# Part III — Advanced Probability
## Theorems

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The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

**Review of measure and integration:** sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
**Conditional expectation:** Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
**Martingales:** Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
**Stochastic processes in continuous time:** Kolmogorov's criterion, regularization of paths; martingales in continuous time.
**Weak convergence:** Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
**Sums of independent random variables:** Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
**Brownian motion:** Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
**Poisson random measures:** Construction and properties; integrals.
**Lévy processes:** Lévy-Khinchin theorem.

### Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

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# 0 Introduction

# 1 Some measure theory

## 1.1 Review of measure theory

**Theorem.** Let $(E, \mathcal{E}, \mu)$ be a measure space. Then there exists a unique function $\tilde{\mu} : m\mathcal{E}^+ \to [0, \infty]$ satisfying

- $\tilde{\mu}(\mathbf{1}_A) = \mu(A)$, where $\mathbf{1}_A$ is the indicator function of $A$.
- Linearity: $\tilde{\mu}(\alpha f + \beta g) = \alpha\tilde{\mu}(f) + \beta\tilde{\mu}(g)$ if $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $f, g \in m\mathcal{E}^+$.
- Monotone convergence: iff $f_1, f_2, \ldots \in m\mathcal{E}^+$ are such that $f_n \nearrow f \in m\mathcal{E}^+$ pointwise a.e. as $n \to \infty$, then
$$\lim_{n\to\infty} \tilde{\mu}(f_n) = \tilde{\mu}(f).$$

We call $\tilde{\mu}$ the *integral* with respect to $\mu$, and we will write it as $\mu$ from now on.

**Lemma** (Fatou's lemma)**.** Let $f_i \in m\mathcal{E}^+$. Then
$$\mu\left(\liminf_n f_n\right) \leq \liminf_n \mu(f_n).$$

**Theorem** (Dominated convergence theorem)**.** If $f_i \in m\mathcal{E}$ and $f_i \to f$ a.e., such that there exists $g \in L^1$ such that $|f_i| \leq g$ a.e. Then
$$\mu(f) = \lim \mu(f_n).$$

**Theorem.** If $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ are $\sigma$-finite measure spaces, then there exists a unique measure $\mu$ on $\mathcal{E}_1 \otimes \mathcal{E}_2)$ satisfying
$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$
for all $A_i \in \mathcal{E}_i$.

This is called the *product measure*.

**Theorem** (Fubini's/Tonelli's theorem)**.** If $f = f(x_1, x_2) \in m\mathcal{E}^+$ with $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, then the functions
$$x_1 \mapsto \int f(x_1, x_2)\mathrm{d}\mu_2(x_2) \in m\mathcal{E}_1^+$$
$$x_2 \mapsto \int f(x_1, x_2)\mathrm{d}\mu_1(x_1) \in m\mathcal{E}_2^+$$
and
$$\begin{aligned}\int_E f\;\mathrm{d}u &= \int_{E_1}\left(\int_{E_2} f(x_1, x_2)\;\mathrm{d}\mu_2(x_2)\right)\mathrm{d}\mu_1(x_1)\\ &= \int_{E_2}\left(\int_{E_1} f(x_1, x_2)\;\mathrm{d}\mu_1(x_1)\right)\mathrm{d}\mu_2(x_2)\end{aligned}$$

## 1.2 Conditional expectation

**Lemma.** The conditional expectation $Y = \mathbb{E}(X \mid \mathcal{G})$ satisfies the following properties:

- $Y$ is $\mathcal{G}$-measurable
- We have $Y \in L^1$, and
$$\mathbb{E}Y\mathbf{1}_A = \mathbb{E}X\mathbf{1}_A$$
for all $A \in \mathcal{G}$.

**Theorem** (Existence and uniqueness of conditional expectation)**.** Let $X \in L^1$, and $\mathcal{G} \subseteq \mathcal{F}$. Then there exists a random variable $Y$ such that

- $Y$ is $\mathcal{G}$-measurable
- $Y \in L^1$, and $\mathbb{E}X\mathbf{1}_A = \mathbb{E}Y\mathbf{1}_A$ for all $A \in \mathcal{G}$.

Moreover, if $Y'$ is another random variable satisfying these conditions, then $Y' = Y$ almost surely.

We call $Y$ a (version of) the conditional expectation given $\mathcal{G}$.

**Lemma.** If $Y$ is $\sigma(Z)$-measurable, then there exists $h : \mathbb{R} \to \mathbb{R}$ Borel-measurable such that $Y = h(Z)$. In particular,
$$\mathbb{E}(X \mid Z) = h(Z) \text{ a.s.}$$
for some $h : \mathbb{R} \to \mathbb{R}$.

**Proposition.**

(i) $\mathbb{E}(X \mid \mathcal{G}) = X$ iff $X$ is $\mathcal{G}$-measurable.

(ii) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}X$

(iii) If $X \geq 0$ a.s., then $\mathbb{E}(X \mid \mathcal{G}) \geq 0$

(iv) If $X$ and $\mathcal{G}$ are independent, then $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}[X]$

(v) If $\alpha, \beta \in \mathbb{R}$ and $X_1, X_2 \in L^1$, then
$$\mathbb{E}(\alpha X_1 + \beta X_2 \mid \mathcal{G}) = \alpha\mathbb{E}(X_1 \mid \mathcal{G}) + \beta\mathbb{E}(X_2 \mid \mathcal{G}).$$

(vi) Suppose $X_n \nearrow X$. Then
$$\mathbb{E}(X_n \mid \mathcal{G}) \nearrow \mathbb{E}(X \mid \mathcal{G}).$$

(vii) *Fatou's lemma*: If $X_n$ are non-negative measurable, then
$$\mathbb{E}\left(\liminf_{n\to\infty} X_n \mid \mathcal{G}\right) \leq \liminf_{n\to\infty} \mathbb{E}(X_n \mid \mathcal{G}).$$

(viii) *Dominated convergence theorem*: If $X_n \to X$ and $Y \in L^1$ such that $Y \geq |X_n|$ for all $n$, then
$$\mathbb{E}(X_n \mid \mathcal{G}) \to \mathbb{E}(X \mid \mathcal{G}).$$

(ix) *Jensen's inequality*: If $c : \mathbb{R} \to \mathbb{R}$ is convex, then
$$\mathbb{E}(c(X) \mid \mathcal{G}) \geq c(\mathbb{E}(X) \mid \mathcal{G}).$$

(x) *Tower property*: If $\mathcal{H} \subseteq \mathcal{G}$, then
$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H}).$$

(xi) For $p \geq 1$,
$$\|\mathbb{E}(X \mid \mathcal{G})\|_p \leq \|X\|_p.$$

(xii) If $Z$ is bounded and $\mathcal{G}$-measurable, then
$$\mathbb{E}(ZX \mid \mathcal{G}) = Z\mathbb{E}(X \mid \mathcal{G}).$$

(xiii) Let $X \in L^1$ and $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$. Assume that $\sigma(X, \mathcal{G})$ is independent of $\mathcal{H}$. Then
$$\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H})).$$

**Lemma.** If $X \in L^1$, then the family of random variables $Y_\mathcal{G} = \mathbb{E}(X \mid \mathcal{G})$ for all $\mathcal{G} \subseteq \mathcal{F}$ is uniformly integrable.

In other words, for all $\varepsilon > 0$, there exists $\lambda > 0$ such that
$$\mathbb{E}(Y_\mathcal{G} \mathbf{1}_{|Y_\mathcal{G} > \lambda|}) < \varepsilon$$
for all $\mathcal{G}$.

# 2 Martingales in discrete time

## 2.1 Filtrations and martingales

## 2.2 Stopping time and optimal stopping

**Proposition.**

(i) If $T, S, (T_n)_{n \geq 0}$ are all stopping times, then
$$T \vee S, T \wedge S, \sup_n T_n, \inf T_n, \limsup T_n, \liminf T_n$$
are all stopping times.

(ii) $\mathcal{F}_T$ is a $\sigma$-algebra

(iii) If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(iv) $X_T \mathbf{1}_{T<\infty}$ is $\mathcal{F}_T$-measurable.

(v) If $(X_n)$ is an adapted process, then so is $(X_n^T)_{n \geq 0}$ for any stopping time $T$.

(vi) If $(X_n)$ is an integrable process, then so is $(X_n^T)_{n \geq 0}$ for any stopping time $T$. □

**Theorem** (Optional stopping theorem)**.** Let $(X_n)_{n \geq 0}$ be a super-martingale and $S \leq T$ *bounded* stopping times. Then
$$\mathbb{E} X_T \leq \mathbb{E} X_S.$$

**Theorem.** The following are equivalent:

(i) $(X_n)_{n \geq 0}$ is a super-martingale.

(ii) For any bounded stopping times $T$ and any stopping time $S$,
$$\mathbb{E}(X_T \mid \mathcal{F}_S) \leq X_{S \wedge T}.$$

(iii) $(X_n^T)$ is a super-martingale for any stopping time $T$.

(iv) For bounded stopping times $S, T$ such that $S \leq T$, we have
$$\mathbb{E} X_T \leq \mathbb{E} X_S.$$

## 2.3 Martingale convergence theorems

**Theorem** (Almost sure martingale convergence theorem)**.** Suppose $(X_n)_{n \geq 0}$ is a super-martingale that is bounded in $L^1$, i.e. $\sup_n \mathbb{E}|X_n| < \infty$. Then there exists an $\mathcal{F}_\infty$-measurable $X_\infty \in L^1$ such that
$$X_n \to X_\infty \text{ a.s. as } n \to \infty.$$

**Lemma.** Let $(x_n)_{n \geq 0}$ be a sequence of numbers. Then $x_n$ converges in $\mathbb{R}$ if and only if

(i) $\liminf |x_n| < \infty$.

(ii) For all $a, b \in \mathbb{Q}$ with $a < b$, we have $U[a, b, (x_n)] < \infty$.

**Lemma** (Doob's upcrossing lemma)**.** If $X_n$ is a super-martingale, then

$$(b - a)\mathbb{E}(U_n[a, b(X_n)]) \leq \mathbb{E}(X_n - a)^-$$

**Lemma** (Maximal inequality)**.** Let $(X_n)$ be a sub-martingale that is non-negative, or a martingale. Define

$$X_n^* = \sup_{k \leq n} |X_k|, \quad X^* = \lim_{n \to \infty} X_n^*.$$

If $\lambda \geq 0$, then

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}[|X_n| \mathbf{1}_{X_n^* \geq \lambda}].$$

In particular, we have

$$\lambda \mathbb{P}(X_n^* \geq \lambda) \leq \mathbb{E}[|X_n|].$$

**Lemma** (Doob's $L^p$ inequality)**.** For $p > 1$, we have

$$\|X_n^*\|_p \leq \frac{p}{p-1} \|X_n\|_p$$

for all $n$.

**Theorem** ($L^p$ martingale convergence theorem)**.** Let $(X_n)_{n \geq 0}$ be a martingale, and $p > 1$. Then the following are equivalent:

(i) $(X_n)_{n \geq 0}$ is bounded in $L^p$, i.e. $\sup_n \mathbb{E}|X_i|^p < \infty$.

(ii) $(X_n)_{n \geq 0}$ converges as $n \to \infty$ to a random variable $X_\infty \in L^p$ almost surely and in $L^p$.

(iii) There exists a random variable $Z \in L^p$ such that

$$X_n = \mathbb{E}(Z \mid \mathcal{F}_n)$$

Moreover, in (iii), we always have $X_\infty = \mathbb{E}(Z \mid \mathcal{F}_\infty)$.

**Theorem** (Convergence in $L^1$)**.** Let $(X_n)_{n \geq 0}$ be a martingale. Then the following are equivalent:

(i) $(X_n)_{n \geq 0}$ is uniformly integrable.

(ii) $(X_n)_{n \geq 0}$ converges almost surely and in $L^1$.

(iii) There exists $Z \in L^1$ such that $X_n = \mathbb{E}(Z \mid \mathcal{F}_n)$ almost surely.

Moreover, $X_\infty = \mathbb{E}(Z \mid \mathcal{F}_\infty)$.

**Theorem.** If $(X_n)_{n \geq 0}$ is a uniformly integrable martingale, and $S, T$ are arbitrary stopping times, then $\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{S \wedge T}$. In particular $\mathbb{E}X_T = X_0$.

## 2.4 Applications of martingales

**Theorem.** Let $Y \in L^1$, and let $\hat{\mathcal{F}}_n$ be a backwards filtration. Then

$$\mathbb{E}(Y \mid \hat{\mathcal{F}}_n) \to \mathbb{E}(Y \mid \hat{\mathcal{F}}_\infty)$$

almost surely and in $L^1$.

**Theorem** (Kolmogorov 0-1 law)**.** Let $(X_n)_{n\geq 0}$ be independent random variables. Then, let

$$\hat{\mathcal{F}}_n = \sigma(X_{n+1}, X_{n+2}, \ldots).$$

Then the *tail $\sigma$-algebra* $\hat{\mathcal{F}}_\infty$ is trivial, i.e. $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \hat{\mathcal{F}}_\infty$.

**Theorem** (Strong law of large numbers)**.** Let $(X_n)_{n\geq 1}$ be iid random variables in $L^1$, with $\mathbb{E}X_1 = \mu$. Define

$$S_n = \sum_{i=1}^n X_i.$$

Then

$$\frac{S_n}{n} \to \mu \text{ as } n \to \infty$$

almost surely and in $L^1$.

**Theorem** (Radon–Nikodym)**.** Let $(\Omega, \mathcal{F})$ be a measurable space, and $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on $(\Omega, \mathcal{F})$. Then the following are equivalent:

(i) $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, i.e. for any $A \in \mathcal{F}$, if $\mathbb{P}(A) = 0$, then $\mathbb{Q}(A) = 0$.

(ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $A \in \mathcal{F}$, if $\mathbb{P}(A) \leq \delta$, then $\mathbb{Q}(A) \leq \varepsilon$.

(iii) There exists a random variable $X \geq 0$ such that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(X \mathbf{1}_A).$$

In this case, $X$ is called the *Radon–Nikodym derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$, and we write $X = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.

**Proposition.** If $F$ is harmonic and bounded, and $(X_n)_{n\geq 0}$ is Markov, then $(f(X_n))_{n\geq 0}$ is a martingale.

# 3 Continuous time stochastic processes

**Theorem** (Kolmogorov's criterion)**.** Let $(\rho_t)_{t\in I}$ be random variables, where $I \subseteq [0,1]$ is dense. Assume that for some $p > 1$ and $\beta > \frac{1}{p}$, we have

$$\|\rho_t - \rho_s\|_p \leq C|t-s|^\beta \text{ for all } t, s \in I. \tag{$*$}$$

Then there exists a continuous process $(X_t)_{t\in I}$ such that for all $t \in I$,

$$X_t = \rho_t \text{ almost surely,}$$

and moreover for any $\alpha \in [0, \beta - \frac{1}{p})$, there exists a random variable $K_\alpha \in L^p$ such that

$$|X_s - X_t| \leq K_\alpha |s-t|^\alpha$$

for all $s, t \in [0,1]$.

**Proposition.** Let $(X_t)_{t\geq 0}$ be a cadlag adapted process and $S, T$ stopping times. Then

(i) $S \wedge T$ is a stopping time.

(ii) If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(iii) $X_T \mathbf{1}_{T<\infty}$ is $\mathcal{F}_T$-measurable.

(iv) $(X_t^T)_{t\geq 0} = (X_{T\wedge t})_{t\geq 0}$ is adapted.

**Lemma.** A random variable $Z$ is $\mathcal{F}_T$-measurable iff $Z\mathbf{1}_{\{T\leq t\}}$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

**Proposition.** Let $A \subseteq \mathbb{R}$ be a closed set and $(X_t)_{t\geq 0}$ be continuous. Then $T_A$ is a stopping time.

**Proposition.** Let $(X_t)_{t\geq 0}$ be an adapted process (to $(\mathcal{F}_t)_{t\geq 0}$) that is cadlag, and let $A$ be an open set. Then $T_A$ is a stopping time with respect to $\mathcal{F}_t^+$.

**Theorem** (Optional stopping theorem)**.** Let $(X_t)_{t\geq 0}$ be an adapted cadlag process in $L^1$. Then the following are equivalent:

(i) For any bounded stopping time $T$ and any stopping time $S$, we have $X_T \in L^1$ and
$$\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{T\wedge S}.$$

(ii) For any stopping time $T$, $(X_t^T)_{t\geq 0} = (X_{T\wedge t})_{t\geq 0}$ is a martingale.

(iii) For any bounded stopping time $T$, $X_T \in L^1$ and $\mathbb{E}X_T = \mathbb{E}X_0$.

**Theorem.** Let $(X_t)_{t\geq 0}$ be a super-martingale bounded in $L^1$. Then it converges almost surely as $t \to \infty$ to a random variable $X_\infty \in L^1$.

**Lemma** (Maximal inequality)**.** Let $(X_t)_{t\geq 0}$ be a cadlag martingale or a non-negative sub-martingale. Then for all $t \geq 0$, $\lambda \geq 0$, we have

$$\lambda \mathbb{P}(X_t^* \geq \lambda) \leq \mathbb{E}|X_t|.$$

**Lemma** (Doob's $L^p$ inequality). Let $(X_t)_{t \geq 0}$ be as above. Then

$$\|X_t^*\|_p \leq \frac{p}{p-1} \|X_t\|_p.$$

**Theorem** (Regularization of martingales). Let $(X_t)_{t \geq 0}$ be a martingale with respect to $(\mathcal{F}_t)$, and suppose $\mathcal{F}_t$ satisfies the usual conditions. Then there exists a version $(\tilde{X}_t)$ of $(X_t)$ which is cadlag.

**Theorem** ($L^p$ convergence of martingales). Let $(X_t)_{t \geq 0}$ be a cadlag martingale. Then the following are equivalent:

(i) $(X_t)_{t \geq 0}$ is bounded in $L^p$.

(ii) $(X_t)_{t \geq 0}$ converges almost surely and in $L^p$.

(iii) There exists $Z \in L^p$ such that $X_t = \mathbb{E}(Z \mid \mathcal{F}_t)$ almost surely.

**Theorem** ($L^1$ convergence of martingales). Let $(X_t)_{t \geq 0}$ be a cadlag martingale. Then the folloiwng are equivalent:

(i) $(X_t)_{t \geq 0}$ is uniformly integrable.

(ii) $(X_t)_{t \geq 0}$ converges almost surely and in $L^1$ to $X_\infty$.

(iii) There exists $Z \in L^1$ such that $\mathbb{E}(Z \mid \mathcal{F}_t) = X_t$ almost surely.

**Theorem** (Optional stopping theorem). Let $(X_t)_{t \geq 0}$ be a uniformly integrable martingale, and let $S, T$ b e any stopping times. Then

$$\mathbb{E}(X_T \mid \mathcal{F}_s) = X_{S \wedge T}.$$

# 4 Weak convergence of measures

**Proposition.** Let $(\mu_n)_{n\geq 0}$ be as above. Then, the following are equivalent:

(i) $(\mu_n)_{n\geq 0}$ converges weakly to $\mu$.

(ii) For all open $G$, we have

$$\liminf_{n\to\infty} \mu_n(G) \geq \mu(G).$$

(iii) For all closed $A$, we have

$$\limsup_{n\to\infty} \mu_n(A) \leq \mu(A).$$

(iv) For all $A$ such that $\mu(\partial A) = 0$, we have

$$\lim_{n\to\infty} \mu_n(A) = \mu(A)$$

(v) (when $M = \mathbb{R}$) $F_{\mu_n}(x) \to F_\mu(x)$ for all $x$ at which $F_\mu$ is continuous, where $F_\mu$ is the *distribution function* of $\mu$, defined by $F_\mu(x) = \mu_n((-\infty, t])$.

**Theorem** (Prokhorov's theorem)**.** If $(\mu_n)_{n\geq 0}$ is a sequence of tight probability measures, then there is a subsequence $(\mu_{n_k})_{k\geq 0}$ and a measure $\mu$ such that $\mu_{n_k} \Rightarrow \mu$.

**Proposition.** If $\varphi_X = \varphi_Y$, then $\mu_X = \mu_Y$.

**Theorem** (Lévy's convergence theroem)**.** Let $(X_n)_{n\geq 0}$, $X$ be random variables taking values in $\mathbb{R}^d$. Then the following are equivalent:

(i) $\mu_{X_n} \Rightarrow \mu_X$ as $n \to \infty$.

(ii) $\varphi_{X_n} \to \varphi_X$ pointwise.

**Theorem** (Lévy)**.** Let $(X_n)_{n\geq 0}$ be as above, and let $\varphi_{X_n}(t) \to \psi(t)$ for all $t$. Suppose $\psi$ is continuous at 0 and $\psi(0) = 1$. Then there exists a random variable $X$ such that $\varphi_X = \psi$ and $\mu_{X_n} \Rightarrow \mu_X$ as $n \to \infty$.

**Lemma.** Let $X$ be a real random variable. Then for all $\lambda > 0$,

$$\mu_X(|x| \geq \lambda) \leq c\lambda \int_0^{1/\lambda} (1 - \operatorname{Re} \varphi_X(t))\, \mathrm{d}t,$$

where $C = (1 - \sin 1)^{-1}$.

# 5 Brownian motion

## 5.1 Basic properties of Brownian motion

**Theorem** (Wiener's theorem)**.** There exists a Brownian motion on some probability space.

**Lemma.** Brownian motion is a Gaussian process, i.e. for any $0 \leq t_1 < t_2 < \cdots < t_m \leq 1$, the vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ is Gaussian with covariance

$$\operatorname{cov}(B_{t_1}, B_{t_2}) = t_1 \wedge t_2.$$

**Proposition** (Invariance properties)**.** Let $(B_t)_{t \geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$.

(i) If $U$ is an orthogonal matrix, then $(UB_t)_{t \geq 0}$ is a standard Brownian motion.

(ii) *Brownian scaling*: If $a > 0$, then $(a^{-1/2} B_{at})_{t \geq 0}$ is a standard Brownian motion. This is known as a *random fractal property*.

(iii) (*Simple*) *Markov property*: For all $s \geq 0$, the sequence $(B_{t+s} - B_s)_{t \geq 0}$ is a standard Brownian motion, independent of $(\mathcal{F}_s^B)$.

(iv) *Time inversion*: Define a process

$$X_t = \begin{cases} 0 & t = 0 \\ tB_{1/t} & t > 0 \end{cases}.$$

Then $(X_t)_{t \geq 0}$ is a standard Brownian motion.

**Theorem.** For all $s \geq t$, the process $(B_{t+s} - B_s)_{t \geq 0}$ is independent of $\mathcal{F}_s^+$.

**Theorem** (Blumenthal's 0-1 law)**.** The $\sigma$-algebra $\mathcal{F}_0^+$ is trivial, i.e. if $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) \in \{0, 1\}$.

**Proposition.**

(i) If $d = 1$, then

$$\begin{aligned} 1 &= \mathbb{P}(\inf\{t \geq 0 : B_t > 0\} = 0) \\ &= \mathbb{P}(\inf\{t \geq 0 : B_t < 0\} = 0) \\ &= \mathbb{P}(\inf\{t > 0 : B_t = 0\} = 0) \end{aligned}$$

(ii) For any $d \geq 1$, we have

$$\lim_{t \to \infty} \frac{B_t}{t} = 0$$

almost surely.

(iii) If we define

$$S_t = \sup_{0 \leq s \leq t} B_t, \quad I_t = \inf_{0 \leq s \leq t} B_t,$$

then $S_\infty = \infty$ and $I_\infty = -\infty$ almost surely.

(iv) If $A$ is open an $\mathbb{R}^d$, then the cone of $A$ is $C_A = \{tx : x \in A, t > 0\}$. Then $\inf\{t \geq 0 : B_t \in C_A\} = 0$ almost surely.

**Theorem** (Strong Markov property)**.** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$, and let $T$ be an almost-surely finite stopping time with respect to $(\mathcal{F}_t^+)_{t\geq 0}$. Then

$$\tilde{B}_t = B_{T+t} - B_T$$

is a standard Brownian motion with respect to $(\mathcal{F}_{T+t}^+)_{t\geq 0}$ that is independent of $\mathcal{F}_T^+$.

**Theorem** (Reflection principle)**.** Let $(B_t)_{T\geq 0}$ and $T$ be as above. Then the *reflected process* $(\tilde{B}_t)_{t\geq 0}$ defined by

$$\tilde{B}_t = B_t \mathbf{1}_{t<T} + (2B_T - B_t)\mathbf{1}_{t\geq T}$$

is a standard Brownian motion.

**Corollary.** Let $(B_t)_{T\geq 0}$ be a standard Brownian motion in $d = 1$. Let $b > 0$ and $a \leq b$. Let

$$S_t = \sup_{0\leq s\leq t} B_t.$$

Then

$$\mathbb{P}(S_t \geq b, B_t \leq a) = \mathbb{P}(B_t \geq 2b - a).$$

**Corollary.** The law of $S_t$ is equal to the law of $|B_t|$.

**Proposition.** Let $d = 1$ and $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then the following processes are $(\mathcal{F}_t^+)_{t\geq 0}$ martingales:

(i) $(B_t)_{t\geq 0}$

(ii) $(B_t^2 - t)_{t\geq 0}$

(iii) $\left(\exp\left(uB_t - \frac{u^2 t}{2}\right)\right)_{t\geq 0}$ for $u \in \mathbb{R}$.

## 5.2 Harmonic functions and Brownian motion

**Lemma.** Let $u : D \to \mathbb{R}$ be measurable and locally bounded. Then the following are equivalent:

(i) $u$ is twice continuously differentiable and $\Delta u = 0$.

(ii) For any $x \in D$ and $r > 0$ such that $B(x, r) \subseteq D$, we have

$$u(x) = \frac{1}{\mathrm{Vol}(B(x,r))} \int_{B(x,r)} u(y)\;\mathrm{d}y$$

(iii) For any $x \in D$ and $r > 0$ such that $B(x, r) \subseteq D$, we have

$$u(x) = \frac{1}{\mathrm{Area}(\partial B(x,r))} \int_{\partial B(x,r)} u(y)\;\mathrm{d}y.$$

**Theorem.** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$, and $u : \mathbb{R}^d \to \mathbb{R}$ be harmonic such that

$$\mathbb{E}|u(x + B_t)| < \infty$$

for any $x \in \mathbb{R}^d$ and $t \geq 0$. Then the process $(u(B_t))_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t^+)_{t\geq 0}$.

**Lemma.** If $X$ and $Y$ are independent random variables in $\mathbb{R}^d$, and $X$ is $\mathcal{G}$-measurable. If $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that $f(X, Y)$ is integrable, then

$$\mathbb{E}(f(X, Y) \mid \mathcal{G}) = \mathbb{E} f(z, Y)|_{z=X}.$$

**Theorem.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable with bounded derivatives. Then, the processes $(X_t)_{t \geq 0}$ defined by

$$X_t = f(B_t) - \frac{1}{2} \int_0^t \Delta f(B_s) \, \mathrm{d}s$$

is a martingale with respect to $(\mathcal{F}_t^+)_{t \geq 0}$.

**Corollary.** If $u$ and $u'$ solve $\Delta u = \Delta u' = 0$, and $u$ and $u'$ agree on $\partial D$, then $u = u'$.

**Lemma.** Let $C$ be an open cone in $\mathbb{R}^d$ based at 0. Then there exists $0 \leq a < 1$ such that if $|x| \leq \frac{1}{2^k}$, then

$$\mathbb{P}_x(T_{\partial B(0,1)} < T_C) \leq a^k.$$

**Theorem.** Let $D$ be a bounded domain satisfying the Poincaré cone condition, and let $\varphi : \partial D \to \mathbb{R}$ be continuous. Let

$$T_{\partial D} = \inf\{t \geq 0 : B_t \in \partial D\}.$$

This is a *bounded* stopping time. Then the function $u : \bar{D} \to \mathbb{R}$ defined by

$$u(x) = \mathbb{E}_x(\varphi(B_{T_{\partial D}})),$$

where $\mathbb{E}_x$ is the expectation if we start at $x$, is the unique continuous function such that $u(x) = \varphi(x)$ for $x \in \partial D$, and $\Delta u = 0$ for $x \in D$.

## 5.3 Transience and recurrence

**Theorem.** Let $(B_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^d$.

- If $d = 1$, then $(B_t)_{t \geq 0}$ is *point recurrent*, i.e. for each $x, z \in \mathbb{R}$, the set $\{t \geq 0 : B_t = z\}$ is unbounded $\mathbb{P}_x$-almost surely.
- If $d = 2$, then $(B_t)_{t \geq 0}$ is *neighbourhood recurrent*, i.e. for each $x \in \mathbb{R}^2$ and $U \subseteq \mathbb{R}^2$ open, the set $\{t \geq 0 : B_t \in U\}$ is unbounded $\mathbb{P}_x$-almost surely. However, the process does not visit points, i.e. for all $x, z \in \mathbb{R}^d$, we have

  $$\mathbb{P}_X(B_t = z \text{ for some } t > 0) = 0.$$

- If $d \geq 3$, then $(B_t)_{t \geq 0}$ is *transient*, i.e. $|B_t| \to \infty$ as $t \to \infty$ $\mathbb{P}_x$-almost surely.

## 5.4 Donsker's invariance principle

**Theorem** (Donsker's invariance principle)**.** Let $(X_n)_{n\geq 0}$ be iid random variables with mean 0 and variance 1, and set $S_n = X_1 + \cdots + X_n$. Define

$$S_t = (1 - \{t\})s_{\lfloor t\rfloor} + \{t\}S_{\lfloor t\rfloor + 1}.$$

where $\{t\} = t - \lfloor t\rfloor$.

Define

$$(S_t^{[N]})_{t\geq 0} = (N^{-1/2}S_{t\cdot N})_{t\in[0,1]}.$$

As $(S_t^{[N]})_{t\in[0,1]}$ converges in distribution to the law of standard Brownian motion on $[0, 1]$.

**Theorem** (Skorokhod embedding theorem)**.** Let $\mu$ be a probability measure on $\mathbb{R}$ with mean 0 and variance $\sigma^2$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ on which there is a standard Brownian motion $(B_t)_{t\geq 0}$ and a sequence of stopping times $(T_n)_{n\geq 0}$ such that, setting $S_n = B_{T_n}$,

(i) $T_n$ is a random walk with steps of mean $\sigma^2$

(ii) $S_n$ is a random walk with step distribution $\mu$.

**Lemma.** Let $x, y > 0$. Then

$$\mathbb{P}_0(T_{-x} < T_y) = \frac{y}{x+y}, \quad \mathbb{E}_0 T_{-x} \wedge T_y = xy.$$

# 6 Large deviations

**Lemma** (Fekete)**.** If $b_n$ is a non-negative sub-additive sequence, then $\lim_n \frac{b_n}{n}$ exists.

**Theorem** (Cramér's theorem)**.** For $a > \bar{x}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -\psi^*(a).$$