow{q_*} H_0(X, A) \longrightarrow 0$$

where i_* is induced by $i : C_\bullet(A) \to C_\bullet(X)$ and q_* is induced by the quotient $q : C_\bullet(X) \to C_\bullet(X, A)$.

3.4 Excision theorem

Theorem (Excision theorem)**.** Let (X, A) be a pair of spaces, and $Z \subseteq A$ be such that $\overline{Z} \subseteq \mathring{A}$ (the closure is taken in X). Then the map

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

is an isomorphism.

3.5 Applications

Theorem. We have

$$H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

Theorem. For any $n \geq 1$, we have

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

Corollary. If $n \neq m$, then $S^{n-1} \not\simeq S^{m-1}$, since they have different homology groups.

Corollary. If $n \neq m$, then $\mathbb{R}^n \not\cong \mathbb{R}^m$.

Proposition.

(i) $\deg(\mathrm{id}_{S^n}) = 1$.

(ii) If f is not surjective, then $\deg(f) = 0$.

(iii) We have $\deg(f \circ g) = (\deg f)(\deg g)$.

(iv) Homotopic maps have equal degrees.

Corollary (Brouwer's fixed point theorem)**.** Any map $f : D^n \to D^n$ has a fixed point.

Proposition. A reflection $r : S^n \to S^n$ about a hyperplane has degree -1. As before, we cover S^n by

$$\begin{aligned} A &= S^n \setminus \{N\} \cong \mathbb{R}^n \simeq *, \\ B &= S^n \setminus \{S\} \cong \mathbb{R}^n \simeq *, \end{aligned}$$

where we suppose the north and south poles lie in the hyperplane of reflection. Then both A and B are invariant under the reflection. Consider the diagram

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow[\sim]{\partial_{MV}} & H_{n-1}(A \cap B) & \xleftarrow[\sim]{} & H_{n-1}(S^{n-1}) \\ \downarrow r_* & & \downarrow r_* & & \downarrow r_* \\ H_n(S^n) & \xrightarrow[\sim]{\partial_{MV}} & H_{n-1}(A \cap B) & \xleftarrow[\sim]{} & H_{n-1}(S^{n-1}) \end{array}$$

where the S^{n-1} on the right most column is given by contracting $A \cap B$ to the equator. Note that r restricts to a reflection on the equator. By tracing through the isomorphisms, we see that $\deg(r) = \deg(r|_{\text{equator}})$. So by induction, we only have to consider the case when $n = 1$. Then we have maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1(S^1) & \xrightarrow[\sim]{\partial_{MV}} & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \\ & & \downarrow r_* & & \downarrow r_* & & \downarrow r_* \oplus r_* \\ 0 & \longrightarrow & H_1(S^1) & \xrightarrow[\sim]{\partial_{MV}} & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \end{array}$$

Now the middle vertical map sends $p \mapsto q$ and $q \mapsto p$. Since $H_1(S^1)$ is given by the kernel of $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$, and is generated by $p - q$, we see that this sends the generator to its negation. So this is given by multiplication by -1. So the degree is -1.

Corollary. The antipodal map $a : S^n \to S^n$ given by

$$a(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, -x_{n+1})$$

has degree $(-1)^{n+1}$ because it is a composition of $(n + 1)$ reflections.

Corollary (Hairy ball theorem)**.** S^n has a nowhere 0 vector field iff n is odd. More precisely, viewing $S^n \subseteq \mathbb{R}^{n+1}$, a vector field on S^n is a map $v : S^n \to \mathbb{R}^{n+1}$ such that $\langle v(x), x\rangle = 0$, i.e. $v(x)$ is perpendicular to x.

Lemma. Let M be a d-dimensional manifold (i.e. a Hausdorff, second-countable space locally homeomorphic to $\mathbb{R}^d$). Then

$$H_n(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & n = d \\ 0 & \text{otherwise} \end{cases}.$$

This is known as the *local homology*.

Theorem. Let $f : S^d \to S^d$ be a map. Suppose there is a $y \in S^d$ such that

$$f^{-1}(y) = \{x_1, \cdots, x_k\}$$

is finite. Then

$$\deg(f) = \sum_{i=1}^{k} \deg(f)_{x_i}.$$

3.6 Repaying the technical debt

Theorem (Snake lemma)**.** Suppose we have a short exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{i_{\bullet}} B_{\bullet} \xrightarrow{q_{\bullet}} C_{\bullet} \longrightarrow 0\ .$$

Then there are maps

$$\partial : H_n(C_{\bullet}) \to H_{n-1}(A_{\bullet})$$

such that there is a long exact sequence

$$\begin{array}{c} \cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{q_*} H_n(C) \xrightarrow{\partial_*} \\ H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{q_*} H_{n-1}(C) \longrightarrow \cdots \end{array}.$$

Lemma (Five lemma)**.** Consider the following commutative diagram:

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E' \end{array}$$

If the two rows are exact, m and p are isomorphisms, q is injective and ℓ is surjective, then n is also an isomorphism.

Corollary. Let $f : (X, A) \to (Y, B)$ be a map of pairs, and that any two of $f_* : H_*(X, A) \to H_*(Y, B)$, $H_*(X) \to H_*(Y)$ and $H_*(A) \to H_*(B)$ are isomorphisms. Then the third is also an isomorphism.

Lemma. If $f_{\bullet}$ and $g_{\bullet}$ are chain homotopic, then $f_* = g_* : H_*(C_{\bullet}) \to H_*(D_{\bullet})$.

Theorem (Small simplices theorem)**.** The natural map $H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.

Lemma. $\rho_{\bullet}^X$ is a natural chain map.

Lemma. $\rho_{\bullet}^X$ is chain homotopic to the identity.

Lemma. The diameter of each subdivided simplex in $(\rho_n^{\Delta^n})^k(\iota_n)$ is bounded by $\left(\frac{n}{n+1}\right)^k \operatorname{diam}(\Delta^n)$.

Proposition. If $c \in C_n^{\mathcal{U}}(X)$, then $p^X(c) \in C_n^{\mathcal{U}}(X)$.

Moreover, if $c \in C_n(X)$, then there is some k such that $(\rho_n^X)^k(c) \in C_n^{\mathcal{U}}(X)$.

Theorem (Small simplices theorem)**.** The natural map $U : H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.

4 Reduced homology

Theorem. If (X, A) is good, then the natural map

$$H_*(X, A) \longrightarrow H_*(X/A, A/A) = \tilde{H}_*(X/A)$$

is an isomorphism.

5 Cell complexes

Lemma. If $A \subseteq X$ is a subcomplex, then the pair (X, A) is *good*.

Corollary. If $A \subseteq X$ is a subcomplex, then

$$H_n(X, A) \xrightarrow{\sim} \tilde{H}_n(X/A)$$

is an isomorphism.

Lemma. Let X be a cell complex. Then

(i)
$$H_i(X^n, X^{n-1}) = \begin{cases} 0 & i \neq n \\ \bigoplus_{i \in I_n} \mathbb{Z} & i = n \end{cases}.$$

(ii) $H_i(X^n) = 0$ for all $i > n$.

(iii) $H_i(X^n) \to H_i(X)$ is an isomorphism for $i < n$.

Theorem.
$$H_n^{\text{cell}}(X) \cong H_n(X).$$

Corollary. If X is a finite cell complex, then $H_n(X)$ is a finitely-generated abelian group for all n, generated by at most $|I_n|$ elements. In particular, if there are no n-cells, then $H_n(X)$ vanishes.

If X has a cell-structure with cells in even-dimensional cells only, then $H_*(X)$ are all free.

Lemma. The coefficients $d_{\alpha\beta}$ are given by the degree of the map

$$\underbrace{S_\alpha^{n-1} = \partial D_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} = \bigvee_{\gamma \in I_{n-1}} S_\gamma^{n-1} \longrightarrow S_\beta^{n-1}}_{f_{\alpha\beta}},$$

where the final map is obtained by collapsing the other spheres in the wedge.

In the case of cohomology, the maps are given by the transposes of these.

6 (Co)homology with coefficients

7 Euler characteristic

Theorem. We have
$$\chi = \chi_{\mathbb{Z}} = \chi_{\mathbb{F}}.$$

8 Cup product

Lemma. If $\phi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, then

$$d(\phi \smile \psi) = (d\phi) \smile \psi + (-1)^k \phi \smile (d\psi).$$

Corollary. The cup product induces a well-defined map

$$\begin{aligned} \smile : H^k(X;R) \times H^\ell(X;R) &\longrightarrow H^{k+\ell}(X;R) \\ ([\phi],[\psi]) &\longmapsto [\phi \smile \psi] \end{aligned}$$

Proposition. $(H^*(X;R), \smile, [1])$ is a unital ring.

Proposition. Let R be a commutative ring. If $\alpha \in H^k(X;R)$ and $\beta \in H^\ell(X;R)$, then we have

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

Proposition. The cup product is natural, i.e. if $f : X \to Y$ is a map, and $\alpha, \beta \in H^*(Y;R)$, then

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

So f^* is a homomorphism of unital rings.

9 Künneth theorem and universal coefficients theorem

Theorem (Künneth's theorem)**.** Let R be a commutative ring, and suppose that $H^n(Y;R)$ is a free R-module for each n. Then the cross product map

$$\bigoplus_{k+\ell=n} H^k(X;R)\otimes H^\ell(Y;R) \xrightarrow{\times} H^n(X\times Y;R)$$

is an isomorphism for every n, for every finite cell complex X.

It follows from the five lemma that the same holds if we have a relative complex (Y,A) instead of just Y.

Theorem (Universal coefficients theorem for (co)homology)**.** Let R be a PID and M an R-module. Then there is a natural map

$$H_*(X;R)\otimes M \to H_*(X;M).$$

If $H_*(X;R)$ is a free module for each n, then this is an isomorphism. Similarly, there is a natural map

$$H^*(X;M)\to \mathrm{Hom}_R(H_*(X;R),M),,$$

which is an isomorphism again if $H^*(X;R)$ is free.

10 Vector bundles

10.1 Vector bundles

Theorem (Tubular neighbourhood theorem)**.** Let $M \subseteq N$ be a smooth submanifold. Then there is an open neighbourhood U of M and a homeomorphism $\nu_{M \subseteq N} \to U$, and moreover, this homeomorphism is the identity on M (where we view M as a submanifold of $\nu_{M \subseteq N}$ by the image of the zero section).

Proposition. Partitions of unity exist for any open cover.

Lemma. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is a continuous family of inner products on E. In other words, there is a map $E \otimes E \to \mathbb{R}$ which restricts to an inner product on each E_x.

Lemma. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some N such that E is a vector subbundle of $X \times \mathbb{R}^N$.

Corollary. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some $p : F \to X$ such that $E \oplus F \cong X \times \mathbb{R}^n$. In particular, E embeds as a subbundle of a trivial bundle.

Theorem. There is a correspondence

$$\left\{\begin{array}{c}\text{homotopy classess}\\ \text{of maps}\\ f : X \to \mathrm{Gr}_d(\mathbb{R}^\infty)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}d\text{-dimensional}\\ \text{vector bundles}\\ \pi : E \to X\end{array}\right\}$$

$$[f] \longmapsto f^*\gamma_d^{\mathbb{R}}$$

$$[f_\pi] \longleftarrow\!\shortmid \pi$$

10.2 Vector bundle orientations

Lemma. Every vector bundle is $\mathbb{F}_2$-orientable.

Lemma. If $\{U_\alpha\}_{\alpha \in I}$ is a family of covers such that for each $\alpha, \beta \in I$, the homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^d \xleftarrow[\cong]{\varphi_\alpha} E|_{U_\alpha \cap U_\beta} \xrightarrow[\varphi_\beta]{\cong} (U_\alpha \cap U_\beta) \times \mathbb{R}^d$$

gives an orientation preserving map from $(U_\alpha \cap U_\beta) \times \mathbb{R}^d$ to itself, i.e. has a positive determinant on each fiber, then E is orientable for any R.

10.3 The Thom isomorphism theorem

Theorem (Thom isomorphism theorem)**.** Let $\pi : E \to X$ be a d-dimensional vector bundle, and $\{\varepsilon_x\}_{x \in X}$ be an R-orientation of E. Then

(i) $H^i(E, E^\#; R) = 0$ for $i < d$.

(ii) There is a unique class $u_E \in H^d(E, E^\#; R)$ which restricts to ε_x on each fiber. This is known as the *Thom class*.

(iii) The map Φ given by the composition

$$H^i(X;R) \xrightarrow{\pi^*} H^i(E;R) \xrightarrow{-\smile u_E} H^{i+d}(E, E^\#; R)$$

is an isomorphism.

Note that (i) follows from (iii), since $H^i(X;R) = 0$ for $i < 0$.

Theorem. If there is a section $s : X \to E$ which is nowhere zero, then $e(E) = 0 \in H^d(X;R)$.

Theorem. We have

$$u_E \smile u_E = \Phi(e(E)) = \pi^*(e(E)) \smile u_E \in H^*(E, E^\#; R).$$

Lemma. If $\pi : E \to X$ is a d-dimensional R-module vector bundle with d odd, then $2e(E) = 0 \in H^d(X;R)$.

10.4 Gysin sequence

11 Manifolds and Poincaré duality

11.1 Compactly supported cohomology

Theorem. For any space X, we let

$$\mathcal{K}(X) = \{K \subseteq X : K \text{ is compact}\}.$$

This is a directed set under inclusion, and the map

$$K \mapsto H^n(X, X \setminus K)$$

gives a direct system of abelian groups indexed by $\mathcal{K}(X)$, where the maps ρ are given by restriction.

Then we have

$$H_c^*(X) \cong \varinjlim_{\mathbf{K}(X)} H^n(X, X \setminus K).$$

Lemma. We have

$$H_c^i(\mathbb{R}^d; R) \cong \begin{cases} R & i = d \\ 0 & \text{otherwise} \end{cases}.$$

Proposition. Let $K, L \subseteq X$ be compact. Then there is a long exact sequence

$$H^n(X \mid K \cap L) \longrightarrow H^n(X \mid K) \oplus H^n(X \mid L) \longrightarrow H^n(X \mid K \cup L) \xrightarrow{\partial} H^{n+1}(X \mid K \cap L) \to H^{n+1}(X \mid K) \oplus H^{n+1}(X \mid L) \longrightarrow \cdots,$$

where the unlabelled maps are those induced by inclusion.

Corollary. Let X be a manifold, and $X = A \cup B$, where A, B are open sets. Then there is a long exact sequence

$$H_c^n(A \cap B) \longrightarrow H_c^n(A) \oplus H_c^n(B) \longrightarrow H_c^n(X) \xrightarrow{\partial} H_c^{n+1}(A \cap B) \longrightarrow H_c^{n+1}(A) \oplus H_c^{n+1}(B) \longrightarrow \cdots$$

11.2 Orientation of manifolds

Lemma.

(i) If $R = \mathbb{F}_2$, then every manifold is R-orientable.

(ii) If $\{\varphi_\alpha : \mathbb{R}^d \to U_\alpha \subseteq M\}$ is an open cover of M by Euclidean space such that each homeomorphism

$$\mathbb{R}^d \supseteq \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \xleftarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^d$$

is orientation-preserving, then M is R-orientable.

Theorem. Let M be an R-oriented manifold and $A \subseteq M$ be compact. Then

(i) There is a unique class $\mu_A \in H_d(M \mid A; R)$ which restricts to $\mu_x \in H_d(M \mid x; R)$ for all $x \in A$.

(ii) $H_i(M \mid A; R) = 0$ for $i > d$.

Corollary. If M is compact, then we get a unique class $[M] = \mu_M \in H_n(M; R)$ such that it restricts to μ_x for each $x \in M$. Moreover, $H^i(M; R) = 0$ for $i > d$.

11.3 Poincaré duality

Theorem (Poincaré duality)**.** Let M be a d-dimensional R-oriented manifold. Then there is a map

$$D_M : H_c^k(M; R) \to H_{d-k}(M; R)$$

that is an isomorphism.

Lemma. We have

$$d(\sigma \frown \varphi) = (-1)^d((d\sigma) \frown \varphi - \sigma \frown (d\varphi)).$$

Lemma. If $f : X \to Y$ is a map, and $x \in H_k(X; R)$ And $y \in H^\ell(Y; R)$, then we have

$$f_*(x) \frown y = f_*(x \frown f^*(y)) \in H_{k-\ell}(Y; R).$$

In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & H_k(Y;R) \times H^\ell(Y;R) & \xrightarrow{\frown} & H_{k-\ell}(Y;R) \\ & \nearrow^{f_* \times \mathrm{id}} & & & \uparrow f_* \\ H_k(X;R) \times H^\ell(Y;R) & & & & \\ & \searrow_{\mathrm{id} \times f^*} & & & \\ & & H_k(X;R) \times H^\ell(X;R) & \xrightarrow{\frown} & H_{k-\ell}(X;R) \end{array}$$

Corollary. For any compact d-dimensional R-oriented manifold M, the map

$$[M] \frown \cdot : H^\ell(M; R) \to H_{d-\ell}(M; R)$$

is an isomorphism.

Corollary. Let M be an odd-dimensional compact manifold. Then the Euler characteristic $\chi(M) = 0$.

Theorem. Let M be a d-dimensional compact R-oriented manifold, and consider the following pairing:

$$\begin{aligned} \langle \cdot, \cdot \rangle : H^k(M; R) \otimes H^{d-k}(M, R) &\longrightarrow R \\ [\varphi] \otimes [\psi] &\longmapsto (\varphi \smile \psi)[M] \end{aligned}.$$

If $H_*(M; R)$ is free, then $\langle \cdot, \cdot \rangle$ is non-singular, i.e. both adjoints are isomorphisms, i.e. both

$$\begin{aligned} H^k(M; R) &\longrightarrow \mathrm{Hom}(H^{d-k}(M; R), R) \\ [\varphi] &\longmapsto ([\psi] \mapsto \langle \varphi, \psi \rangle) \end{aligned}$$

and the other way round are isomorphisms.

11.4 Applications

Corollary. Let $f : M \to N$ be a map between manifolds. If $\mathbb{F}$ is a field and $\deg(f) \neq 0 \in \mathbb{F}$, then then the induced map

$$f^* : H^*(N, \mathbb{F}) \to H^*(M, \mathbb{F})$$

is injective.

11.5 Intersection product

Lemma. Let X be a space and V a vector bundle over X. If $V = U \oplus W$, then orientations for any two of U, W, V give an orientation for the third.

Theorem. The Poincaré dual of a submanifold is (the extension by zero of) the normal Thom class.

11.6 The diagonal

Theorem. We have

$$\delta = \sum_i (-1)^{|a_i|} a_i \otimes b_i.$$

Corollary. We have

$$\Delta^*(\delta)[M] = \chi(M),$$

the Euler characteristic.

Corollary. We have

$$e(TM)[M] = \chi(M).$$

Corollary. If M has a nowhere-zero vector field, then $\chi(M) = 0$.

Lemma. Suppose we have R-oriented vector bundles $E \to X$ and $F \to X$ with Thom classes u_E, u_F. Then the Thom class for $E \oplus F \to X$ is $u_E \smile u_F$. Thus

$$e(E \oplus F) = e(E) \smile e(F).$$

Corollary. TS^{2n} has no proper subbundles.

11.7 Lefschetz fixed point theorem

Theorem (Lefschetz fixed point theorem)**.** Let M be a compact d-dimensional $\mathbb{Z}$-oriented manifold, and let $f : M \to M$ be a map such that the graph Γ_f and diagonal Δ intersect transversely. Then Then we have

$$\sum_{x \in \mathrm{fix}(f)} \operatorname{sgn} \det(I - \mathrm{D}_x f) = \sum_k (-1)^k \operatorname{tr}(f^* : H^i(M; \mathbb{Q}) \to H^k(M; \mathbb{Q})).$$