Michaelmas Term 2016 O. Randal-Williams

Part III Algebraic Topology // Example Sheet 1

1. (i) Prove that homotopy equivalence is an equivalence relation on topological spaces.
   (ii) Which of the following are homotopy equivalent to S^1? (a) the annulus $\{1 < |z| < r\}$, (b) a bagel, (c) a genus two surface with a disc sewn across one of the holes, (d) the complement of a point in the real projective plane $\mathbb{RP}^2$.

2. Compute $H^0(X)$ for a topological space X. Give an example of a space X for which $H_0(X)$ and $H^0(X)$ are not isomorphic.

3. What can you say about the group G and/or the homomorphism α in an exact sequence of the shape
   (i) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$;
   (ii) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$;
   (iii) $0 \to \mathbb{Z}/4 \xrightarrow{\alpha} G \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$?

4. (i) The *suspension* ΣX of a space X is the quotient of $X \times [0,1]$ by the map which collapses each end of the cylinder to a point: $X \times \{0\} \sim p_0$ and $X \times \{1\} \sim p_1$. Observe $\Sigma S^n \cong S^{n+1}$. Hence or otherwise prove there are maps $f : S^n \to S^n$ of any degree, for any $n > 0$.
   (ii) Suppose A is a closed manifold. Is ΣA necessarily homeomorphic to a closed manifold? Justify your answer.

5. If $f : S^n \to S^n$ has no fixed points, show that it is homotopic to the antipodal map. Hence show that if a group G acts freely on S^{2n} then $|G| \leq 2$.

6. (i) Finish the proof (begun in lectures) of the theorem that a short exact sequence of chain complexes has an associated long exact sequence on homology, by showing that the sequence obtained really is exact.
   (ii) Finish the proof (begun in lectures) of the 5-lemma.

7. If $X \subset \mathbb{R}^n$ is convex, show (*without using homotopy invariance!*) that $H_i(X) = 0$ for $i > 0$.

8. (i) Compute the homology groups of the closed orientable surface Σ_g of genus g.
   (ii) Compute $H_*(\Sigma_2, A)$ where A is a simple closed curve which: (a) separates Σ_2 into two genus one pieces with one boundary component each; (b) is a non-separating simple closed curve cutting along which gives a genus one surface with two holes, and (c) bounds an embedded disc.

9. Using Mayer–Vietoris, compute the cohomology groups of complex projective space $\mathbb{CP}^k$. For each n, construct a closed connected four-dimensional manifold X_n with $H^1(X_n) = 0$ and $H^2(X_n) \cong \mathbb{Z}^n$. [*Hint: look up the "connect sum".*]

10. (i) Define relative cohomology $H^*(X, A)$ in such a way that there is a long exact sequence
$$\cdots \to H^i(X, A) \to H^i(X) \to H^i(A) \to H^{i+1}(X, A) \to \cdots$$
    (ii) Compute the relative cohomology $H^*(D, \{p_1, \ldots, p_k\})$ of the closed disc in $\mathbb{C}$ relative to k points.

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Part III Algebraic Topology // Example Sheet 2

1. Say a map $f : X \to Y$ between cell complexes is *cellular* if $f(X^n) \subset Y^n$ for every n. Show how to associate to such an f a chain map $f^{cell}_{\#} : C^{cell}_{\bullet}(X) \to C^{cell}_{\bullet}(Y)$ and show that the induced map $f^{cell}_* : H^{cell}_*(X) \to H^{cell}_*(Y)$ agrees with $f_* : H_*(X) \to H_*(Y)$ under a suitable identification of the homology groups.

2. Let $X = S^n \cup_f D^{n+1}$ be given by gluing an $(n+1)$-cell to S^n by a map $f : S^n \to S^n$ of degree $m > 1$. Show that the natural map $X \to X/S^n \cong S^{n+1}$ is trivial on homology $H_{*>0}$, but is non-trivial on cohomology $H^{*>0}$. What happens if we instead consider the inclusion map $S^n \hookrightarrow X$?

3. (i) Let X be a cell complex and $A \subset X$ be a subcomplex. Prove that the pair (X, A) is good.
   (ii) Let X be a cell complex and $K \subset X$ a compact subspace. Prove that K intersects only finitely many open cells in X. Hence show that any element of $H_i(X)$ lies in the image of $H_i(X^m) \to H_i(X)$ for some $m \gg 0$.

4. Show that for each $m \in \mathbb{Z}$ and any space X there are short exact sequences of chain complexes

$$0 \longrightarrow C^\bullet(X) \longrightarrow C^\bullet(X) \longrightarrow C^\bullet(X; \mathbb{Z}/m) \longrightarrow 0$$

$$0 \longrightarrow C^\bullet(X; \mathbb{Z}/m) \longrightarrow C^\bullet(X; \mathbb{Z}/m^2) \longrightarrow C^\bullet(X; \mathbb{Z}/m) \longrightarrow 0$$

and hence describe "Bockstein operations"

$$\tilde{\beta} : H^i(X; \mathbb{Z}/m) \longrightarrow H^{i+1}(X) \qquad \text{and} \qquad \beta : H^i(X; \mathbb{Z}/m) \longrightarrow H^{i+1}(X; \mathbb{Z}/m).$$

How are these two operations related? Show that $\beta(x \smile y) = \beta(x) \smile y + (-1)^{|x|} x \smile \beta(y)$. Compute the effect of β and $\tilde{\beta}$ for $m = 2$ and $X = \mathbb{RP}^n$, and hence compare $H^*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}/4)$ with $H^*(\mathbb{RP}^2; \mathbb{Z}/4) \otimes H^*(\mathbb{RP}^2; \mathbb{Z}/4)$.

5. Compute the cohomology ring of $S^1 \times S^1$. Hence compute the cohomology ring of the closed oriented surface Σ_g of genus g.

6. Recall that $H^2(\Sigma_g) \cong \mathbb{Z}$ for every $g \geq 0$, and define the degree of a map of oriented surfaces to be the induced map on H^2. For which g is there a map $\Sigma_g \to \Sigma_1$ of positive degree? For which g is there a map $\Sigma_1 \to \Sigma_g$ of positive degree?

7. A map $\pi : E \to B$ is called a *covering map* if there is an open cover $\{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha)$ is a disjoint union $\coprod V_{\alpha,\beta}$ with each $\pi|_{V_{\alpha,\beta}} : V_{\alpha,\beta} \to U_\alpha$ a homeomorphism.
   (i) If $\pi : E \to B$ is a covering map with finite fibres of cardinality N, show how to construct a map $\pi^! : H_*(B) \to H_*(E)$ such that $\pi_* \circ \pi^!$ is multiplication by N.
   (ii) In the same situation, show that $\chi(E) = N \cdot \chi(B)$.
   (iii) Show there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g = kh - k + 1$ for some $k \in \mathbb{N}$.

8. If $f : \mathbb{C}^n \to \mathbb{C}^n$ has components the elementary symmetric functions

$$(z_1, \ldots, z_n) \mapsto (\sigma_i(\underline{z})) \qquad \sigma_1 = \sum_j z_j \qquad \sigma_2 = \sum_{i<j} z_i z_j \quad \cdots \quad \sigma_n = \prod_j z_j$$

then prove that f extends to a map $\psi : S^{2n} \to S^{2n}$ of degree $n!$.

Hence construct a map $\phi : (\mathbb{CP}^1)^n \to \mathbb{CP}^n$ of degree $n!$, and compute the effect of the map $\phi^* : H^2(\mathbb{CP}^n) \to H^2((\mathbb{CP}^1)^n)$. Deduce that there is a $x \in H^2(\mathbb{CP}^n)$ such that x^n is a generator of the abelian group $H^{2n}(\mathbb{CP}^n)$, and hence that $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$ as a ring.

[*Hint: relate* $\mathbb{CP}^k$ *to the space of degree* k *homogeneous polynomials in two variables.*]

9. By considering a map to the wedge (one-point-union) of two copies of $\mathbb{CP}^2$, or otherwise, compute $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$ as a ring. Deduce that $\mathbb{CP}^2 \# \mathbb{CP}^2$ is not homotopy equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$, even though they have the same (co)homology groups additively.

10. Show that there is a *relative cup product*

$$\smile : H^i(X, A) \times H^j(X, B) \to H^{i+j}(X, A \cup B)$$

[*Hint: it may be helpful to consider a cochain complex $C^*_{A+B}(X)$ of cochains vanishing on simplices lying wholly in A or B, and use the Small Simplices Theorem.*] Using this, show that if X has a cover by n contractible (i.e. homotopy equivalent to a point) open sets, then the *cup-length*

$$\max\left\{k \mid \exists\, a_1, \ldots, a_k \in H^{*>0}(X),\ a_1 \smile \ldots \smile a_k \neq 0\right\}$$

is strictly smaller than n. What does this say about the ring $H^*(\Sigma X)$, where Σ is the suspension operation?

11. (i) Let $e : [0,1]^k \to S^n$ be a map which is a homeomorphism onto its image $D \subset S^n$. By considering the open sets

$$A = S^n \setminus e([0,1]^{k-1} \times [0,1/2]) \qquad B = S^n \setminus e([0,1]^{k-1} \times [1/2,1])$$

in S^n, show by induction on k that $\widetilde{H}_i(S^n \setminus D) = 0$.

(ii) If $e : S^k \to S^n$ is a map which is a homeomorphism onto its image $S \subset S^n$, compute $\widetilde{H}_i(S^n \setminus S)$. Think about the consequence of this in the case $(n,k) = (2,1)$.

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Part III Algebraic Topology // Example Sheet 3

1. If X is a finite cell complex, by showing that $C_\bullet^{cell}(X)$ is (unnaturally) isomorphic to a direct sum of chain complexes of the form $0 \to B_n(X) \overset{A_n}{\to} Z_n(X) \to 0$, show that
$$H^n(X) \cong \frac{H_n(X)}{\mathrm{Tors}(H_n(X))} \oplus \mathrm{Tors}(H_{n-1}(X)),$$
where $\mathrm{Tors}(A) \leq A$ denotes the subgroup of elements of finite order.

2. Let $E \to X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. Let $F \subset E$ be a subbundle. Prove that the orthogonal complement bundle $F^\perp$ is locally trivial.

3. (i) Explain how to view an open Möbius band as a 1-dimensional real bundle over S^1. Show that it is a non-trivial bundle.

   (ii) Show that a 1-dimensional real bundle over S^n with $n > 1$ is trivial. Hence show that 1-dimensional real bundles over a finite cell complex X up to isomorphism are naturally in 1-1 correspondence with elements of $H^1(X; \mathbb{Z}/2)$. [*Hint: Think about an associated double cover.*]

4. Show that a complex vector bundle has a canonical orientation.

5. If $\pi : E \to X$ is a d-dimensional real vector bundle which is not necessarily R-orientable, show that we still have $H^i(E, E^\#; R) = 0$ for $i < d$. If X is path-connected show that restriction to the fibre at $x \in X$ still gives an injective map $H^d(E, E^\#; R) \to H^d(E_x, E_x^\#; R) \cong R$.

   Give an example to show that $H^{i+d}(E, E^\#; R)$ need not be isomorphic to $H^i(X; R)$ in general.

6. (i) Show that any map $f : \mathbb{RP}^n \to \mathbb{RP}^m$ induces a trivial map on reduced cohomology if $n > m$. What about if $n < m$?

   (ii) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$ although they have additively isomorphic (co)homology.

7. (i) If $f : S^n \to S^n$ is odd (i.e. $f(-x) = -f(x)$) show that it induces a map $\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. By considering the Gysin sequence show that f has odd degree.

   (ii) Show that any $g : S^n \to \mathbb{R}^n$ satisfies $g(x) = g(-x)$ for some $x \in S^n$.

8. (i) Let $L = \gamma_{1,n+1}^{\mathbb{C}} \to \mathbb{CP}^n$ be the canonical 1-dimensional complex bundle. By considering $\pi_1^* L \otimes_{\mathbb{C}} \pi_2^* L \to \mathbb{CP}^n \times \mathbb{CP}^n$, with the $\pi_i : \mathbb{CP}^n \times \mathbb{CP}^n \to \mathbb{CP}^n$ being projections to the factors, prove that the Euler class of $L \otimes_{\mathbb{C}} L$ is equal to twice the Euler class of L.

   (ii) Show that the unit circle bundle in $L \otimes_{\mathbb{C}} L$ is homeomorphic to $\mathbb{RP}^{2n+1}$. Hence, compute the cohomology of $\mathbb{RP}^{2n+1}$ from knowledge of the cohomology of $\mathbb{CP}^n$.

9. Let $V_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k$ be the subspace of k-tuples of orthonormal vectors in $\mathbb{C}^n$ (a *Stiefel manifold*). Show there is a vector bundle $E_k \to V_k(\mathbb{C}^n)$ with fibre over $(v_1, \ldots, v_k)$ given by the vector space $\mathrm{span}(v_1, \ldots, v_k) \leq \mathbb{C}^n$.

   Show that the forgetful map $(v_1, \ldots, v_k) \mapsto (v_1, \ldots, v_{k-1}) : V_k(\mathbb{C}^n) \to V_{k-1}(\mathbb{C}^n)$ exhibits $V_k(\mathbb{C}^n)$ as the sphere bundle of a certain vector bundle $F \to V_{k-1}(\mathbb{C}^n)$. Hence compute $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$ as a ring.

   Deduce that the unitary group $U(n)$ has the same cohomology ring as $S^1 \times S^3 \times S^5 \times \cdots \times S^{2n-1}$, and hence that
$$\sum_{j \geq 0} \mathrm{rk}\, H^j(U(n); \mathbb{Z})\, t^j \;=\; \prod_{i=1}^{n} (1 + t^{2i-1}).$$

10. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let X be a compact Hausdorff space, and $Gr_k = Gr_k(\mathbb{F}^\infty) = \bigcup_n Gr_k(\mathbb{F}^n)$ be the infinite Grassmannian. The bundles $\gamma^{\mathbb{F}}_{k,n} \to Gr_k(\mathbb{R}^n)$ assemble to a bundle $\gamma^{\mathbb{F}}_k \to Gr_k$. To a map $f : X \to Gr_k$ we associate the pullback $f^*\gamma^{\mathbb{R}}_k$. Fix the standard inner product on $\mathbb{F}^\infty$ throughout.

(i) Suppose $f_0, f_1 : X \to Gr_k$ are maps with image in $Gr_k(\mathbb{F}^N)$ for some N. Let $U \subset Gr_k(\mathbb{F}^N) \times Gr_k(\mathbb{F}^N)$ be the following open neighbourhood of the diagonal:

$$U = \{(v_1, v_2) \mid v_1 \cap v_2^\perp = \{0\}\}.$$

Show that if $f_0(x)$ and $f_1(x)$ belong to U for every $x \in X$ then $f_0^*\gamma^{\mathbb{F}}_k \cong f_1^*\gamma^{\mathbb{F}}_k$.

(ii) By splitting the homotopy into many small intervals, deduce that if $f_0, f_1 : X \to Gr_k$ are homotopic then $f_0^*\gamma^{\mathbb{F}}_k$ and $f_1^*\gamma^{\mathbb{F}}_k$ are isomorphic.

(iii) Let $i_j : V_j \hookrightarrow \mathbb{F}^N$ be the inclusion of k-dimensional subspaces V_j, for $j = 0, 1$, and let $\alpha : V_0 \to V_1$ be a linear isomorphism. Show that

$$\gamma : t \longmapsto (t \cdot (i_0 \oplus \{0\}^n) + (1-t) \cdot (\{0\}^n \oplus i_1 \circ \alpha))(V_0)$$

is a continuous path from $V_0 \oplus \{0\}$ to $\{0\} \oplus V_1$ in $Gr_k(\mathbb{F}^{2N})$.

(iv) Let $f_0, f_1 : X \to Gr_k$ have image in $Gr_k(\mathbb{F}^N)$ and $f_0^*\gamma^{\mathbb{F}}_k \cong f_1^*\gamma^{\mathbb{F}}_k$. Let $T : \mathbb{F}^N \oplus \mathbb{F}^N \to \mathbb{F}^N \oplus \mathbb{F}^N$ be the map $(\xi, \eta) \mapsto (-\eta, \xi)$. Show that f_0 and $T \circ f_1$ are homotopic as maps from X to $Gr_k(\mathbb{F}^{2N})$, and deduce that $f_0 \simeq f_1 : X \to Gr_k$.

Conclude that the set of isomorphism classes $\mathrm{Vect}_k(X)$ of k-dimensional vector bundles over X is in bijection with the set $[X, Gr_k]$ of homotopy classes of maps.

11. (i) Show that $Gr_k(\mathbb{C}^n)$ is a smooth manifold of dimension $2k(n-k)$. Show that the map $j : Gr_{k-1}(\mathbb{C}^n) \to Gr_k(\mathbb{C}^{n+1})$, which adds on the last coordinate direction to a $(k-1)$-dimensional subspace of $\mathbb{C}^n$, is the inclusion of a submanifold. Show that the complement U of the image of j is homotopy equivalent to the subspace $Gr_k(\mathbb{C}^n) \subset Gr_k(\mathbb{C}^{n+1})$, and hence deduce that $H^i(Gr_k(\mathbb{C}^{n+1}), Gr_k(\mathbb{C}^n); \mathbb{Z}) = 0$ for $i < 2(n+1-k)$. [*Hint: Tubular neighbourhood theorem.*]

(ii) For the canonical bundle $\gamma^{\mathbb{C}}_{n,k} \to Gr_k(\mathbb{C}^n)$, show that $S(\gamma^{\mathbb{C}}_{k,n}) \cong S((\gamma^{\mathbb{C}}_{k-1,n})^\perp)$, and hence deduce that there is an exact sequence

$$\cdots \longrightarrow H^{i-2k}(Gr_k(\mathbb{C}^n)) \overset{-\cdot e(\gamma^{\mathbb{C}}_k(\mathbb{C}^n))}{\longrightarrow} H^i(Gr_{k,n}) \longrightarrow H^i(Gr_{k-1}(\mathbb{C}^n)) \longrightarrow H^{i-2k+1}(Gr_k(\mathbb{C}^n)) \longrightarrow \cdots$$

defined for $i \leq 2(n-k)$. Hence show by induction on k that the infinite complex Grassmannian Gr_k has cohomology ring $H^*(Gr_k; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_k]$ for certain classes c_i of degree $2i$ (the *Chern classes*).

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Part III Algebraic Topology // Example Sheet 4

1. If $\{C_\bullet(a), \rho_{ab}\}_{a\in I}$ is a direct system of chain complexes, show that $H_k(\varinjlim C_\bullet(a)) = \varinjlim H_k(C_\bullet(a))$. Deduce that a direct limit of exact sequences is exact.

2. (a) Which of the following are $\mathbb{Z}$-orientable? (i) $\mathbb{RP}^3$ (ii) $\mathbb{RP}^2 \times \mathbb{CP}^2$ (iii) $K\#T^2$, where K is the Klein bottle (# denotes connect sum).

   (b) Prove that any manifold has a $\mathbb{Z}$-orientable double cover.

3. If M is a connected compact d-manifold and $x \in M$, show that $H_d(M \setminus x; \mathbb{F}_2) = 0$.

   If $H_d(M;\mathbb{Z}) \cong \mathbb{Z}$, deduce that the restriction map $\mathrm{res}_x : \mathbb{Z} \cong H_d(M;\mathbb{Z}) \to H_d(M|x;\mathbb{Z}) \cong \mathbb{Z}$ is injective, and that the index of its image is independent of x. [*Hint: Show it is locally constant as a function of x.*] Hence show that M is $\mathbb{Z}$-orientable.

4. (i) Let M be a compact connected $\mathbb{Z}$-oriented d-manifold. Show that there is a degree one map $M \to S^d$.

   (ii) If M and N are compact connected $\mathbb{Z}$-oriented manifolds of the same dimension and $f : M \to N$ is a map of non-zero degree, is $f^* : H^*(N;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ necessarily injective?

   (iii) Prove that if a finite group G acts freely on S^n then some G-orbit is not contained in any open hemisphere. [*Hint: Construct a map $S^n/G \to S^n$.*]

5. (a) If M is a smooth manifold, show that it is equivalent to give an R-orientation of the manifold M and an R-orientation of the vector bundle TM.

   (b) Let V be a real n-dimensional vector space. Show that a $\mathbb{Z}$-orientation of V, meaning a choice of generator of $H^n(V, V - \{0\}) \cong \mathbb{Z}$, is equivalent to an orientation in the sense of linear algebra, i.e. a choice of ordered basis, where bases differing by a positive determinant matrix are equivalent.

   (c) If M is R-oriented and $Y \subset M$ is a compact submanifold, show an R-orientation of Y determines an R-co-orientation of Y (i.e. an R-orientation of its normal bundle).

   (d) If M is R-oriented and $Y, Z \subset M$ are compact R-oriented submanifolds which meet transversely, show that an ordering of Y and Z defines a R-co-orientation of $Y \cap Z$.

6. Show that the only non-trivial cup-products in $(S^2 \times S^8)\#(S^4 \times S^6)$ are those forced by Poincaré duality. Give an example of a space in which that conclusion would not be true.

7. Let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be a map of degree 8. What can you say about n ?

8. (a) Show that there is no map from $\mathbb{CP}^2$ to itself of degree -1.

   (b) Show that there is no map from $\mathbb{CP}^2 \times \mathbb{CP}^2$ to itself of degree -1.

9. (a) Suppose $Y \subset X$ is a smooth compact submanifold of a smooth compact manifold. Using the tubular neighbourhood theorem, prove $H_c^*(X \backslash Y) \cong H^*(X, Y)$.

   (b) Suppose $M \subset S^d$ is a compact $(d-1)$-dimensional smooth submanifold. Show that the complement $S^d \backslash M$ has one more path component than M does.

   (c) Suppose $M \subset \mathbb{R}^d$ is a compact $(d-1)$-dimensional smooth submanifold. Show that $\mathbb{R}^d \setminus M$ consists of a bounded and an unbounded region, and hence that the 1-dimensional normal bundle of $M \subset \mathbb{R}^d$ is trivial. Describe the degree of the map $\nu : M \to S^d$ which assigns to each point its unit outward-pointing normal vector. [*Hint: relate the degree of ν to a vector field on M.*]

10. (i) Show by induction on the dimension that a non-degenerate skew-symmetric bilinear form over $\mathbb{R}$ is equivalent to a direct sum of copies of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence show that any oriented closed 6-manifold M has $\dim_{\mathbb{Q}} H_3(M;\mathbb{Q})$ even.

    (ii) Let V be a vector space with a non-degenerate skew form as above. If $W \subset V$ is *isotropic*, meaning $\langle \cdot, \cdot \rangle|_{W \times W} \equiv 0$, show that $\dim(W) \leq \frac{\dim(V)}{2}$. What does this say about the cohomology classes defined by a collection of pairwise disjoint 3-dimensional submanifolds of a closed oriented six-manifold?

11. Consider the manifold $S^m \times \mathbb{CP}^1$ with the free involution τ defined by $\tau(x, [z_0, z_1]) := (-x, [\bar{z}_0, \bar{z}_1])$. Let $P(m)$ be the quotient space under this involution. Compute the groups $H^*(P(m);\mathbb{Z})$ and the ring $H^*(P(m);\mathbb{F}_2)$. [*Hint: find a cell structure to compute the cohomology groups, and use the intersection product to compute the cohomology ring.*]

12. Let M be a compact $\mathbb{Z}$-oriented smooth d-manifold.

    (a) If $f : M \to M$ be an orientation-preserving smooth map such that $f^p = \mathrm{Id}_M$, and the fixed-points of f form a discrete set, show that

$$\#\{\text{fixed points of } f\} = \sum_{k=0}^{d} (-1)^k \operatorname{Tr}(f^* : H^k(M;\mathbb{Q}) \to H^k(M;\mathbb{Q}))$$

    and if p is prime show that $\#\{\text{fixed points of } f\} \equiv \chi(M) \mod p$. [*Hint: Rational canonical form.*]

    (b) If the circle group S^1 acts smoothly on M with discrete fixed set M^{S^1}, show that $\#M^{S^1} = \chi(M)$.

13. Let $n > 1$. For a continuous map $\phi : S^{2n-1} \to S^n$, let Y_ϕ be the space obtained by attaching a $(2n)$-cell to S^n via ϕ. Compute $H^*(Y_\phi)$. Fixing $\alpha_i \in H^i(Y_\phi)$ to be generators for $i \in \{n, 2n\}$, define $h(\phi)$ by $\alpha_n^2 = h(\phi)\alpha_{2n}$.

    (i) If ϕ is homotopic to a constant map, then show that $h(\phi) = 0$.

    (ii) Let n be even. Fix a base-point $e \in S^n$. By considering the quotient $(S^n \times S^n)/\sim$ for $\sim$ the equivalence relation $(x, e) \sim (e, x)\ \forall x$, show that there is a map $\phi : S^{2n-1} \to S^n$ with $h(\phi) = \pm 2$. Hence show that there are infinitely-many non-homotopic maps from S^{2n-1} to S^n.