Example Sheet 1

[You may submit questions 2 and 4 to be marked.]

1. The four-dimensional 4×4 Dirac matrices are defined uniquely up to an equivalence by $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}1$, with 1 the unit matrix. We may also require that if $\gamma^\mu = (\gamma^0,\boldsymbol{\gamma})$ then $\gamma^{\mu\dagger} = (\gamma^0,-\boldsymbol{\gamma})$. If $[X,\gamma^\mu]=0$ for all μ then $X \propto 1$ and if γ^μ, γ'^μ both obey the Dirac algebra then $\gamma'^\mu = S\gamma^\mu S^{-1}$ for some S. Define the charge conjugation matrix C by $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$, where T denotes the matrix transpose. Show that $[C^T C^{-1},\gamma^\mu]=0$ and hence that $C^T = cC,\ c=\pm 1$. Derive the results
$$\big(\gamma^\mu C\big)^T = -\,c\gamma^\mu C\,, \quad \big(\gamma^5 C\big)^T = c\gamma^5 C$$
$$\big(\gamma^\mu\gamma^5 C\big)^T = c\gamma^\mu\gamma^5 C\,, \quad \big([\gamma^\mu,\gamma^\nu]C\big)^T = -c[\gamma^\mu,\gamma^\nu]C\,.$$
Hence, since there are 6 independent antisymmetric and 10 symmetric 4×4 matrices, show that we must take $c=-1$.

   Using the assumed Hermiticity properties of the Dirac matrices, show $[\gamma^\mu, CC^\dagger]=0$ so that we may take $CC^\dagger = 1$.

   The matrix B is defined by $B^{-1}\gamma^{\mu *}B = (\gamma^0,-\boldsymbol{\gamma})$. Show that $B^{-1}\gamma^{5*}B = \gamma^5$. With the assumed form for $\gamma^{\mu\dagger}$ verify that we may take $B^{-1} = \pm\gamma^5 C$. [This is consistent with lectures because in our setup there we had $C^{-1} = -C$ and $C\gamma^5 = \gamma^5 C$.]

   *Generalise the above argument for finding c to $2n$ dimensions when the Dirac matrices are $2^n\times 2^n$ and we may take as a linearly independent basis 1 and $\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$, where [ ] denotes antisymmetrisation of indices, for $r=1,\ldots 2n$ ($\gamma^{\mu_1\cdots\mu_r}$ has $\binom{2n}{r}$ independent components). Show that $C(\gamma^{\mu_1\cdots\mu_r})^T C^{-1} = (-1)^{\frac{1}{2}r(r+1)}\gamma^{\mu_1\cdots\mu_r}$ and hence $c = (-1)^{\frac{1}{2}n(n+1)}$. Generalise γ^5 by taking $\hat\gamma = -i^{n-1}\gamma^0\gamma^1\ldots\gamma^{2n-1}$ and show that $\hat\gamma$ is Hermitian and $\hat\gamma^2 = 1$. Show that
$$\psi^c = C\bar\psi^T\,,\quad \psi' = \hat\gamma\psi \quad\Rightarrow\quad \psi = -c\,C\bar\psi^{c\,T}\,,\quad \psi'^c = -(-1)^n\hat\gamma\psi^c\,.$$
In what dimensions is it possible to have Majorana-Weyl spinors, so that $\psi^c = \pm\psi' = \psi$?

2. A Dirac quantum field transforms under parity so that
$$\hat P\psi(x)\hat P^{-1} = \gamma^0\psi(x_P)\,,\quad x_P^\mu = (x^0,-\mathbf{x})\,,$$
and has an interaction with a scalar field $\phi(x)$
$$\mathcal{L}_I(x) = g\,\bar\psi(x)\psi(x)\phi(x) + g'\bar\psi(x)i\gamma^5\psi(x)\phi(x)\,.$$
Obtain the necessary form for $\hat P\phi(x)\hat P^{-1}$ to ensure that the theory is invariant under parity if $g'=0$. What are the transformation properties of $\phi(x)$ for parity invariance when instead $g=0$? Can parity be conserved in a theory if both g, g' are non zero?

   How does the axial current $j_5^\mu(x) = \bar\psi(x)\gamma^\mu\gamma^5\psi(x)$ transform under parity?

3. For a free operator Dirac field $\hat{\psi}(x)$ assume $\hat{\psi}(x) = \sum_r a_r \, \psi_r(x)$ where $\{\psi_r(x)\}$ forms a basis for solutions of the Dirac equation and a_r are operators. Explain why a basis may be chosen so that $B^{-1}\psi_r^*(x) = \psi_{r'}(x_T)$ where $x_T^\mu = (-x^0, \mathbf{x})$ and $B^{-1}\gamma^{\mu *}B = (\gamma^0, -\boldsymbol{\gamma})$. Assume that the time-reversal operator is defined so that $\hat{T} a_r \hat{T}^{-1} = a_{r'}$. What is $\hat{T}\hat{\psi}(x)\hat{T}^{-1}$?

4. Under charge conjugation and time reversal a Dirac field ψ transforms as

$$\hat{C}\psi(x)\hat{C}^{-1} = C\bar{\psi}^T(x)\,, \qquad \hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)\,, \quad x_T^\mu = (-x^0, \mathbf{x})\,.$$

with $\hat{C}$, $\hat{T}$ the unitary, anti-unitary operators implementing these operations (recall that if $\hat{T}|\phi\rangle = |\phi_T\rangle$ then $\langle\phi'|\phi\rangle = \langle\phi_T|\phi'_T\rangle$). The matrices C, B are defined in question 1 and note that $C^\dagger C = B^\dagger B = 1$. Show that, if X is a matrix acting on Dirac spinors,

$$\hat{C}\,\bar{\psi}(x)X\psi(x)\hat{C}^{-1} = \bar{\psi}(x)X_C\psi(x)\,, \quad \hat{T}\,\bar{\psi}(x)X\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)X_T\psi(x_T)\,,$$

where $X_C = CX^TC^{-1}$ (take ψ and $\bar{\psi}$ to anti-commute) and $X_T = B^{-1}X^*B$. Hence determine the transformation properties under charge conjugation and time reversal of

$$\bar{\psi}(x)\psi(x)\,, \qquad \bar{\psi}(x)i\gamma^5\psi(x)\,, \qquad \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\,.$$

If $|\pi\rangle$ is a boson with momentum p and $\langle 0|\bar{\psi}(0)i\gamma^5\psi(0)|\pi(p)\rangle \neq 0$ show that, in a theory in which parity and charge conjugation are conserved, then the boson must have negative intrinsic parity and also positive charge-conjugation parity.

5. From Maxwell's equation $\partial_\nu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, derive the required transformation properties of $A_\mu(x)$ to ensure that Maxwell's equation is invariant under parity, charge conjugation and time reversal. Show that $\int d^4x\, \epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$ is odd under both parity and time reversal.

6. For a Dirac field ψ define $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$. Show that $\bar{\psi}_\pm\gamma^5 = \mp\bar{\psi}_\pm$. Let $\Psi_\pm = \begin{pmatrix} \psi_\pm \\ C\bar{\psi}_\mp^T \end{pmatrix}$ and show that then $\bar{\Psi}_\pm = \left(\bar{\psi}_\pm, -\psi_\mp^T C^{-1}\right)$. [Hint: it is easier to keep the new 2-dimensional "super-spin" space separate from the 4-dimensional spinor space of ψ.] A generalized Lorentz-invariant mass term can be written as

$$\mathcal{L}_m = \frac{1}{2}\Psi_+^T C^{-1}\mathcal{M}\Psi_+ \; - \; \frac{1}{2}\bar{\Psi}_+\mathcal{M}^* C\bar{\Psi}_+^T$$

where $\mathcal{M}$ is a symmetric 2×2 matrix which commutes with C and γ^μ. [The notation can be confusing but it is conventional. You can read the matrices more explicitly as $\mathbb{1}_4 \otimes \mathcal{M}$, $C \otimes \mathbb{1}_2$ and $\gamma^\mu \otimes \mathbb{1}_2$]. Verify that $\mathcal{L}_m^\dagger = \mathcal{L}_m$.

(i) If $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$ show that by absorbing any phase into $\psi_\pm$ we can take m real and positive, and that this reduces to the conventional Dirac mass term $\mathcal{L}_m = -m\bar{\psi}\psi$. Show that the kinetic term $\mathcal{L}_K = \bar{\psi}i\partial\!\!\!/\psi = \bar{\Psi}_+ i\partial\!\!\!/\Psi_+$. Regarding Ψ_+ and $\bar{\Psi}_+$ as independent and assuming $\mathcal{L} = \mathcal{L}_K + \mathcal{L}_m$, derive the equations

$$i\partial\!\!\!/\Psi_+ - \mathcal{M}^* C\bar{\Psi}_+^T = 0, \quad i\partial\!\!\!/ C\bar{\Psi}_+^T - \mathcal{M}\Psi_+ = 0$$

Hence show that the mass-squared eigenvalues are found by solving

$$\det\left(p^2 1 - \mathcal{M}^* \mathcal{M}\right) = 0\,.$$

(ii) Requiring $\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)$ and $\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$, with $B = \pm(\gamma^5 C)^{-1} = \pm C^{-1}\gamma^5$ as in question 1, show that $\hat{T}\Psi_+(x)\hat{T}^{-1} = B\Psi_+(x_T)$. Hence demonstrate that $\mathcal{M}$ should be real in order to have $\hat{T}\mathcal{L}_m(x)\hat{T}^{-1} = \mathcal{L}_m(x_T)$.

(iii) If $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$ with m real and positive and $|M| \gg m$, show that the masses are approximately $|M|$ and $m^2/|M|$.

Comments/corrections to `C.E.Thomas@damtp.cam.ac.uk`.

Standard Model
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Mathematical Tripos, Part III
Lent Term 2017

Example Sheet 2

[You may submit questions 1.i and 3 to be marked.]

1. A field theory is described in terms of the elements of a complex $N \times N$ matrix M by a Lagrangian

$$\mathcal{L} = \mathrm{Tr}(\partial^\mu M^\dagger \partial_\mu M) - \tfrac{1}{2}\lambda\,\mathrm{Tr}(M^\dagger M M^\dagger M) - k\,\mathrm{Tr}(M^\dagger M)\,,$$

where Tr denotes the matrix trace and $\lambda > 0$.

(i) Show that this theory is invariant under the symmetry group $\big(U(N)\times U(N)\big)/U(1)$ for transformations given by $M \mapsto AMB^{-1}$ for $A, B \in U(N)$ and where the $U(1)$ corresponds to $A = B = e^{\mathrm{i}\theta} I$. [Note that if H is a subgroup of G then G/H is a group if H belongs to the centre of G, i.e. $hg = gh$ for all $h \in H$, $g \in G$.] Show that if $k < 0$ spontaneous symmetry breakdown occurs and that in the ground state ${M_0}^\dagger M_0 = v^2 I$ for some v. What is the unbroken symmetry group and how many Goldstone modes are there?

(ii) If $\mathcal{L} \to \mathcal{L} + \mathcal{L}'$ where

$$\mathcal{L}' = h\left(\det M + \det M^\dagger\right)\,,$$

what is the symmetry group and how many Goldstone modes are there now after spontaneous symmetry breakdown? [Assume the ground state still satisfies ${M_0}^\dagger M_0 = v^2 I$.]

[Note $U(N) = \big(SU(N) \times U(1)\big)/Z_N$ where Z_N is the finite group corresponding to the complex numbers $e^{2\pi \mathrm{i}k/N}$, $k = 0, \ldots N-1$, under multiplication.]

2. A field theory has 5 real scalar fields ϕ_a which are expressed in terms of a symmetric traceless 3×3 matrix $\Phi = \sum_{a=1}^{5} \phi_a t_a$ where t_a are a basis of symmetric traceless matrices with $\mathrm{Tr}(t_a t_b) = \delta_{ab}$. The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}(\partial^\mu \Phi \partial_\mu \Phi) - V(\Phi)\,, \quad V(\Phi) = g\big(\tfrac{1}{4}\mathrm{Tr}(\Phi^4) + \tfrac{1}{3}b\,\mathrm{Tr}(\Phi^3) + \tfrac{1}{2}c\,\mathrm{Tr}(\Phi^2)\big)\,,$$

where $g > 0$. Show that this theory has an $SO(3)$ symmetry. Let $\mathcal{M}_0 = \{\Phi_0 : V(\Phi_0) = V_{\min}\}$. Assume $SO(3)$ acts transitively on $\mathcal{M}_0$, i.e. all points in $\mathcal{M}_0$ can be linked by an $SO(3)$ transformation. Show that then all $\Phi_0 \in \mathcal{M}_0$ have the same eigenvalues, which add up to zero, and that we may choose Φ_0 so that it is diagonal. Describe how the eigenvalues of Φ_0 determine the unbroken subgroup of $SO(3)$.

For this theory show that $\mathcal{M}_0$ is determined by the equation

$$\Phi_0^3 + b\,\Phi_0^2 + c\,\Phi_0 = \mu\, I\,, \quad 3\mu = \mathrm{Tr}(\Phi_0^3) + b\,\mathrm{Tr}(\Phi_0^2)\,.$$

[Here μ may be regarded as a Lagrange multiplier for the condition $\mathrm{Tr}(\Phi) = 0$ when varying $V(\Phi)$.] Verify that there is a potential solution in which the unbroken subgroup is $SO(2)$ if $b^2 > 12c$. [Note that in this case Φ_0 may be given in terms of a single eigenvalue.]

For 3×3 traceless matrices $\mathrm{Tr}(M^4) = \frac{1}{2}\big(\mathrm{Tr}(M^2)\big)^2$. Show that if $b = 0$ the initial symmetry is in fact $SO(5)$ and that $V_{\mathrm{min}} = -\frac{1}{2}\,gc^2$ with an unbroken group $SO(4)$.

How do the results on possible unbroken symmetry groups generalise to the analogous theory with an $SO(N)$ symmetry defined in terms of $N\times N$ symmetric traceless matrices?

3. Consider an $SU(2)$ gauge theory coupled to a two component complex scalar field ϕ. The $SU(2)$ generators acting on ϕ are represented by $\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}$ the usual Pauli matrices,

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\!\mathbf{F}_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \tfrac{1}{2}\lambda\big(\phi^\dagger\phi - \tfrac{1}{2}v^2\big)^2\,,$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - g\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,, \quad D_\mu\phi = \partial_\mu\phi + \mathrm{i}g\,\mathbf{A}_\mu\cdot\boldsymbol{\tau}\phi\,,$$

$\mathbf{A}_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$ and $\mathbf{F}_{\mu\nu} = (F^1_{\mu\nu}, F^2_{\mu\nu}, F^3_{\mu\nu})$. [The cross product above arises because the $SU(2)$ structure constant is the Levi-Civita symbol: $[t^a, t^b] = i\epsilon^{abc}t^c$.] Explain why we may choose $\phi = (0, v{+}h)^T/\sqrt{2}$ and why the $SU(2)$ gauge symmetry is completely broken. Neglecting quantum corrections, what are the masses of the elementary particle states?

4. (i) A triplet gauge field $\mathbf{A}_\mu$ is coupled to a real triplet field $\boldsymbol{\phi}$ with the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\!\mathbf{F}_{\mu\nu} + \tfrac{1}{2}(D^\mu\boldsymbol{\phi})\cdot(D_\mu\boldsymbol{\phi}) - \tfrac{1}{8}\lambda\big(\boldsymbol{\phi}^2 - v^2\big)^2\,,$$
$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - e\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,, \quad D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} - e\,\mathbf{A}_\mu\times\boldsymbol{\phi}\,.$$

[I.e. $\boldsymbol{\phi}$ transforms in the adjoint representation of $SU(2)$. The cross product above arises from writing the $SU(2)$ generators in the adjoint representation as $(t^a)_{jk} = -i\epsilon_{ajk}$.] Show that this theory is invariant under $SU(2)$ gauge transformations but that this is broken by the ground state to $U(1)$. Rewrite the theory in terms of physical fields and determine their masses and couplings.

(ii) For a complex triplet field $\boldsymbol{\phi}$ suppose the Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\!\mathbf{F}_{\mu\nu} + (D^\mu\boldsymbol{\phi})^*\cdot(D_\mu\boldsymbol{\phi}) + \tfrac{1}{2}g^2(\boldsymbol{\phi}^*\times\boldsymbol{\phi})^2\,.$$

Show that in the classical ground state the potential may be minimised, up to a freedom of gauge transformations, by choosing $\boldsymbol{\phi}_0 = v\,\mathbf{e}_3/\sqrt{2}$ for any complex v where $\mathbf{e}_3$ is the unit vector in the 3-direction. Explain why $v \sim -v$ under residual gauge transformations. Why is it possible to impose the conditions $\mathrm{Re}(v^*\boldsymbol{\phi}\!\cdot\!\mathbf{e}_1) = \mathrm{Re}(v^*\boldsymbol{\phi}\!\cdot\!\mathbf{e}_2) = 0$? Determine the masses of the physical fields. Why are theories with different values of v^2 inequivalent?

5. A gauge theory for the group G is described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}{}_a F_{\mu\nu a} + \tfrac{1}{2}(D^\mu\phi)\cdot(D_\mu\phi) - V(\phi)\,,$$
$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - g\, c_{abc} A_{\mu b} A_{\nu c}\,, \quad D_\mu\phi = \partial_\mu\phi + ig\, A_{\mu a}\theta_a\phi\,,$$

where ϕ is a multiplet of complex scalar fields, $a = 1,\ldots,\dim G$, and θ_a are matrices representing the Lie algebra of G, $[\theta_a, \theta_b] = ic_{abc}\theta_c$, where c_{abc} is completely antisymmetric. Assuming $V'(\phi)\cdot\theta_a\phi = 0$ and $\tilde{\phi}\cdot(\theta_a\phi) = (\theta_a\tilde{\phi})\cdot\phi$, show that $\mathcal{L}$ is invariant under G gauge transformations. [Recall that $\tilde{\phi}\cdot\phi = \tilde{\phi}^\dagger\phi$.]

Suppose $V(\phi)$ is minimised at $\phi = \phi_0$ and that we add a gauge fixing term of the form

$$\mathcal{L}_{\text{g.f.}} = -\tfrac{1}{2}\big(\partial^\mu A_{\mu a} - ig(\theta_a\phi_0)\cdot\phi\big)\big(\partial^\nu A_{\nu a} - ig(\theta_a\phi_0)\cdot\phi\big)\,.$$

If $\phi = \phi_0 + f$ derive the decoupled linearised equations of motion for the vector and scalar fields,

$$\partial^2 A_{\mu a} - g^2(\theta_a\phi_0)\cdot(\theta_b\phi_0)\, A_{\mu b} = 0\,, \quad \partial^2 f + \mathcal{M}\cdot f - g^2(\theta_a\phi_0)\,(\theta_a\phi_0)\cdot f = 0\,,$$

where $\mathcal{M}$ is a matrix determined by the second derivatives of $V(\phi)$ at $\phi = \phi_0$. Show that the mass eigenstates form multiplets of the unbroken gauge group H (for which the corresponding gauge fields are massless). [It is sufficient to show that the mass matrices appearing in the linear field equations commute with the generators of H in the appropriate representation.]

6. Let $\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - \frac{1}{2}g(\phi^*\phi - \frac{1}{2}v^2)^2$ be the Lagrangian for a complex scalar field ϕ. Writing $\phi = (v + f + \mathrm{i}\alpha)/\sqrt{2}$, show that the α field is massless whereas the f field has a mass $\sqrt{gv^2}$. Consider the scattering amplitude $\mathcal{M}$ for α particle scattering which is defined by $\langle\alpha(p_3)\alpha(p_4)|T|\alpha(p_1)\alpha(p_2)\rangle = (2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2)\mathcal{M}$ where the scattering S-matrix is $S = 1 - \mathrm{i}T$. Neglecting any Feynman diagrams with loops, show that

$$\mathcal{M} = g^2v^2\left(\frac{1}{s - gv^2} + \frac{1}{t - gv^2} + \frac{1}{u - gv^2}\right) + 3g\,,$$

where

$$s = (p_1 + p_2)^2\,,\ t = (p_3 - p_1)^2\,,\ u = (p_4 - p_1)^2\,.$$

Verify that $s + t + u = 0$ and hence show that for α particles with low energies E we have $\mathcal{M} = \mathrm{O}(E^4)$.

Comments/corrections to `C.E.Thomas@damtp.cam.ac.uk`.

Standard Model
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Mathematical Tripos, Part III
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Example Sheet 3

[You may submit questions 2 and 3 to be marked.]

1. The Lagrangian density for the Weinberg-Salam theory with gauge fields $\mathbf{W}_\mu, B_\mu$, a complex scalar field ϕ and fermion fields ψ may be written as

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \tfrac{1}{4}\lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)^2 + \bar\psi i\gamma^\mu D_\mu\psi\,,\\ &\quad -\left\{\bar\psi\Gamma_2\phi\,\tfrac{1}{2}(1+\gamma^5)\psi_2 + \bar\psi\Gamma_1\phi^c\,\tfrac{1}{2}(1+\gamma^5)\psi_1 + \text{Hermitian conjugate}\right\},\end{aligned}$$

where $\boldsymbol{\sigma}$ are the usual Pauli matrices, Γ_1 and Γ_2 are Yukawa couplings, and

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu\,,\quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$
$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\quad \phi^c = i\sigma_2\phi^*\,,\quad D_\mu\phi = \partial_\mu\phi + i(g\mathbf{W}_\mu\cdot\tfrac{1}{2}\boldsymbol{\sigma} + g'B_\mu Y)\phi\,,$$
$$D_\mu\psi = \partial_\mu\psi + i(g\mathbf{W}_\mu\cdot\tfrac{1}{2}\boldsymbol{\sigma} + g'B_\mu y)\tfrac{1}{2}(1-\gamma^5)\psi + ig'B_\mu(y+Y\sigma_3)\tfrac{1}{2}(1+\gamma^5)\psi\,.$$

Identify the gauge group of $\mathcal{L}$ and show how each term in $\mathcal{L}$ is gauge invariant [note that Y only enters in the product $g'Y$ so its value is essentially arbitrary]. Show that a mass term for the gauge fields of the form

$$m_W^2 W^{\mu\dagger}W_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu$$

is produced where $W_\mu = \frac{1}{\sqrt{2}}(W_{1\mu} - iW_{2\mu})$ and $Z_\mu = \cos\theta_W W_{3\mu} - \sin\theta_W B_\mu$ for a suitable choice of the angle θ_W. How is m_Z/m_W related to θ_W? Explain why it is impossible to introduce mass terms for the fermion fields into $\mathcal{L}$ that are compatible with gauge invariance. What are the fermion charges which give the coupling to the photon?

2. Show that the decay rate of a Higgs boson to a $\ell\bar\ell$ pair is given by

$$\Gamma_{H\to\ell\bar\ell} = \frac{G_F}{\sqrt{2}}\frac{1}{4\pi}\frac{m_\ell^2}{m_H^2}\left(m_H^2 - 4m_\ell^2\right)^{\frac{3}{2}}\,,$$

assuming $m_H > 2m_\ell$.

3. For two generations of quarks with masses m_d, m_s, m_u, m_c, suppose the part of the Lagrangian containing mass terms is given by

$$\mathcal{L}_m = -\big\{ \overline{q}_+ m_+ \tfrac{1}{2}(1+\gamma_5) q_+ + \overline{q}_- m_- \tfrac{1}{2}(1+\gamma_5) q_- + \text{h.c.} \big\}$$

with

$$q_+ = \begin{pmatrix} u' \\ c' \end{pmatrix} , \quad q_- = \begin{pmatrix} d' \\ s' \end{pmatrix} , \quad m_+ = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} , \quad m_- = \begin{pmatrix} 0 & c \\ c^* & d \end{pmatrix} , \ (b, d \text{ are real}) .$$

where u', d', c', s' are quark fields (the prime indicates they diagonalize the weak charged-current interaction). Given $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, define θ_+ by

$$R(\theta_+) \begin{pmatrix} 0 & |a| \\ |a| & b \end{pmatrix} R(\theta_+)^{-1} = \begin{pmatrix} m_u & 0 \\ 0 & -m_c \end{pmatrix}$$

and define θ_- similarly for m_s and m_d. Show that, after suitable rephasing of quark fields, one can use R to diagonalize $m_\pm$. [This is just a toy model so do not be concerned about the minus sign in front of m_c.] Hence, from the mixing matrix generated by the quarks' weak charged-current interactions, show that the Cabbibo angle in this case is given by

$$\theta_C = \theta_- - \theta_+ = \tan^{-1}\sqrt{\frac{m_d}{m_s}} - \tan^{-1}\sqrt{\frac{m_u}{m_c}} .$$

4. In the μ^- decay process,

$$\mu^-(p) \to e^-(k) + \bar{\nu}_e(q) + \nu_\mu(q') ,$$

suppose the initial μ is polarised. The spin polarisation may be represented by a 4-vector s^μ, with $s \cdot p = 0$, where in the muon rest frame $s^\mu = (0, \mathbf{s})$, and the corresponding Dirac spinor then satisfies $u_\mu(p)\overline{u}_\mu(p) = (\not{p} + m_\mu)\frac{1}{2}(1 + \gamma^5 \not{s})$. Show that if $\mathcal{M}$ is the matrix element of $\mathcal{L}_W(0)$ for this process then,

$$\sum_{\text{spins } e, \bar{\nu}_e, \nu_\mu} |\mathcal{M}|^2 = 64\, G_F^{\,2}\, [q' \cdot k]\, [q \cdot (p - m_\mu\, s)] .$$

Hence, neglecting the electron mass, obtain the differential decay rate in the μ^- rest frame for the final electron of energy E_e emitted into a solid angle $\mathrm{d}\Omega(\hat{\mathbf{k}})$ about the direction $\hat{\mathbf{k}}$

$$\mathrm{d}\Gamma = \frac{G_F^{\,2} m_\mu^5}{24(2\pi)^4}\, x^2 \big(3 - 2x - (2x-1)\, \hat{\mathbf{k}} \cdot \mathbf{s}\big)\, \mathrm{d}x\, \mathrm{d}\Omega(\hat{\mathbf{k}}) ,$$

where $x = 2E_e/m_\mu$ and $0 \leqslant x \leqslant 1$. Explain why this decay distribution is a direct indication of the breakdown of parity invariance.

5. If $A_\alpha(x)$ is the $\Delta S = \Delta Q = 1$ weak hadronic current, the kaon decay constant F_K is defined by

$$\langle 0|A_\alpha(0)|K^-(p)\rangle = i\sqrt{2}F_K p_\alpha \ .$$

Use the theory of weak decays to show that the decay rate for a kaon decaying to a muon and an antineutrino is

$$\Gamma_{K^-\to\mu^-\bar{\nu}_\mu} = \frac{G_F^{\,2}F_K^2\sin^2\theta_C}{4\pi} m_\mu^2 m_K \left(1-\frac{m_\mu^2}{m_K^2}\right)^2 \ .$$

6. Show that the interaction of the Z boson with a lepton field ℓ can be written as

$$\mathcal{L}_I = -\frac{g}{2\cos\theta_W}\,\bar{\ell}\gamma^\mu(v-a\gamma^5)\ell\, Z_\mu \ ,$$

where for the electron $v = 2\sin^2\theta_W - \frac{1}{2}$ and $a = -\frac{1}{2}$, while the ν_e has $v = a = \frac{1}{2}$. Calculate the decay rate $\Gamma_{Z\to\ell\bar{\ell}}$ neglecting the lepton mass.

7. Neglecting lepton masses, show that including Z-mediated processes as well as photon-mediated processes, the differential cross section for $e^-e^+\to\ell\bar{\ell}$, where $\ell=\mu$ or τ, has the form,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\Big\{(1+\cos^2\theta)\big(1+2v^2D+(v^2+a^2)^2D^2\big)+4\cos\theta\big(a^2D+2v^2a^2D^2\big)\Big\} \ ,$$

where $v = 2\sin^2\theta_W - \frac{1}{2}$, $a = -\frac{1}{2}$ and $D = \frac{G_F}{\sqrt{2}}\frac{m_Z^2}{q^2-m_Z^2}\Big/\frac{2\pi\alpha}{q^2}$. [The differential cross section here is the cross section per unit solid angle, $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.] For $q^2\approx m_Z^2$ the cross section behaves like

$$\sigma_{e^-e^+\to\ell\bar{\ell}} \sim 12\pi\,\frac{\Gamma_{Z\to e^-e^+}\Gamma_{Z\to\ell\bar{\ell}}}{(q^2-m_Z^2)^2+m_Z^2\Gamma^2} \ ,$$

where Γ is the total decay width. Show that the formula for the differential cross section is compatible with the result for $\Gamma_{Z\to\ell\bar{\ell}}$.

Comments/corrections to `C.E.Thomas@damtp.cam.ac.uk`.

Standard Model
Dr C E Thomas

Mathematical Tripos, Part III
Lent Term 2017

Example Sheet 4

[You may submit questions 1 and 3 to be marked.]

1. In QCD let the strong coupling $a = g^2/(4\pi)^2$. The running coupling $a(\mu^2)$ is defined by
$$\mu^2 \frac{\mathrm{d}a}{\mathrm{d}\mu^2} = \beta(a)\,, \qquad \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 + O(a^5)\,.$$
Show that, for a suitable choice of Λ,
$$\frac{1}{a(\mu^2)} = \beta_0 \log \frac{\mu^2}{\Lambda^2} + \frac{\beta_1}{\beta_0} \log\log \frac{\mu^2}{\Lambda^2} + O\left(1 \Big/ \log \frac{\mu^2}{\Lambda^2}\right).$$
If the coupling is redefined so that
$$\bar{a} = f(a) = a + v_1 a^2 + v_2 a^3 + O(a^4)\,,$$
show that $\bar{\beta}(\bar{a}) = \beta(a) f'(a)$ and hence $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_1 = \beta_1$, $\bar{\beta}_2 = \beta_2 + \beta_0(v_2 - v_1^2) - \beta_1 v_1$. If $\bar{a}(\mu^2) = f(a(\mu^2))$ is written in terms of $\bar{\Lambda}$ in the same form as a in terms of Λ above, show that
$$\log \frac{\bar{\Lambda}^2}{\Lambda^2} = \frac{v_1}{\beta_0}\,.$$
Suppose a physical observable R has a perturbative expansion in terms of the coupling $a(\mu^2)$ of the form $R = a^N\big[r_0 + r_1 a + r_2 a^2 + \ldots\big]$. Under the above redefinition $r_1 \to \bar{r}_1 = r_1 + N v_1 r_0$, $r_2 \to \bar{r}_2 = r_2 + N v_2 r_0 + \frac{1}{2}N(N-1)v_1^2 r_0 + (N+1)v_1 r_1$. Show that
$$\hat{r}_1 = r_1 - \beta_0 N r_0 \log \frac{\mu^2}{\Lambda^2}\,, \qquad \hat{r}_2 = r_2 + N\frac{\beta_2}{\beta_0} r_0 - \frac{N+1}{2N}\frac{r_1^2}{r_0} - \frac{\beta_1}{\beta_0} r_1\,,$$
are invariant under this redefinition of the coupling.

2. Use the interaction $\mathcal{L}_W = -(G_F/\sqrt{2})\, J_\alpha^{\text{had}\dagger}\, \bar{\nu}_\tau \gamma^\alpha (1-\gamma_5)\tau$ to show that the total decay rate for $\tau^- \to \nu_\tau$ + hadrons is
$$\Gamma_{\tau^- \to \nu_\tau + \text{hadrons}} = \frac{G_F{}^2 m_\tau^3}{16\pi} \int_0^{m_\tau^2} \mathrm{d}\sigma \left(1 - \frac{\sigma}{m_\tau^2}\right)^2 \left[\rho_0(\sigma) + \left(1 + \frac{2\sigma}{m_\tau^2}\right)\rho_1(\sigma)\right],$$
where
$$\sum_X (2\pi)^3 \delta^4(P_X - k) \langle 0 | J_\alpha^{\text{hadrons}} | X \rangle \langle X | J_\beta^{\text{hadrons}\dagger} | 0 \rangle = k_\alpha k_\beta \rho_0(k^2) + (-g_{\alpha\beta} k^2 + k_\alpha k_\beta)\rho_1(k^2)\,,$$
and X covers all possible hadronic final states. If X is restricted to the π^- show that $\rho_0(\sigma) = 2F_\pi^2 \cos^2\theta_C\, \delta(\sigma - m_\pi^2)$ and $\rho_1(\sigma) = 0$. Hence find $\Gamma_{\tau^- \to \nu_\tau \pi^-}$.

3. Using light-cone coordinates, the longitudinal components forward/backward along the light cone in the $\mathbf{e}_3$ direction for an arbitrary 4-vector V are $V^\pm = V^0 \pm V^3$ and the transverse components are $\mathbf{V}_\perp = (V^1, V^2)$. Show that the Minkowski metric has components $g_{+-} = g_{-+} = \frac{1}{2}$ and $g_{11} = g_{22} = -1$ with the others zero.

   Consider deep inelastic scattering off a hadron where P is the initial-state hadron momentum and q is the photon momentum. Using a frame where $\mathbf{P}_\perp = \mathbf{q}_\perp = 0$ show that,

$$Q^2 = -q^+ q^- \, , \qquad \nu = \tfrac{1}{2}(q^+ P^- + q^- P^+) \, .$$

   Taking the DIS limit to be $q^- \to \infty$ with $q^+ = O(P^+)$ show that,

$$\nu \sim \tfrac{1}{2} q^- P^+ \qquad x \sim -\frac{q^+}{P_+} \, .$$

   Obtain the following expressions,

$$W_H^{+-}(q,P) = -W_1 + \left(M^2 + \frac{\nu^2}{Q^2} \right) W_2 \equiv F_L(x,Q^2) \, ,$$

$$W_H^{++}(q,P) = \frac{(q^+)^2}{Q^2} F_L(x,Q^2) \, , \qquad W_H^{--}(q,P) = \frac{(q^-)^2}{Q^2} F_L(x,Q^2) \, ,$$

   and show that in the DIS limit,

$$F_L(x,Q^2) \sim -F_1(x,Q^2) + \frac{1}{2x} F_2(x,Q^2) \, .$$

4. For a fundamental complex scalar field ϕ the electromagnetic current has the form $J^\mu = i[\phi^* (\partial^\mu \phi) - (\partial^\mu \phi^*) \phi]$. Assuming that this field corresponds to the charged constituents of a hadron H and treating the constituents as if they were free, obtain

$$W_H{}^{\mu\nu}(q,P) \sim \int d^4k \; W_\phi{}^{\mu\nu}(q,k) \big[\Gamma_H(P,k) + \overline{\Gamma}_H(P,k) \big] \, ,$$

   where

$$W_\phi{}^{\mu\nu}(q,k) = \tfrac{1}{2}(2k+q)^\mu (2k+q)^\nu \, \delta\big((k+q)^2\big) \, ,$$

   the momentum of the initial-state hadron is P and the photon momentum is q. Hence obtain for $F_2(x,Q^2) = \nu \, W_2(\nu,Q^2)$ and $F_1(x,Q^2) = W_1(\nu,Q^2)$, where $x = \frac{Q^2}{2\nu}$ and $\nu = P \cdot q$, the asymptotic forms in the deep inelastic limit $Q^2 = -q^2 \to \infty$

$$F_2(x,Q^2) \sim x \big[f(x) + \overline{f}(x) \big] \, , \qquad F_1(x,Q^2) \sim 0 \, ,$$

   where, for $0 < x < 1$ and taking $k = \xi P + k'$ with $k' \cdot q$ bounded

$$f(x) = x \int d^4k \, \delta(\xi - x) \, \Gamma_H(P,k) \, , \quad \overline{f}(x) = x \int d^4k \, \delta(\xi - x) \, \overline{\Gamma}_H(P,k) \, .$$

5. [From Donoghue, Golowich, Holstein, Chapter IV.] In describing the decay $\mu \to e\gamma$, one may try to use an effective Lagrangian $\mathcal{L}_{3,4}$ which contains terms of dimension 3 and 4

$$\mathcal{L}_{3,4} \;=\; a_3(\bar{e}\mu + \bar{\mu}e) \;+\; ia_4(\bar{e}\not{D}\mu \;+\; \bar{\mu}\not{D}e)$$

where $D_\mu = \partial_\mu + ieQA_\mu$ and a_3, a_4 are constants. Show by direct calculation that $\mathcal{L}_{3,4}$ does not lead to $\mu \to e\gamma$. If $\mathcal{L}_{3,4}$ is added to the QED Lagrangian for muons and electrons, show that one can define new fields μ' and e' to yield a Lagrangian which is diagonal in flavour. Thus, even in the presence of $\mathcal{L}_{3,4}$ there are two conserved fermion numbers. Finally, at dimension 5, show that $\mu \to e\gamma$ can be described by including in the Lagrangian

$$\mathcal{L}_5 \;=\; \bar{e}\sigma^{\alpha\beta}(c + d\gamma^5)\mu F_{\alpha\beta} \;+\; \text{h.c.}$$

where c and d are constants and $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$.

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