Part III — Algebraic Topology
Definitions

Based on lectures by O. Randal-Williams
Notes taken by Dexter Chua

Michaelmas 2016

Algebraic Topology assigns algebraic invariants to topological spaces; it permeates modern pure mathematics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular homology and cohomology, vector bundles and the Thom Isomorphism theorem, and the cohomology of manifolds up to Poincaré duality. Time permitting, there will also be some discussion of characteristic classes and cobordism, and conceivably some homotopy theory.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. The course will not assume any knowledge of Algebraic Topology, but will go quite fast in order to reach more interesting material, so some previous exposure to simplicial homology or the fundamental group would be helpful. The Part III Differential Geometry course will also contain useful, relevant material.

Hatcher's book is especially recommended for the course, but there are many other suitable texts.

Contents

1 Homotopy

Definition (Homotopy)**.** Let X, Y be topological spaces. A *homotopy* between $f_0, f_1 : X \to Y$ is a map $F : [0,1] \times X \to Y$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$. If such an F exists, we say f_0 is *homotopic* to f_1, and write $f_0 \simeq f_1$.

This $\simeq$ defines an equivalence relation on the set of maps from X to Y.

Definition (Homotopy equivalence)**.** A map $f : X \to Y$ is a *homotopy equivalence* if there is some $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We call g a *homotopy inverse* to f.

2 Singular (co)homology

2.1 Chain complexes

Definition (Chain complex)**.** A *chain complex* is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

such that

$$d_i \circ d_{i+1} = 0$$

for all i.

Definition (Cochain complex)**.** A *cochain complex* is a sequence of abelian groups and homomorphisms

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow \cdots$$

such that

$$d^{i+1} \circ d^i = 0$$

for all i.

Definition (Differentials)**.** The maps d^i and d_i are known as *differentials*.

Definition (Homology)**.** The *homology* of a chain complex $C_\bullet$ is

$$H_i(C_\bullet) = \frac{\ker(d_i : C_i \to C_{i-1})}{\operatorname{im}(d_{i+1} : C_{i+1} \to C_i)}.$$

An element of $H_i(C_\bullet)$ is known as a *homology class*.

Definition (Cohomology)**.** The *cohomology* of a cochain complex $C^\bullet$ is

$$H^i(C^\bullet) = \frac{\ker(d^i : C^i \to C^{i+1})}{\operatorname{im}(d^{i-1} : C^{i-1} \to C^i)}.$$

An element of $H^i(C^\bullet)$ is known as a *cohomology class*.

Definition (Cycles and cocycles)**.** The elements of $\ker d_i$ are the *cycles*, and the elements of $\ker d^i$ are the *cocycles*.

Definition (Boundaries and coboundaries)**.** The elements of $\operatorname{im} d_i$ are the *boundaries*, and the elements of $\operatorname{im} d^i$ are the *coboundaries*.

Definition (Chain map)**.** If $(C_\bullet, d_\bullet^C)$ and $(D_\bullet, d_\bullet^D)$ are chain complexes, then a *chain map* $C_\bullet \to D_\bullet$ is a collection of homomorphisms $f_n : C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$. In other words, the following diagram has to commute for all n:

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow{\scriptstyle d_n^C} & & \downarrow{\scriptstyle d_n^D} \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

2.2 Singular (co)homology

Definition (Standard n-simplex)**.** The standard n-simplex is

$$\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1\right\}.$$

Definition (Face of standard simplex)**.** The *ith face* of Δ^n is

$$\Delta^n_i = \{(t_0, \cdots, t_n) \in \Delta^n : t_i = 0\}.$$

Definition (Singular n-simplex)**.** Let X be a space. Then a *singular n-simplex* in X is a map $\sigma : \Delta^n \to X$.

Definition (Singular chain complex)**.** We let $C_n(X)$ be the free abelian group on the set of singular n-simplices in X. More explicitly, we have

$$C_n(X) = \left\{\sum n_\sigma \sigma : \sigma : \Delta^n \to X, n_\sigma \in \mathbb{Z}, \text{only finitely many } n_\sigma \text{ non-zero}\right\}.$$

We define $d_n : C_n(X) \to C_{n-1}(X)$ by

$$\sigma \mapsto \sum_{i=0}^n (-1)^i \sigma \circ \delta_i,$$

and then extending linearly.

Definition (Singular homology)**.** The *singular homology* of a space X is the homology of the chain complex $C_\bullet(X)$:

$$H_i(X) = H_i(C_\bullet(X), d_\bullet) = \frac{\ker(d_i : C_i(X) \to C_{i-1}(X))}{\operatorname{im}(d_{i+1} : C_{i+1}(X) \to C_i(X))}.$$

Definition (Singular cohomology)**.** We define the dual cochain complex by

$$C^n(X) = \operatorname{Hom}(C_n(X), \mathbb{Z}).$$

We let

$$d^n : C^n(X) \to C^{n+1}(X)$$

be the adjoint to d_{n+1}, i.e.

$$(\varphi : C_n(X) \to \mathbb{Z}) \mapsto (\varphi \circ d_{n+1} : C_{n+1}(X) \to \mathbb{Z}).$$

We observe that

$$0 \longrightarrow C^0(X) \xrightarrow{d^0} C^1(X) \longrightarrow \cdots$$

is indeed a cochain complex, since

$$d^{n+1}(d^n(\varphi)) = \varphi \circ d_{n+1} \circ d_{n+2} = \varphi \circ 0 = 0.$$

The *singular cohomology* of X is the cohomology of this cochain complex, i.e.

$$H^i(X) = H^i(C^\bullet(X), d^\bullet) = \frac{\ker(d^i : C^i(X) \to C^{i+1}(X))}{\operatorname{im}(d^{i-1} : C^{i-1}(X) \to C^i(X))}.$$

3 Four major tools of (co)homology

3.1 Homotopy invariance

3.2 Mayer-Vietoris

Definition (Exact sequence)**.** We say a pair of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact at B* if $\operatorname{im}(f) = \ker(g)$.

We say a sequence

$$\cdots \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

is *exact* if it is exact at each X_n.

Definition (Short exact sequence)**.** A *short exact sequence* is a sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\ .$$

3.3 Relative homology

Definition (Relative homology)**.** Let $A \subseteq X$. We write $i : A \to X$ for the inclusion map. Then the map $i_n : C_n(A) \to C_n(X)$ is injective as well, and we write

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}.$$

The differential $d_n : C_n(X) \to C_{n-1}(X)$ restricts to a map $C_n(A) \to C_{n-1}(A)$, and thus gives a well-defined differential $d_n : C_n(X, A) \to C_{n-1}(X, A)$, sending $[c] \mapsto [d_n(c)]$. The *relative homology* is given by

$$H_n(X, A) = H_n(C_\bullet(X, A)).$$

Definition (Map of pairs)**.** Let (X, A) and (Y, B) be topological spaces with $A \subseteq X$ and $B \subseteq Y$. A *map of pairs* is a map $f : X \to Y$ such that $f(A) \subseteq B$.

3.4 Excision theorem

3.5 Applications

Definition (Degree of a map)**.** Let $f : S^n \to S^n$ be a map. The *degree* $\deg(f)$ is the unique integer such that under the identification $H_n(S^n) \cong \mathbb{Z}$, the map f_* is given by multiplication by $\deg(f)$.

Definition (Local degree)**.** Let $f : S^d \to S^d$ be a map, and $x \in S^d$. Then f induces a map

$$f_* : H_d(S^d, S^d \setminus \{x\}) \to H_d(S^d, S^d \setminus \{f(x)\}).$$

We identify $H_d(S^d, S^d \setminus \{x\}) \cong H_d(S^d) \cong \mathbb{Z}$ for any x via the inclusion $H_d(S^d) \to H_d(S^d, S^d \setminus \{x\})$, which is an isomorphism by the long exact sequence. Then f_* is given by multiplication by a constant $\deg(f)_x$, called the *local degree* of f at x.

3.6 Repaying the technical debt

Definition (Chain homotopy). A *chain homotopy* between chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ is a collection of homomorphisms $F_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d^D_{n+1} \circ F_n + F_{n-1} \circ d^C_n : C_n \to D_n$$

for all n.

Notation. From now on, we will just write d for d^C_n.

For $f : X \to Y$, we will write $f_\# : C_n(X) \to C_n(Y)$ for the map $\sigma \mapsto f \circ \sigma$, i.e. what we used to call f_n.

Definition ($C^{\mathcal{U}}_n(X)$ and $H^{\mathcal{U}}_n(X)$). We let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of subspaces of X such that their interiors cover X, i.e.

$$\bigcup_{\alpha \in I} \mathring{U}_\alpha = X.$$

Let $C^{\mathcal{U}}_n(X) \subseteq C_n(X)$ be the subgroup generated by those singular n-simplices $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq U_\alpha$ for some α. It is clear that if σ lies in U_α, then so do its faces. So $C^{\mathcal{U}}_n(X)$ is a sub-chain complex of $C_\bullet(X)$.

We write $H^{\mathcal{U}}_n(X) = H_n(C^{\mathcal{U}}_\bullet(X))$.

4 Reduced homology

Definition (Reduced homology)**.** Let X be a space, and $x_0 \in X$ a basepoint. We define the *reduced homology* to be $\tilde{H}_*(X) = H_*(X, \{x_0\})$.

Definition (Good pair)**.** We say a pair (X, A) is *good* if there is an open set U containing $\bar{A}$ such that the inclusion $A \hookrightarrow U$ is a deformation retract, i.e. there exists a homotopy $H : [0, 1] \times U \to U$ such that

$$\begin{aligned} H(0, x) &= x \\ H(1, x) &\in A \\ H(t, a) &= a \text{ for all } a \in A, t \in [0, 1]. \end{aligned}$$

5 Cell complexes

Definition (Cell complex)**.** A *cell complex* is any space built out of the following procedure:

(i) Start with a discrete space X^0. The set of points in X^0 are called I_0.

(ii) If X^{n-1} has been constructed, then we may choose a family of maps $\{\varphi_\alpha : S^{n-1} \to X^{n-1}\}_{\alpha \in I_n}$, and set

$$X^n = \left(X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} D^n_\alpha\right)\right) / \{x \in \partial D^n_\alpha \sim \varphi_\alpha(x) \in X^{n-1}\}.$$

We call X^n the *n-skeleton* of X We call the image of $D^n_\alpha \setminus \partial D^n_\alpha$ in X^n the *open cell* e_α.

(iii) Finally, we define

$$X = \bigcup_{n \geq 0} X^n$$

with the *weak topology*, namely that $A \subseteq X$ is open if $A \cap X^n$ is open in X^n for all n.

Definition (Finite-dimensional cell complex)**.** If $X = X^n$ for some n, we say X is *finite-dimensional.*

Definition (Finite cell complex)**.** If X is finite-dimensional and I_n are all finite, then we say X is *finite.*

Definition (Subcomplex)**.** A subcomplex A of X is a simplex obtained by using a subset $I_n' \subseteq I_n$.

Definition (Cellular cohomology)**.** We define *cellular cohomology* by

$$C^n_{\mathrm{cell}}(X) = H^n(X^n, X^{n-1})$$

and let d^n_{cell} be the composition

$$H^n(X^n, X^{n-1}) \xrightarrow{q^*} H^n(X^n) \xrightarrow{\partial} H^{n+1}(X^{n+1}, X^n).$$

This defines a cochain complex $C^\bullet_{\mathrm{cell}}(X)$ with cohomology $H^*_{\mathrm{cell}}(X)$, and we have

$$H^*_{\mathrm{cell}}(X) \cong H^*(X).$$

One can directly check that

$$C^\bullet_{\mathrm{cell}}(X) \cong \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(X), \mathbb{Z}).$$

6 (Co)homology with coefficients

Definition ((Co)homology with coefficients)**.** Let A be an abelian group, and X be a topological space. We let

$$C_\bullet(X; A) = C_\bullet(X) \otimes A$$

with differentials $d \otimes \mathrm{id}_A$. In other words $C_\bullet(X; A)$ is the abelian group obtained by taking the direct sum of many copies of A, one for each singular simplex.

We let

$$H_n(X; A) = H_n(C_\bullet(X; A), d \otimes \mathrm{id}_A).$$

We can also define

$$H_n^{\mathrm{cell}}(X; A) = H_n(C_\bullet^{\mathrm{cell}}(X) \otimes A),$$

and the same proof shows that $H_n^{\mathrm{cell}}(X; A) = H_n(X; A)$.

Similarly, we let

$$C^\bullet(X; A) = \mathrm{Hom}(C_\bullet(X), A),$$

with the usual (dual) differential. We again set

$$H^n(X; A) = H^n(C^\bullet(X; A)).$$

We similarly define cellular cohomology.

If A is in fact a commutative ring, then these are in fact R-modules.

7 Euler characteristic

Definition (Euler characteristic)**.** Let X be a cell complex. We let

$$\chi(X) = \sum_n (-1)^n \text{ number of } n\text{-cells of } X \in \mathbb{Z}.$$

8 Cup product

Definition (Cup product)**.** Let R be a commutative ring, and $\phi \in C^k(X; R)$, $\psi \in C^\ell(X; R)$. Then $\phi \smile \psi \in C^{k+\ell}(X; R)$ is given by

$$(\phi \smile \psi)(\sigma : \Delta^{k+\ell} \to X) = \phi(\sigma|_{[v_0, \ldots, v_k]}) \cdot \psi(\sigma|_{[v_k, \ldots, v_{k+\ell}]}).$$

Here the multiplication is multiplication in R, and $v_0, \cdots, v_\ell$ are the vertices of $\Delta^{k+\ell} \subseteq \mathbb{R}^{k+\ell+1}$, and the restriction is given by

$$\sigma|_{[x_0, \ldots, x_i]}(t_0, \ldots, t_i) = \sigma\left(\sum t_j x_j\right).$$

This is a bilinear map.

Notation. We write

$$H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R).$$

Definition (Cross product)**.** Let $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ be the projection maps. Then we have a *cross product*

$$\begin{aligned} \times : H^k(X; R) \otimes_R H^\ell(Y; R) &\longrightarrow H^{k+\ell}(X \times Y; R) \\ a \otimes b &\longmapsto (\pi_X^* a) \smile (\pi_Y^* b) \end{aligned}.$$

9 Künneth theorem and universal coefficients theorem

10 Vector bundles

10.1 Vector bundles

Definition (Vector bundle)**.** Let X be a space. A (real) *vector bundle* of dimension d over X is a map $\pi : E \to X$, with a (real) vector space structure on each *fiber* $E_x = \pi^{-1}(\{x\})$, subject to the local triviality condition: for each $x \in X$, there is a neighbourhood U of x and a homeomorphism $\varphi : E|_U = \pi^{-1}(U) \to U \times \mathbb{R}^d$ such that the following diagram commutes

$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & U \times \mathbb{R}^d \\ & \searrow^{\pi} \quad \swarrow_{\pi_1} & \\ & U & \end{array},$$

and for each $y \in U$, the restriction $\varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^d$ is a *linear* isomorphism for each $y \in U$. This maps is known as a *local trivialization*.

Definition (Section)**.** A *section* of a vector bundle $\pi : E \to X$ is a map $s : X \to E$ such that $\pi \circ s = \mathrm{id}$. In other words, $s(x) \in E_x$ for each x.

Definition (Zero section)**.** The *zero section* of a vector bundle is $s_0 : X \to E$ given by $s_0(x) = 0 \in E_x$.

Definition (Pullback of vector bundles)**.** Let $\pi : E \to X$ be a vector bundle, and $f : Y \to X$ a map. We define the *pullback*

$$f^*E = \{(y, e) \in Y \times E : f(y) = \pi(e)\}.$$

This has a map $f^*\pi : f^*E \to Y$ given by projecting to the first coordinate. The vector space structure on each fiber is given by the identification $(f^*E)_y = E_{f(y)}$. It is a little exercise in topology to show that the local trivializations of $\pi : E \to X$ induce local trivializations of $f^*\pi : f^*E \to Y$.

Definition (Whitney sum of vector bundles)**.** Let $\pi : E \to F$ and $\rho : F \to X$ be vector bundles. The *Whitney sum* is given by

$$E \oplus F = \{(e, f) \in E \times F : \pi(e) = \rho(f)\}.$$

This has a natural map $\pi \oplus \rho : E \oplus F \to X$ given by $(\pi \oplus \rho)(e, f) = \pi(e) = \rho(f)$. This is again a vector bundle, with $(E \oplus F)_x = E_x \oplus F_x$ and again local trivializations of E and F induce one for $E \oplus F$.

Definition (Vector subbundle)**.** Let $\pi : E \to X$ be a vector bundle, and $F \subseteq E$ a subspace such that for each $x \in X$ there is a local trivialization (U, φ)

$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & U \times \mathbb{R}^d \\ & \searrow^{\pi} \quad \swarrow_{\pi_1} & \\ & U & \end{array},$$

such that φ takes $F|_U$ to $U \times \mathbb{R}^k$, where $\mathbb{R}^k \subseteq \mathbb{R}^d$. Then we say F is a *vector sub-bundle*.

Definition (Quotient bundle). Let F be a sub-bundle of E. Then E/F, given by the fiberwise quotient, is a vector bundle and is given by the *quotient bundle*.

Definition (Partition of unity). Let X be a compact Hausdorff space, and $\{U_\alpha\}_{\alpha \in I}$ be an open cover. A *partition of unity subordinate to* $\{U_\alpha\}$ is a collection of functions $\lambda_\alpha : X \to [0, \infty)$ such that

(i) $\operatorname{supp}(\lambda_\alpha) = \overline{\{x \in X : \lambda_\alpha(x) > 0\}} \subseteq U_\alpha$.

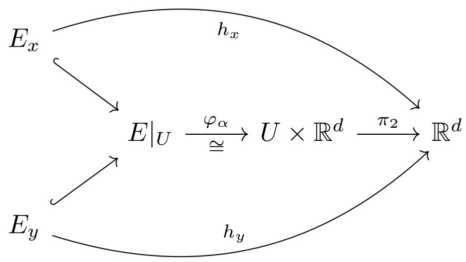
(ii) Each $x \in X$ lies in finitely many of these $\operatorname{supp}(\lambda_\alpha)$.

(iii) For each x, we have
$$\sum_{\alpha \in I} \lambda_\alpha(x) = 1.$$

10.2 Vector bundle orientations

Definition (R-orientation). A *local R-orientation* of E at $x \in X$ is a choice of R-module generator $\varepsilon_x \in H^d(E_x, E_x^\#; R)$.

An *R-orientation* is a choice of local R-orientation $\{\varepsilon_x\}_{x \in X}$ which are compatible in the following way: if $U \subseteq X$ is open on which E is trivial, and $x, y \in U$, then under the homeomorphisms (and in fact linear isomorphisms):



the map
$$h_y^* \circ (h_x^{-1})^* : H^d(E_x, E_x^\#; R) \to H^d(E_y, E_y^\#; R)$$
sends ε_x to ε_y. Note that this definition does not depend on the choice of φ_U, because we used it twice, and they cancel out.

10.3 The Thom isomorphism theorem

Definition (Euler class). Let $\pi : E \to X$ be a vector bundle. We define the *Euler class* $e(E) \in H^d(X; R)$ by the image of u_E under the composition
$$H^d(E, E^\#; R) \longrightarrow H^d(E; R) \xrightarrow{s_0^*} H^d(X; R)\ .$$

10.4 Gysin sequence

Definition (Sphere bundle). Let $\pi : E \to X$ be a vector bundle, and let $\langle\,\cdot\,,\,\cdot\,\rangle : E \otimes E \to \mathbb{R}$ be an inner product, and let
$$S(E) = \{v \in E; \langle v, v\rangle = 1\} \subseteq E.$$
This is the *sphere bundle* associated to E.

11 Manifolds and Poincaré duality

11.1 Compactly supported cohomology

Definition (Support of cochain)**.** Let $\varphi \in C^n(X)$ be a cochain. We say φ has *support* in $S \subseteq X$ if whenever $\sigma : \Delta^n \hookrightarrow X \setminus S \subseteq X$, then $\varphi(\sigma) = 0$. In this case, $\mathrm{d}\varphi$ also has support in S.

Definition (Compactly-supported cochain)**.** Let $C_c^\bullet(X) \subseteq C^\bullet(X)$ be the sub-chain complex consisting of these φ which has support in *some* compact $K \subseteq X$.

Definition (Compactly-supported cohomology)**.** The *compactly supported cohomology* of X is

$$H_c^*(X) = H^*(C_c^\bullet(X)).$$

Definition (Directed set)**.** A *directed set* is a partial order $(I, \leq)$ such that for all $i, j \in I$, there is some $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition (Direct limit)**.** Let I be a directed set. An *direct system* of abelian groups indexed by I is a collection of abelian groups G_i for each $i \in I$ and homomorphisms

$$\rho_{ij} : G_i \to G_j$$

for all $i, j \in I$ such that $i \leq j$, such that

$$\rho_{ii} = \mathrm{id}_{G_i}$$

and

$$\rho_{ik} = \rho_{jk} \circ \rho_{ij}$$

whenever $i \leq j \leq k$.

We define the *direct limit* on the system (G_i, ρ_{ij}) to be

$$\varinjlim_{i \in I} G_i = \left(\bigoplus_{i \in I} G_i\right) / \langle x - \rho_{ij}(x) : x \in G_i \rangle.$$

The underlying set of it is

$$\left(\coprod_{i \in I} G_i\right) / \{x \sim \rho_{ij}(x) : x \in G_i\}.$$

Definition (Proper map)**.** A map $f : X \to Y$ of spaces is *proper* if the preimage of a compact space is compact.

11.2 Orientation of manifolds

Definition (Local R-orientation of manifold)**.** Fr a d-manifold M, a local R-orientation of M at x is an R-module generator $\mu_x = H_d(M \mid x; R)$.

Definition (R-orientation)**.** An *R-orientation* of M is a collection $\{\mu_x\}_{x \in M}$ of local R-orientations such that if

$$\varphi : \mathbb{R}^d \to U \subseteq M$$

is a chart of M, and $p, q \in \mathbb{R}^d$, then the composition of isomorphisms

$$\begin{array}{ccccc} H_d(M \mid \varphi(p)) & \xrightarrow{\sim} & H_d(U \mid \varphi(p)) & \xleftarrow[\varphi_*]{\sim} & H_d(\mathbb{R}^d \mid p) \\ & & & & \downarrow{\sim} \\ H_d(M \mid \varphi(q)) & \xrightarrow{\sim} & H_d(U \mid \varphi(q)) & \xleftarrow[\varphi_*]{\sim} & H_d(\mathbb{R}^d \mid q) \end{array}$$

sends μ_x to μ_y, where the vertical isomorphism is induced by a translation of $\mathbb{R}^d$.

Definition (Orientation-preserving homeomorphism)**.** For a homomorphism $f : U \to V$ with $U, V \in \mathbb{R}^d$ open, we say f is R-orientation-preserving if for each $x \in U$, and $y = f(x)$, the composition

$$\begin{array}{ccccc} H_d(\mathbb{R}^d \mid 0; R) & \xrightarrow{\text{translation}} & H_d(\mathbb{R}^d \mid x; R) & \xrightarrow{\text{excision}} & H_d(U \mid x; R) \\ & & & & \downarrow{f_*} \\ H_d(\mathbb{R}^d \mid 0; R) & \xrightarrow{\text{translation}} & H_d(\mathbb{R}^d \mid y; R) & \xrightarrow{\text{excision}} & H_d(V \mid y; R) \end{array}$$

is the identity $H_d(\mathbb{R}^d \mid 0; R) \to H_d(\mathbb{R}^d \mid 0; R)$.

Definition (Fundamental class)**.** The *fundamental class* of an R-oriented manifold is the unique class $[M]$ that restricts to μ_x for each $x \in M$.

11.3 Poincaré duality

Definition (Cap product)**.** The *cap product* is defined by

$$\begin{aligned} \cdot \frown \cdot : C_k(X; R) \times C^\ell(X; R) &\to C_{k-\ell}(X; R) \\ (\sigma, \varphi) &\mapsto \varphi(\sigma|_{[v_0, \ldots, v_\ell]})\sigma|_{[v_\ell, \ldots, v_k]}. \end{aligned}$$

11.4 Applications

Definition (Signature of manifold)**.** Let M be a $4k$-dimensional $\mathbb{Z}$-oriented manifold. Then the signature is the number of positive eigenvalues of

$$\langle \cdot , \cdot \rangle : H^{2k}(M; R) \otimes H^{2k}(M; \mathbb{R}) \to \mathbb{R}$$

minus the number of negative eigenvalues. We write this as $\operatorname{sgn}(M)$.

Definition (Degree of map)**.** If M, N are d-dimensional compact connected $\mathbb{Z}$-oriented manifolds, and $f : M \to N$, then

$$f_*([M]) \in H_d(N, \mathbb{Z}) \cong \mathbb{Z}[N].$$

So $f_*([M]) = k[N]$ for some k. This k is called the *degree* of f, written $\deg(f)$.

11.5 Intersection product

Definition (Transverse intersection)**.** We say two submanifolds $N, W \subseteq M$ *intersect transversely* if for all $x \in N \cap W$, we have

$$T_x N + T_x W = T_x M.$$

Definition (Intersection product)**.** The *intersection product* on the homology of a compact manifold is given by

$$\begin{aligned} H_{n-k}(M) \otimes H_{n-\ell}(M) &\longrightarrow H_{n-k-\ell}(M) \\ (a, b) &\longmapsto a \cdot b = D_M(D_M^{-1}(a) \smile D_M^{-1}(b)) \end{aligned}$$

11.6 The diagonal

11.7 Lefschetz fixed point theorem