

Part III — Symplectic Geometry

Based on lectures by A. R. Pires

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Lent 2018

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The first part of the course will be an overview of the basic structures of symplectic geometry, including symplectic linear algebra, symplectic manifolds, symplectomorphisms, Darboux theorem, cotangent bundles, Lagrangian submanifolds, and Hamiltonian systems. The course will then go further into two topics. The first one is moment maps and toric symplectic manifolds, and the second one is capacities and symplectic embedding problems.

Pre-requisites

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

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1 Symplectic manifolds

1.1 Symplectic linear algebra

In symplectic geometry, we study symplectic manifolds. These are manifolds equipped with a certain structure on the tangent bundle. In this section, we first analyze the condition fiberwise.

Definition (Symplectic vector space). A *symplectic vector space* is a real vector space V together with a non-degenerate skew-symmetric bilinear map $\Omega : V \times V \rightarrow \mathbb{R}$.

Recall that

Definition (Non-degenerate bilinear map). We say a bilinear map Ω is non-degenerate if the induced map $\tilde{\Omega} : V \rightarrow V^*$ given by $v \mapsto \Omega(v, \cdot)$ is bijective.

Even if we drop the non-degeneracy condition, there aren't that many symplectic vector spaces around up to isomorphism.

Theorem (Standard form theorem). Let V be a real vector space and Ω a skew-symmetric bilinear form. Then there is a basis $\{u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that

- (i) $\Omega(u_i, v) = 0$ for all $v \in V$
- (ii) $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$.
- (iii) $\Omega(e_i, f_j) = \delta_{ij}$.

The proof is a skew-symmetric version of Gram-Schmidt.

Proof. Let

$$U = \{u \in V : \Omega(u, v) = 0 \text{ for all } v \in V\},$$

and pick a basis u_1, \dots, u_k of this. Choose any W complementary to U .

First pick $e_1 \in W \setminus \{0\}$ arbitrarily. Since $e_1 \notin U$, we can pick f_1 such that $\Omega(e_1, f_1) = 1$. Then define $W_1 = \text{span}\{e_1, f_1\}$, and

$$W_1^\Omega = \{w \in W : \Omega(w, v) = 0 \text{ for all } v \in W_1\}.$$

It is clear that $W_1 \cap W_1^\Omega = \{0\}$. Moreover, $W = W_1 \oplus W_1^\Omega$. Indeed, if $v \in W$, then

$$v = (\Omega(v, f_1)e_1 - \Omega(v, e_1)f_1) + (v - (\Omega(v, f_1)e_1 - \Omega(v, e_1)f_1)),$$

Then we are done by induction on the dimension. □

Here $k = \dim U$ and $2n = \dim V - k$ are invariants of (V, Ω) . The number $2n$ is called the *rank* of Ω . Non-degeneracy is equivalent to $k = 0$.

Exercise. Ω is non-degenerate iff $\Omega^{\wedge n} = \Omega \wedge \dots \wedge \Omega \neq 0$.

By definition, every symplectic vector space is canonically isomorphic to its dual, given by the map Ω . As in the above theorem, a *symplectic basis* $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of V is a basis such that

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_i) = \delta_{ij}.$$

Every symplectic vector space has a symplectic basis, and in particular has even dimension. In such a basis, the matrix representing Ω is

$$\Omega_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We will need the following definitions:

Definition (Symplectic subspace). If (V, Ω) is a symplectic vector space, a *symplectic subspace* is a subspace $W \subseteq V$ such that $\Omega|_{W \times W}$ is non-degenerate.

Definition (Isotropic subspace). If (V, Ω) is a symplectic vector space, an *isotropic subspace* is a subspace $W \subseteq V$ such that $\Omega|_{W \times W} = 0$.

Definition (Lagrangian subspace). If (V, Ω) is a symplectic vector space, an *Lagrangian subspace* is an isotropic subspace W with $\dim W = \frac{1}{2} \dim V$.

Definition (Symplectomorphism). A *symplectomorphism* between symplectic vector spaces $(V, \Omega), (V', \Omega')$ is an isomorphism $\varphi: V \rightarrow V'$ such that $\varphi^* \Omega' = \Omega$.

The standard form theorem says any two $2n$ -dimensional symplectic vector space (V, Ω) are symplectomorphic to each other.

1.2 Symplectic manifolds

We are now ready to define symplectic manifolds.

Definition (Symplectic manifold). A *symplectic manifold* is a manifold M of dimension $2n$ equipped with a 2-form ω that is closed (i.e. $d\omega = 0$) and non-degenerate (i.e. $\omega^{\wedge n} \neq 0$). We call ω the *symplectic form*.

The closedness condition is somewhat mysterious. There are some motivations from classical mechanics, for examples, but they are not very compelling, and the best explanation for the condition is “these manifolds happen to be quite interesting”.

Observe that by assumption, the form $\omega^{\wedge n}$ is nowhere-vanishing, and so it is a volume form. If our manifold is compact, then pairing the volume form with the fundamental class (i.e. integrating it over the manifold) gives the volume, and in particular is non-zero. Thus, $\omega^{\wedge n} \neq 0 \in H_{\text{dR}}^{2n}(M)$. Since the wedge is well-defined on cohomology, an immediate consequence of this is that

Proposition. If a compact manifold M^{2n} is such that $H_{\text{dR}}^{2k}(M) = 0$ for some $k < n$, then M does not admit a symplectic structure.

A more down-to-earth proof of this result would be that if $\omega^{2k} = d\alpha$ for some α , then

$$\int \omega^n = \int \omega^{2k} \wedge \omega^{2(n-k)} = \int d(\alpha \wedge \omega^{2(n-k)}) = 0$$

by Stokes’ theorem.

Example. S^n does not admit a symplectic structure unless $n = 2$ (or $n = 0$, if one insists).

On the other hand, S^2 is a symplectic manifold.

Example. Take $M = S^2$. Take the standard embedding in \mathbb{R}^3 , and for $p \in S^2$, the tangent space consists of the vectors perpendicular to p . We can take the symplectic form to be

$$\omega_p(u, v) = p \cdot (u \times v).$$

Anti-symmetry and non-degeneracy is IA Vectors and Matrices.

Definition (Symplectomorphism). Let (X_1, ω_1) and (X_2, ω_2) be symplectic manifolds. A *symplectomorphism* is a diffeomorphism $f : X_1 \rightarrow X_2$ such that $f^*\omega_2 = \omega_1$.

If we have a single fixed manifold, and two symplectic structures on it, there are other notions of “sameness” we might consider:

Definition (Strongly isotopic). Two symplectic structures on M are *strongly isotopic* if there is an isotopy taking one to the other.

Definition (Deformation equivalent). Two symplectic structures ω_0, ω_1 on M are *deformation equivalent* if there is a family of symplectic forms ω_t that start and end at ω_0 and ω_1 respectively.

Definition (Isotopic). Two symplectic structures ω_0, ω_1 on M are *isotopic* if there is a family of symplectic forms ω_t that start and end at ω_0 and ω_1 respectively, and further the cohomology class $[\omega_t]$ is independent of t .

A priori, it is clear that we have the following implications:

$$\text{symplectomorphic} \Leftarrow \text{strongly isotopic} \Rightarrow \text{isotopic} \Rightarrow \text{deformation equivalent}.$$

It turns out when M is compact, isotopic implies strongly isotopic:

Theorem (Moser). If M is compact with a family ω_t of symplectic forms with $[\omega_t]$ constant, then there is an isotopy $\rho_t : M \rightarrow M$ with $\rho_t^*\omega_t = \omega_0$.

The key idea is to express the condition $\rho_t^*\omega_t = \omega_0$ as a differential equation for $v_t = \frac{d}{dt}\rho_t$, and ODE theory guarantees the existence of a v_t satisfying the equation. Then compactness allows us to integrate it up to get a genuine ρ_t .

Proof. We set ρ_0 to be the identity. Then the equation $\rho_t^*\omega_t = \omega_0$ is equivalent to $\rho_t^*\omega_t$ being constant. If v_t is the associated vector field to ρ_t , then we need

$$0 = \frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^* \left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} \right).$$

So we want to solve

$$\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0.$$

To solve for this, since $[\frac{d\omega_t}{dt}] = 0$, it follows that there is a family μ_t of 1-forms such that $\frac{d\omega_t}{dt} = d\mu_t$. Then our equation becomes

$$\mathcal{L}_{v_t}\omega_t + d\mu_t = 0.$$

By assumption, $d\omega_t = 0$. So by Cartan's magic formula, we get

$$d\iota_{v_t}\omega_t + d\mu_t = 0.$$

To solve this equation, it suffices to solve Moser's equation,

$$\iota_{v_t}\omega_t + \mu_t = 0,$$

which can be solved since ω_t is non-degenerate. □

There is also a relative version of this.

Theorem (Relative Moser). Let $X \subseteq M$ be a compact manifold of a manifold M , and ω_0, ω_1 symplectic forms on M agreeing on X . Then there are neighbourhoods U_0, U_1 of X and a diffeomorphism $\varphi : U_0 \rightarrow U_1$ fixing X such that $\varphi^*\omega_1 = \omega_0$.

Proof. We set

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Then this is certainly closed, and since this is constantly ω_0 on X , it is non-degenerate on X , hence, by compactness, non-degenerate on a small tubular neighbourhood U_0 of X (by compactness). Now

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0,$$

and we know this vanishes on X . Since the inclusion of X into a tubular neighbourhood is a homotopy equivalence, we know $[\omega_1 - \omega_0] = 0 \in H_{\text{DR}}^1(U_0)$. Thus, we can find some μ such that $\omega_1 - \omega_0 = d\mu$, and by translation by a constant, we may suppose μ vanishes on X . We then solve Moser's equation, and the resulting ρ will be constant on X since μ vanishes. □

We previously had the standard form theorem for symplectic *vector spaces*, which is not surprising — we have the same for Riemannian metrics, for example. Perhaps more surprisingly, give a symplectic *manifold*, we can always pick coordinate charts where the symplectic form looks like the standard symplectic form throughout:

Theorem (Darboux theorem). If (M, ω) is a symplectic manifold, and $p \in M$, then there is a chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ about p on which

$$\omega = \sum dx_i \wedge dy_i.$$

Proof. ω can certainly be written in this form at p . Then relative Moser with $X = \{p\}$ promotes this to hold in a neighbourhood of p . □

Thus, symplectic geometry is a global business, unlike Riemannian geometry, where it is interesting to talk about local properties such as curvature.

A canonical example of a symplectic manifold is the *cotangent bundle* of a manifold X . In physics, X is called the *configuration space*, and $M = T^*X$ is called the *phase space*.

On a coordinate chart (U, x_1, \dots, x_n) for X , a generic 1-form can be written as

$$\xi = \sum_{i=1}^n \xi_i dx_i.$$

Thus, we have a coordinate chart on T^*X given by $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$. On T^*U , there is the *tautological 1-form*

$$\alpha = \sum_{i=1}^n \xi_i dx_i.$$

Observe that this is independent of the chart chosen, since it can be characterized in the following coordinate-independent way:

Proposition. Let $\pi : M = T^*X \rightarrow X$ be the projection map, and $\pi^* : T^*X \rightarrow T^*M$ the pullback map. Then for $\xi \in M$, we have $\alpha_\xi = \pi^*\xi$.

Proof. On a chart, we have

$$\pi^*\xi \left(\frac{\partial}{\partial x_j} \right) = \xi \left(\frac{\partial}{\partial x_j} \right) = \xi_j = \alpha_\xi \left(\frac{\partial}{\partial x_j} \right),$$

and similarly both vanish on $\frac{\partial}{\partial \xi_j}$. \square

This tautological 1-form is constructed so that the following holds:

Proposition. Let μ be a one-form of X , i.e. a section $s_\mu : X \rightarrow T^*X$. Then $s_\mu^*\alpha = \mu$.

Proof. By definition,

$$\alpha_\xi = \xi \circ d\pi.$$

So for $x \in X$, we have

$$s_\mu^*\alpha_{\mu(x)} = \mu(x) \circ d\pi \circ ds_\mu = \mu(x) \circ d(\pi \circ s_\mu) = \mu(x) \circ d(\text{id}_X) = \mu(x). \quad \square$$

Given this, we can set

$$\omega = -d\alpha = \sum_{i=1}^n dx_i \wedge d\xi_i,$$

the *canonical symplectic form* on T^*X . It is clear that it is anti-symmetric, closed, and non-degenerate, because it looks just like the canonical symplectic form on \mathbb{R}^{2n} .

Example. Take $X = S^1$, parametrized by θ , and $T^*X = S^1 \times \mathbb{R}$ with coordinates (θ, ξ_θ) . Then

$$\omega = d\theta \wedge d\xi_\theta.$$

We can see explicitly how this is independent of coordinate. Suppose we parametrized the circle by $\tau = 2\theta$. Then $d\tau = 2d\theta$. However, since we defined ξ_θ by

$$\xi = \xi_\theta(\xi) d\theta$$

for all $\xi \in T^*S^1$, we know that $\xi_\tau = \frac{1}{2}\xi_\theta$, and hence

$$d\theta \wedge d\xi_\theta = d\tau \wedge d\xi_\tau.$$

Of course, if we have a diffeomorphism $f : X \rightarrow Y$, then the pullback map induces a symplectomorphism $f^* : T^*Y \rightarrow T^*X$. However, not all symplectomorphisms arise this way.

Example. If $X = S^1$ and $T^*X = S^1 \times \mathbb{R}$ is given the canonical symplectic structure, then the vertical translation $g(\theta, \xi) = (\theta, \xi + c)$ is a symplectomorphism that is not a lift of a diffeomorphism $S^1 \rightarrow S^1$.

So when does a symplectomorphism come from a diffeomorphism?

Exercise. A symplectomorphism $y : T^*X \rightarrow T^*X$ is a lift of a diffeomorphism $X \rightarrow X$ iff $g^*\alpha = \alpha$.

1.3 Symplectomorphisms and Lagrangians

We now consider Lagrangian submanifolds. These are interesting for many reasons, but in this section, we are going to use them to understand symplectomorphisms.

Definition (Lagrangian submanifold). Let (M, ω) be a symplectic manifold, and $L \subseteq M$ a submanifold. Then L is a *Lagrangian submanifold* if for all $p \in L$, T_pL is a Lagrangian subspace of T_pM . Equivalently, the restriction of ω to L vanishes and $\dim L = \frac{1}{2} \dim M$.

Example. If (M, ω) is a surface with area form ω , and L is any 1-dimensional submanifold, then L is Lagrangian, since any 1-dimensional subspace of T_pM is Lagrangian.

What do Lagrangian submanifolds of T^*X look like? Let μ be a 1-form on X , i.e. a section of T^*X , and let

$$X_\mu = \{(x, \mu_x) : x \in X\} \subseteq T^*X.$$

The map $s_\mu : X \rightarrow T^*X$ is an embedding, and $\dim X_\mu = \dim X = \frac{1}{2} \dim T^*X$. So this is a good candidate for a Lagrangian submanifold. By definition, X_μ is Lagrangian iff $s_\mu^*\omega = 0$. But

$$s_\mu^*\omega = s_\mu^*(d\alpha) = d(s_\mu^*\alpha) = d\mu.$$

So X_μ is Lagrangian iff μ is closed. In particular, if $h \in C^\infty(X)$, then X_{dh} is a Lagrangian submanifold of T^*X . We call h the *generating function* of the Lagrangian submanifold.

There are other examples of Lagrangian submanifolds of the cotangent bundle. Let S be a submanifold of X , and $x \in X$. We define the *conormal space* of S at x to be

$$N_x^*S = \{\xi \in T_x^*X : \xi|_{T_xS} = 0\}.$$

The *conormal bundle* of S is the union of all N_x^*S .

Example. If $S = \{x\}$, then $N^*S = T_x^*X$.

Example. If $S = X$, then $N^*S = S$ is the zero section.

Note that both of these are n -dimensional submanifolds of T^*X , and this is of course true in general.

So if we are looking for Lagrangian submanifolds, this is already a good start. In the case of $S = X$, this is certainly a Lagrangian submanifold, and so is $S = \{x\}$. Indeed,

Proposition. Let $L = N^*S$ and $M = T^*X$. Then $L \hookrightarrow M$ is a Lagrangian submanifold.

Proof. If L is k -dimensional, then $S \hookrightarrow X$ locally looks like $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$, and it is clear in this case. \square

Our objective is to use Lagrangian submanifolds to understand the following problem:

Problem. Let (M_1, ω_1) and (M_2, ω_2) be $2n$ -dimensional symplectic manifolds. If $f : M_1 \rightarrow M_2$ is a diffeomorphism, when is it a symplectomorphism?

Consider the $4n$ -dimensional manifold $M_1 \times M_2$, with the *product form* $\omega = \text{pr}_1^*\omega_1 + \text{pr}_2^*\omega_2$. We have

$$d\omega = \text{pr}_1^*d\omega_1 + \text{pr}_2^*d\omega_2 = 0,$$

and ω is non-degenerate since

$$\omega^{2n} = \binom{2n}{n} \text{pr}_1^*\omega_1^{\wedge n} \wedge \text{pr}_2^*\omega_2^{\wedge n} \neq 0.$$

So ω is a symplectic form on $M_1 \times M_2$. In fact, for $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, the combination

$$\lambda_1 \text{pr}_1^*\omega_1 + \lambda_2 \text{pr}_2^*\omega_2$$

is also symplectic. The case we are particularly interested in is $\lambda_1 = 1$ and $\lambda_2 = -1$, where we get the *twisted product form*.

$$\tilde{\omega} = \text{pr}_1^*\omega_1 - \text{pr}_2^*\omega_2.$$

Let T_f be the graph of f , namely

$$T_f = \{(p_1, f(p_1)) : p_1 \in M_1\} \subseteq M_1 \times M_2.$$

and let $\gamma_f : M_1 \rightarrow M_1 \times M_2$ be the natural embedding with image T_f .

Proposition. f is a symplectomorphism iff T_f is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.

Proof. The first condition is $f^*\omega_2 = \omega_1$, and the second condition is $\gamma_f^*\tilde{\omega} = 0$, and

$$\gamma_f^*\tilde{\omega} = \gamma_f^*\text{pr}_1^*\omega_1 - \gamma_f^*\text{pr}_2^*\omega_2 = \omega_1 - f^*\omega_2. \quad \square$$

We are particularly interested in the case where M_i are cotangent bundles. If X_1, X_2 are n -dimensional manifolds and $M_i = T^*X_i$, then we can naturally identify

$$T^*(X_1 \times X_2) \cong T^*X_1 \times T^*X_2.$$

The canonical symplectic form on the left agrees with the canonical symplectic form on the right. However, we want the *twisted* symplectic form instead. So we define an involution

$$\sigma : M_1 \times M_2 \rightarrow M_1 \times M_2$$

that fixes M_1 and sends $(x, \xi) \mapsto (x, -\xi)$ on M_2 . Then $\sigma^*\omega = \tilde{\omega}$.

Then, to find a symplectomorphism $M_1 \rightarrow M_2$, we can carry out the following procedure:

- (i) Start with L a Lagrangian submanifold of $(M_1 \times M_2, \omega)$.
- (ii) Twist it to get $L^\sigma = \sigma(L)$.
- (iii) Check if L^σ is the graph of a diffeomorphism.

This doesn't sound like a very helpful specification, until we remember that we have a rather systematic way of producing Lagrangian submanifolds of $(M_1, \times M_2, \omega)$. We pick a generating function $f \in C^\infty(X_1 \times X_2)$. Then df is a closed form on $X_1 \times X_2$, and

$$L = X_{df} = \{(x, y, \partial_x f(x, y), \partial_y f(x, y)) : (x, y) \in X_1 \times X_2\}$$

is a Lagrangian submanifold of $(M_1 \times M_2, \omega)$.

The twist L^σ is then

$$L^\sigma = \{(x, y, d_x f(x, y), -d_y f(x, y)) : (x, y) \in X_1 \times X_2\}$$

We then have to check if L^σ is the graph of a diffeomorphism $\varphi : M_1 \rightarrow M_2$. If so, we win, and we call this the symplectomorphism generated by f . Equivalently, we say f is the *generating function* for φ .

To check if L^σ is the graph of a diffeomorphism, for each $(x, \xi) \in T^*X_1$, we need to find $(y, \eta) \in T^*X_2$ such that

$$\xi = \partial_x f(x, y), \quad \eta = -\partial_y f(x, y).$$

In local coordinates $\{x_i, \xi_i\}$ for T^*X_1 and $\{y_i, \eta_i\}$ for T^*X_2 , we want to solve

$$\begin{aligned} \xi_i &= \frac{\partial f}{\partial x_i}(x, y) \\ \eta_i &= -\frac{\partial f}{\partial y_i}(x, y) \end{aligned}$$

The only equation that requires solving is the first equation for y , and then the second equation tells us what we should pick η_i to be.

For the first equation to have a unique smoothly-varying solution for y , locally, we must require

$$\det \left[\frac{\partial}{\partial y_i} \left(\frac{\partial f}{\partial x_j} \right) \right]_{i,j} \neq 0.$$

This is a necessary condition for f to generate a symplectomorphism φ . If this is satisfied, then we should think a bit harder to solve the global problem of whether ξ_i is uniquely defined.

We can in fact do this quite explicitly:

Example. Let $X_1 = X_2 = \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$(x, y) \mapsto -\frac{|x - y|^2}{2} = -\frac{1}{2} \sum_{i=1}^n (x_i - y_i)^2.$$

What, if any, symplectomorphism is generated by f ? We have to solve

$$\begin{aligned} \xi_i &= \frac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i &= -\frac{\partial f}{\partial y_i} = y_i - x_i \end{aligned}$$

So we find that

$$\begin{aligned}y_i &= x_i + \xi_i, \\ \eta_i &= \xi_i.\end{aligned}$$

So we have

$$\varphi(x, \xi) = (x + \xi, \xi).$$

If we use the Euclidean inner product to identify $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, and view ξ as a vector, then φ is a free translation in \mathbb{R}^n .

That seems boring, but in fact this generalizes quite well.

Example. Let $X_1 = X_2 = X$ be a connected manifold, and g a Riemannian metric on X . We then have the Riemannian distance $d : X \times X \rightarrow \mathbb{R}$, and we can define

$$\begin{aligned}f : X \times X &\rightarrow \mathbb{R} \\ (x, y) &\mapsto -\frac{1}{2}d(x, y)^2.\end{aligned}$$

Using the Riemannian metric to identify $T^*X \cong TX$, we can check that $\varphi : TX \rightarrow TX$ is given by

$$\varphi(x, v) = (\gamma(1), \dot{\gamma}(1)),$$

where γ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Thus, φ is the geodesic flow.

1.4 Periodic points of symplectomorphisms

Symplectic geometers have an unreasonable obsession with periodic point problems.

Definition (Periodic point). Let (M, ω) be a symplectic manifold, and $\varphi : M \rightarrow M$ a symplectomorphism. An n -periodic point of φ is an $x \in M$ such that $\varphi^n(x) = x$. A periodic point is an n -periodic point for some n .

We are particularly interested in the case where $M = T^*X$ with the canonical symplectic form on M , and φ is generated by a function $f : X \rightarrow \mathbb{R}$. We begin with 1-periodic points, namely fixed points of f .

If $\varphi(x_0, \xi_0) = (x_0, \xi_0)$, this means we have

$$\begin{aligned}\xi_0 &= \partial_x f_{(x_0, x_0)} \\ \xi_0 &= -\partial_y f_{(x_0, x_0)}\end{aligned}$$

In other words, we need

$$(\partial_x + \partial_y)f_{(x_0, x_0)} = 0.$$

Thus, if we define

$$\begin{aligned}\psi : X &\rightarrow \mathbb{R} \\ x &\mapsto f(x, x),\end{aligned}$$

then we see that

Proposition. The fixed point of φ are in one-to-one correspondence with the critical points of ψ .

It is then natural to consider the same question for φ^n . One way to think about this is to consider the symplectomorphism $\tilde{\varphi} : M^n \rightarrow M^n$ given by

$$\tilde{\varphi}(m_1, m_2, \dots, m_n) = (\varphi(m_n), \varphi(m_1), \dots, \varphi(m_{n-1})).$$

Then fixed points of $\tilde{\varphi}$ correspond exactly to the n -periodic points of φ .

We now have to find a generating function for this $\tilde{\varphi}$. By inspection, we see that this works:

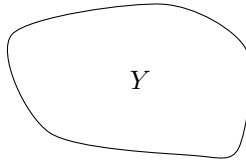
$$\tilde{f}((x_1, \dots, x_n), (y_1, \dots, y_n)) = f(x_1, y_2) + f(x_2, y_3) + \dots + f(x_n, y_1).$$

Thus, we deduce that

Proposition. The n -periodic points of φ are in one-to-one correspondence with the critical points of

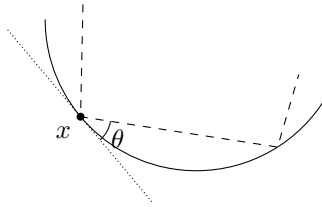
$$\psi_n(x_1, \dots, x_n) = f(x_1, x_2) + f(x_2, x_3) + \dots + f(x_n, x_1).$$

Example. We consider the problem of periodic billiard balls. Suppose we have a bounded, convex region $Y \subseteq \mathbb{R}^2$.



We put a billiard ball in Y and set it in motion. We want to see if it has periodic orbits.

Two subsequent collisions tend to look like this:



We can parametrize this collision as $(x, v = \cos \theta) \in \partial Y \times (-1, 1)$. We can think of this as the unit disk bundle of cotangent bundle on $X = \partial Y \cong S^1$ with canonical symplectic form $dx \wedge dv$, using a fixed parametrization $\chi : S^1 \cong \mathbb{R}/\mathbb{Z} \rightarrow X$. If $\varphi(x, v) = (y, w)$, then v is the projection of the unit vector pointing from x to y onto the tangent line at x . Thus,

$$v = \frac{\chi(y) - \chi(x)}{\|\chi(y) - \chi(x)\|} \cdot \frac{d\chi}{ds} \Big|_{s=x} = \frac{\partial}{\partial x} (-\|\chi(x) - \chi(y)\|),$$

and a similar formula holds for w , using that the angle of incidence is equal to the angle of reflection. Thus, we know that

$$f(x) = -\|\chi(x) - \chi(y)\|$$

is a generating function for φ .

The conclusion is that the N -periodic points are given by the critical points of

$$(x_1, \dots, x_N) \mapsto -|\chi(x_1) - \chi(x_2)| - |\chi(x_2) - \chi(x_3)| - \dots - |\chi(x_N) - \chi(x_1)|.$$

Up to a sign, this is the length of the “generalized polygon” with vertices (x_1, \dots, x_N) .

In general, if X is a compact manifold, and $\varphi : T^*X \rightarrow T^*X$ is a symplectomorphism generated by a function f , then the number of fixed points of φ are the number of fixed points of $\psi(x) = f(x, x)$. By compactness, there is at least a minimum and a maximum, so φ has at least two fixed points.

In fact,

Theorem (Poincaré’s last geometric theorem (Birkhoff, 1925)). Let $\varphi : A \rightarrow A$ be an area-preserving diffeomorphism such that φ preserves the boundary components, and twists them in opposite directions. Then φ has at least two fixed points.

1.5 Lagrangian submanifolds and fixed points

Recall what we have got so far. Let (M, ω) be a symplectic manifold, $\varphi : M \rightarrow M$. Then the graph of φ is a subset of $M \times M$. Again let $\tilde{\omega}$ be the twisted product form on $M \times M$. Then we saw that a morphism $\varphi : M \rightarrow M$ is a symplectomorphism iff the graph of φ is Lagrangian in $(M \times M, \tilde{\omega})$. Moreover, the set of fixed points is exactly $\Delta \cap \text{graph } \varphi$, where $\Delta = \text{graph id}_M$ is the diagonal.

If φ is “close to” the identity, map, then its graph is close to Δ . Thus, we are naturally led to understanding the neighbourhoods of the identity. An important theorem is the following symplectic version of the tubular neighbourhood theorem:

Theorem (Lagrangian neighbourhood theorem). Let (M, ω) be a symplectic manifold, X a compact Lagrangian submanifold, and ω_0 the canonical symplectic form on T^*X . Then there exists neighbourhoods U_0 of X in T^*X and U of X in M and a symplectomorphism $\varphi : U_0 \rightarrow U$ sending X to X .

An equivalent theorem is the following:

Theorem (Weinstein). Let M be a $2n$ -dimensional manifold, X n -dimensional compact submanifold, and $i : X \hookrightarrow M$ the inclusion, and symplectic forms ω_0, ω_1 on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e. X is Lagrangian with respect to both symplectic structures. Then there exists neighbourhoods U_0, U_1 of X in M such that $\rho|_X = \text{id}_X$ and $\rho^*\omega_1 = \omega_0$.

We first prove these are equivalent. This amounts to identifying the (dual of the) cotangent bundle of X with the normal bundle of X .

Proof of equivalence. If (V, Ω) is a symplectic vector space, L a Lagrangian subspace, the bilinear form

$$\begin{aligned} \Omega : V/L \times L &\rightarrow \mathbb{R} \\ ([v], u) &\rightarrow \Omega(v, u). \end{aligned}$$

is non-degenerate and gives a natural isomorphism $V/L \cong L^*$. Taking $V = T_p M$ and $L = T_p X$, we get an isomorphism

$$NX = TM|_X / TX \cong T^*X.$$

Thus, by the standard tubular neighbourhood theorem, there is a neighbourhood \mathcal{N}_0 of X in NX and a neighbourhood \mathcal{N} of X in M , and a diffeomorphism $\psi : \mathcal{N}_0 \rightarrow \mathcal{N}$. We now have two symplectic forms on \mathcal{N}_0 — the one from the cotangent bundle and the pullback of that from M . Then applying the second theorem gives the first.

Conversely, if we know the first theorem, then applying this twice gives us symplectomorphisms between neighbourhoods of X in M under each symplectic structure with a neighbourhood in the cotangent bundle. \square

It now remains to prove the second theorem. This is essentially an application of the relative Moser theorem, which is where the magic happens. The bulk of the proof is to get ourselves into a situation where relative Moser applies.

Proof of second theorem. For $p \in X$, we define $V = T_p M$ and $U = T_p X$, and W any complement of U . By assumption, U is a Lagrangian subspace of both $(V, \omega_0|_p = \Omega_0)$ and $(V, \omega_1|_p = \Omega_1)$. We apply the following linear-algebraic lemma:

Lemma. Let V be a $2n$ -dimensional vector space, Ω_0, Ω_1 symplectic structures on V . Suppose U is a subspace of V Lagrangian with respect to both Ω_0 and Ω_1 , and W is any complement of U . Then we can construct canonically a linear isomorphism $H : V \rightarrow V$ such that $H|_U = \text{id}_U$ and $H^* \Omega_1 = \Omega_0$.

Note that the statement of the theorem doesn't mention W , but the construction of H requires a complement of U , so it is canonical only after we pick a W .

By this lemma, we get canonically an isomorphism $H_p : T_p M \rightarrow T_p M$ such that $H_p|_{T_p X} = \text{id}_{T_p X}$ and $H_p^* \omega_1|_p = \omega_0|_p$. The canonicity implies H_p varies smoothly with p . We now apply the Whitney extension theorem

Theorem (Whitney extension theorem). Let X be a submanifold of M , $H_p : T_p M \rightarrow T_p M$ smooth family of isomorphisms such that $H_p|_{T_p X} = \text{id}_{T_p X}$. Then there exists a neighbourhood \mathcal{N} of X in M and an embedding $h : \mathcal{N} \rightarrow M$ such that $h|_X = \text{id}_X$ and for all $p \in X$, $dh_p = H_p$.

So at $p \in X$, we have $h^* \omega_1|_p = (dh_p)^* \omega_1|_p = h_p^* \omega_1|_p = \omega_0|_p$. So we are done by relative Moser. \square

Example. We can use this result to understand the neighbourhood of the identity in the group $\text{Symp}(M, \omega)$ of symplectomorphisms of a symplectic manifold (M, ω) .

Suppose $\varphi, \text{id} \in \text{Symp}(M, \omega)$. Then the graphs Γ_φ and $\Delta = \Gamma_{\text{id}}$ are Lagrangian submanifolds of $(M \times M, \tilde{\omega})$. Then by our theorem, there is a neighbourhood \mathcal{U} of Δ in $(M \times M, \tilde{\omega})$ that is symplectomorphic to a neighbourhood \mathcal{U}_0 of the zero section of (T^*M, ω_0) .

Suppose φ is sufficiently C^0 -close to id . Then $\Gamma_\varphi \subseteq \mathcal{U}$. If φ is sufficiently C^1 -close to the identity, then the image of Γ_φ in \mathcal{U}_0 is a smooth section X_μ for some 1-form μ .

Now φ is a symplectomorphism iff Γ_φ is Lagrangian iff X_μ is Lagrangian iff μ is closed. So a small C^1 -neighbourhood of id in $\text{Sym}(M, \omega)$ is “the same as” a small C^1 -neighbourhood of the zero-section in the space of closed 1-forms on X .

We can also use this to understand fixed points of symplectomorphisms (again!).

Theorem. Let (M, ω) be a compact symplectic manifold such that $H_{\text{dR}}^1(M) = 0$. Then any symplectomorphism $\varphi : M \rightarrow M$ sufficiently close to the identity has at least two fixed points.

Proof. The graph of φ corresponds to a closed 1-form on M . Since μ is closed and $H_{\text{dR}}^1(M) = 0$, we know $\mu = dh$ for some $h \in C^\infty$. Since M is compact, h has at least two critical points (the global maximum and minimum). Since the fixed points corresponding to the points where μ vanish (so that Γ_φ intersects Δ), we are done. \square

Counting fixed points of symplectomorphisms is a rather popular topic in symplectic geometry, and Arnold made some conjectures about these. A version of this is

Theorem (Arnold conjecture). Let (M, ω) be a compact symplectic manifold of dimension $2n$, and $\varphi : M \rightarrow M$ a symplectomorphism. Suppose φ is *exactly homotopic* to the identity and *non-degenerate*. Then the number of fixed points of φ is at least $\sum_{i=0}^{2n} \dim H^i(M, \mathbb{R})$.

We should think of the sum $\sum_{i=0}^{2n} \dim H^i(M, \mathbb{R})$ as the minimal number of critical points of a function, as Morse theory tells us.

We ought to define the words we used in the theorem:

Definition (Exactly homotopic). We say φ is *exactly homotopic* to the identity if there is isotopy $\rho_t : M \rightarrow M$ such that $\rho_0 = \text{id}$ and $\rho_1 = \varphi$, and further there is some 1-periodic family of functions h_t such that ρ_t is generated by the vector field v_t defined by $\iota_{v_t}^* \omega = dh_t$.

The condition $\iota_{v_t}^* \omega = dh_t$ says v_t is a Hamiltonian vector field, which we will discuss soon after this.

Definition (Non-degenerate function). A endomorphism $\varphi : M \rightarrow M$ is non-degenerate iff all its fixed points are non-degenerate, i.e. if p is a fixed point, then $\det(\text{id} - d\varphi_p) \neq 0$.

Example. In the original statement of the Arnold conjecture, which is the case $(T^2, d\theta_1 \wedge d\theta_2)$, any symplectomorphism has at least 4 fixed points.

In the case where h_t is actually not time dependent, Arnold’s conjecture is easy to prove.

2 Complex structures

2.1 Almost complex structures

Symplectic manifolds are very closely related to complex manifolds. A first (weak) hint of this fact is that they both have to be even dimensional! In general, symplectic manifolds will have *almost* complex structures, but this need not actually come from a genuine complex structure. If it does, then we say it is *Kähler*, which is a very strong structure on the manifold.

We begin by explaining what almost complex structures are, again starting from the linear algebraic side of the story. A complex vector space can be thought of as a real vector space with a linear endomorphism that acts as “multiplication by i ”.

Definition (Complex structure). Let V be a vector space. A *complex structure* is a linear $J : V \rightarrow V$ with $J^2 = -1$.

Here we call it a complex structure. When we move on to manifolds, we will call this “almost complex”, and a complex structure is a stronger condition. It is clear that

Lemma. There is a correspondence between real vector spaces with a complex structure and complex vector spaces, where J acts as multiplication by i . \square

Our symplectic manifolds come with symplectic forms on the tangent space. We require the following compatibility condition:

Definition (Compatible complex structure). Let (V, Ω) be a symplectic vector space, and J a complex structure on V . We say J is *compatible* with Ω if $G_J(u, v) = \Omega(u, Jv)$ is an inner product. In other words, we need

$$\Omega(Ju, Jv) = \Omega(u, v), \quad \Omega(v, Jv) \geq 0$$

with equality iff $v = 0$.

Example. On the standard symplectic vector space $(\mathbb{R}^{2n}, \Omega)$, we set

$$J_0(e_i) = f_i, \quad J_0(f_i) = -e_i.$$

We can then check that this is compatible with the symplectic structure, and in fact gives the standard inner product.

Proposition (Polar decomposition). Let (V, Ω) be a symplectic vector space, and G an inner product on V . Then from G , we can *canonically* construct a compatible complex structure J on (V, Ω) . If $G = G_J$ for some J , then this process returns J .

Note that in general, $G_J(u, v) = \Omega(u, Jv) \neq G(u, v)$.

Proof. Since Ω, G are non-degenerate, we know

$$\Omega(u, v) = G(u, Av)$$

for some $A : V \rightarrow V$. If $A^2 = -1$, then we are done, and set $J = A$. In general, we also know that A is skew-symmetric, i.e. $G(Au, v) = G(u, -Av)$, which is

clear since Ω is anti-symmetric. Since AA^t is symmetric and positive definite, it makes sense to write down $\sqrt{AA^T}$ (e.g. by diagonalizing), and we take

$$J = \sqrt{AA^t}^{-1} A = \sqrt{-A^2}^{-1} A.$$

It is clear that $J^2 = -1$, since A commutes with $\sqrt{AA^t}$, so this is a complex structure. We can write this as $A = \sqrt{AA^T} J$, and this is called the (*left*) *polar decomposition* of A .

We now check that J is a compatible, i.e. $G_J(u, v) = \Omega(u, Jv)$ is symmetric and positive definite. But

$$G_J(u, v) = G(u, \sqrt{AA^t} v),$$

and we are done since $\sqrt{AA^t}$ is positive and symmetric. \square

Notation. Let (V, Ω) be a symplectic vector space. We write $\mathcal{J}(V, \Omega)$ for the space of all compatible complex structures on (V, Ω) .

Proposition. $\mathcal{J}(V, \Omega)$ is path-connected.

Proof. Let $J_0, J_1 \in \mathcal{J}(V, \Omega)$. Then this induces inner products G_{J_0}, G_{J_1} . Let $G_t = (1-t)G_{J_0} + tG_{J_1}$ be a smooth family of inner products on V . Then apply polar decomposition to get a family of complex structures that start from J_0 to J_1 . \square

A quick adaptation of the proof shows it is in fact contractible.

We now move on to the case of manifolds.

Definition (Almost complex structure). An *almost complex structure* J on a manifold is a smooth field of complex structures on the tangent space $J_p : T_p M \rightarrow T_p M$, $J_p^2 = -1$.

Example. Suppose M is a complex manifold with local complex coordinates z_1, \dots, z_n on $U \subseteq M$. We have real coordinates $\{x_i, y_i\}$ given by

$$z_i = x_i + iy_i.$$

The tangent space is spanned by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$. We define J by

$$J_p \left(\frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j}, \quad J_p \left(\frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j}.$$

The Cauchy–Riemann equations imply this is globally well-defined. This J is called the *canonical almost-complex structure* on the complex manifold M .

Definition (Integrable almost complex structure). An almost complex structure on M is called *integrable* if it is induced by a complex structure.

Example. It is a fact that $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ has an almost complex but no complex structure.

Definition (Compatible almost complex structure). An almost complex structure J on M is *compatible* with a symplectic structure ω if J_p is compatible with ω_p for all $p \in M$. In this case, (ω, g_J, J) is called a *compatible triple*.

Any two of the structures determine the three, and this gives rise to the nice fact that the intersection of any two of $O(2n)$, $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ is $\mathrm{U}(n)$.

Performing polar decomposition pointwise gives

Proposition. Let (M, ω) be a symplectic manifold, and g a metric on M . Then from g we can canonically construct a compatible almost complex structure J . \square

As before, in general, $g_J(\cdot, \cdot) \neq g(\cdot, \cdot)$.

Corollary. Any symplectic manifold has a compatible almost complex structure. \square

The converse does not hold. For example, S^6 is almost complex but not symplectic.

Notation. Let (M, ω) be a symplectic manifold. We write $\mathcal{J}(V, \Omega)$ for the space of all compatible almost complex structures on (M, ω) .

The same proof as before shows that

Proposition. $\mathcal{J}(M, \omega)$ is contractible. \square

Proposition. Let J be an almost complex structure on M that is compatible with ω_0 and ω_1 . Then ω_0 and ω_1 are deformation equivalent.

Proof. Check that $\omega_t = (1-t)\omega_0 + t\omega_1$ works, which is non-degenerate since $\omega_t(\cdot, J\cdot)$ is a positive linear combination of inner products, hence is non-degenerate. \square

Proposition. Let (M, ω) be a symplectic manifold, J a compatible almost complex structure. If X is an almost complex submanifold of (M, J) , i.e. $J(TX) = TX$, then X is a symplectic submanifold of (M, ω) .

Proof. We only have to check $\omega|_{TX}$ is non-degenerate, but $\Omega(\cdot, J\cdot)$ is a metric, so is in particular non-degenerate. \square

We shall not prove the following theorem:

Theorem (Gromov). Let (M, J) be an almost complex manifold with M open, i.e. M has no closed connected components. Then there exists a symplectic form ω in any even 2-cohomology class and such that J is homotopic to an almost complex structure compatible with ω . \square

2.2 Dolbeault theory

An almost complex structure on a manifold allows us to discuss the notion of holomorphicity, and this will in turn allow us to stratify our k -forms in terms of “how holomorphic” they are.

Let (M, J) be an almost complex manifold. By complexifying TM to $TM \otimes \mathbb{C}$ and then extending J linearly, we can split $TM \otimes \mathbb{C}$ into its $\pm i$ eigenspace.

Notation. We write $T_{1,0}$ for the $+i$ eigenspace of J and $T_{0,1}$ for the $-i$ eigenspace of J . These are called the *J-holomorphic tangent vectors* and the *J-anti-holomorphic tangent vectors* respectively.

We write $T_{1,0}$ for the $+i$ eigenspace of J , and $T_{0,1}$ the $-i$ -eigenspace of J . Then we have a splitting

$$TM \otimes \mathbb{C} \xrightarrow{\sim} T_{1,0} \oplus T_{0,1}.$$

We can explicitly write down the projection maps as

$$\begin{aligned}\pi_{1,0}(w) &= \frac{1}{2}(w - iJw) \\ \pi_{0,1}(w) &= \frac{1}{2}(w + iJw).\end{aligned}$$

Example. On a complex manifold with local complex coordinates (z_1, \dots, z_n) , the holomorphic tangent vectors $T_{1,0}$ is spanned by the $\frac{\partial}{\partial z_j}$, while $T_{0,1}$ is spanned by the $\frac{\partial}{\partial \bar{z}_j}$.

Similarly, we can complexify the cotangent bundle, and considering the $\pm i$ eigenspaces of (the dual of) J gives a splitting

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\sim} T^{1,0} \oplus T^{0,1}.$$

These are the *complex linear cotangent vectors* and *complex anti-linear cotangent vectors*.

Example. In the complex case, $T^{1,0}$ is spanned by the dz_j and $T^{0,1}$ is spanned by the $d\bar{z}_j$.

More generally, we can decompose

$$\bigwedge^k (T^*M \otimes \mathbb{C}) = \bigwedge^k (T^{1,0} \oplus T^{0,1}) = \bigoplus_{p+q=k} (\bigwedge^p T^{1,0}) \otimes (\bigwedge^q T^{0,1}) \equiv \bigoplus_{p+q=k} \Lambda^{p,q}.$$

We write

$$\begin{aligned}\Omega^k(M, \mathbb{C}) &= \text{sections of } \bigwedge^k (T^*M \otimes \mathbb{C}) \\ \Omega^{p,q}(M, \mathbb{C}) &= \text{sections of } \Lambda^{p,q}.\end{aligned}$$

So

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M, \mathbb{C}).$$

The sections in $\Omega^{p,q}(M, \mathbb{C})$ are called *forms of type (p, q)* .

Example. In the complex case, we have, locally,

$$\Omega_p^{p,q} = \mathbb{C}\{dz_I \wedge dz_K : |I| = \ell, |K| = m\}$$

and

$$\Omega^{p,q} = \left\{ \sum_{|I|=p, |K|=q} b_{I,K} dz_I \wedge d\bar{z}_K : b_{I,K} \in C^\infty(U, \mathbb{C}) \right\}.$$

As always, we have projections

$$\pi^{p,q} : \bigwedge^k(T^*M \otimes \mathbb{C}) \rightarrow \Lambda^{p,q}.$$

Combining the exterior derivative $d : \Omega^k(M, \mathbb{C}) \rightarrow \Omega^{k-1}(M, \mathbb{C})$ with projections yield the ∂ and $\bar{\partial}$ operators

$$\begin{aligned} \partial : \Omega^{p,q} &\rightarrow \Omega^{p+1,q} \\ \bar{\partial} : \Omega^{p,q} &\rightarrow \Omega^{p,q+1}. \end{aligned}$$

Observe that for functions f , we have $df = \partial f + \bar{\partial} f$, but this is not necessarily true in general.

Definition (*J*-holomorphic). We say a function f is *J-holomorphic* if $\bar{\partial} f = 0$, and *J-anti-holomorphic* if $\partial f = 0$.

It would be very nice if the sequence

$$\Omega^{p,q} \xrightarrow{\bar{\partial}} \Omega^{p,q+1} \xrightarrow{\bar{\partial}} \Omega^{p,q+2} \xrightarrow{\bar{\partial}} \dots$$

were a chain complex, i.e. $\bar{\partial}^2 = 0$. This is not always true. However, this is true in the case where J is integrable. Indeed, if M is a complex manifold and $\beta \in \Omega^k(M, \mathbb{C})$, then in local complex coordinates, we can write

$$\beta = \sum_{p+q=k} \left(\sum_{|I|=p, |K|=q} b_{I,K} dz_I \wedge d\bar{z}_K \right).$$

So

$$\begin{aligned} d\beta &= \sum_{p+q=k} \left(\sum_{|I|=p, |K|=q} db_{I,K} \wedge dz_I \wedge d\bar{z}_K \right) \\ &= \sum_{p+q=k} \left(\sum_{|I|=p, |K|=q} (\partial + \bar{\partial})b_{I,K} \wedge dz_I \wedge d\bar{z}_K \right) \\ &= (\partial + \bar{\partial}) \sum_{p+q=k} \left(\sum_{|I|=p, |K|=q} b_{I,K} dz_I \wedge d\bar{z}_K \right) \end{aligned}$$

Thus, on a complex manifold, $d = \partial + \bar{\partial}$.

Thus, if $\beta \in \Omega^{p,q}$, then

$$0 = d^2\beta = \partial^2\beta + (\partial\bar{\partial} + \bar{\partial}\partial)\beta + \bar{\partial}^2\beta.$$

Since the three terms are of different types, it follows that

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

In fact, the converse of this computation is also true, which we will not prove:

Theorem (Newlander–Nirenberg). The following are equivalent:

- $\bar{\partial}^2 = 0$ - J is integrable
- $\partial^2 = 0$ - $\mathcal{N} = 0$
- $d = \partial + \bar{\partial}$

where \mathcal{N} is the *Nijenhuis torsion*

$$\mathcal{N}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]. \quad \square$$

When our manifold is complex, we can then consider the cohomology of the chain complex.

Definition (Dolbeault cohomology groups). Let (M, J) be a manifold with an integrable almost complex structure. The *Dolbeault cohomology groups* are the cohomology groups of the cochain complex

$$\Omega^{p,q} \xrightarrow{\bar{\partial}} \Omega^{p,q+1} \xrightarrow{\bar{\partial}} \Omega^{p,q+2} \xrightarrow{\bar{\partial}} \dots$$

Explicitly,

$$H_{\text{Dolb}}^{p,q}(M) = \frac{\ker(\bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1})}{\text{im}(\bar{\partial} : \Omega^{p,q-1} \rightarrow \Omega^{p,q})}.$$

2.3 Kähler manifolds

In the best possible scenario, we would have a symplectic manifold with a compatible, integrable almost complex structure. Such manifolds are called *Kähler* manifolds.

Definition (Kähler manifold). A *Kähler manifold* is a symplectic manifold equipped with a compatible integrable almost complex structure. Then ω is called a *Kähler form*.

Let (M, ω, J) be a Kähler manifold. What can we say about the Kähler form ω ? We can decompose it as

$$\omega \in \Omega^2(M) \subseteq \Omega^2(M, \mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}.$$

We claim that

Lemma. $\omega \in \Omega^{1,1}$.

Proof. Since $\omega(\cdot, J\cdot)$ is symmetric, we have

$$J^*\omega(u, v) = \omega(Ju, Jv) = \omega(v, JJu) = -\omega(v, -u) = \omega(u, v).$$

So $J^*\omega = \omega$.

On the other hand, J^* acts on holomorphic forms as multiplication by i and anti-holomorphic forms by multiplication by -1 (by definition). So it acts on $\Omega^{2,0}$ and $\Omega^{0,2}$ by multiplication by -1 (locally $\Omega^{2,0}$ is spanned by $dz_i \wedge dz_j$, etc.), while it fixes $\Omega^{1,1}$. So ω must lie in $\Omega^{1,1}$. \square

We can explore what the other conditions on ω tell us. Closedness implies

$$0 = d\omega = \partial\omega + \bar{\partial}\omega = 0.$$

So we have $\partial\omega = \bar{\partial}\omega = 0$. So in particular $\omega \in H_{\text{Dolb}}^{1,1}(M)$.

In local coordinates, we can write

$$\omega = \frac{i}{2} \sum_{j,k} h_{j,k} dz_j \wedge d\bar{z}_k$$

for some $h_{j,k}$. The fact that ω is real valued, so that $\bar{\omega} = \omega$ gives some constraints on what the h_{jk} can be. We compute

$$\bar{\omega} = -\frac{i}{2} \sum_{j,k} \overline{h_{jk}} d\bar{z}_j \wedge dz_k = \frac{i}{2} \sum_{j,k} \overline{h_{jk}} dz_k \wedge \bar{z}_j.$$

So we have

$$\bar{h}_{kj} = h_{jk}.$$

The non-degeneracy condition $\omega^{\wedge n} \neq 0$ is equivalent to $\det h_{jk} \neq 0$, since

$$\omega^n = \left(\frac{i}{2}\right)^n n! \det(h_{jk}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

So h_{jk} is a non-singular Hermitian matrix.

Finally, we take into account the compatibility condition $\omega(v, Jv) > 0$. If we write

$$v = \sum_j a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j},$$

then we have

$$Jv = i \left(\sum_j a_j \frac{\partial}{\partial z_j} - b_j \frac{\partial}{\partial \bar{z}_j} \right).$$

So we have

$$\omega(v, Jv) = \frac{i}{2} \sum h_{jk} (-ia_j b_k - ia_j b_k) = \sum h_{jk} a_j b_k > 0.$$

So the conclusion is that h_{jk} is positive definite.

Thus, the conclusion is

Theorem. A Kähler form ω on a complex manifold M is a ∂ - and $\bar{\partial}$ -closed form of type $(1, 1)$ which on a local chart is given by

$$\omega = \frac{i}{2} \sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k$$

where at each point, the matrix (h_{jk}) is Hermitian and positive definite.

Often, we start with a complex manifold, and want to show that it has a Kähler form. How can we do so? First observe that we have the following proposition:

Proposition. Let (M, ω) be a complex Kähler manifold. If $X \subseteq M$ is a complex submanifold, then $(X, i^*\omega)$ is Kähler, and this is called a *Kähler submanifold*. \square

In particular, if we can construct Kähler forms on \mathbb{C}^n and $\mathbb{C}\mathbb{P}^n$, then we have Kähler forms for a lot of our favorite complex manifolds, and in particular complex projective varieties.

But we still have to construct some Kähler forms to begin with. To do so, we use so-called *strictly plurisubharmonic functions*.

Definition (Strictly plurisubharmonic (spsh)). A function $\rho \in C^\infty(M, \mathbb{R})$ is strictly plurisubharmonic (spsh) if locally, $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)$ is positive definite.

Proposition. Let M be a complex manifold, and $\rho \in C^\infty(M; \mathbb{R})$ strictly plurisubharmonic. Then

$$\omega = \frac{i}{2} \partial \bar{\partial} \rho$$

is a Kähler form.

We call ρ the *Kähler potential* for ω .

Proof. ω is indeed a 2-form of type $(1, 1)$. Since $\partial^2 = \bar{\partial}^2 = 0$ and $\partial \bar{\partial} = -\bar{\partial} \partial$, we know $\partial \omega = \bar{\partial} \omega = 0$. We also have

$$\omega = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k,$$

and the matrix is Hermitian positive definite by assumption. \square

Example. If $M = \mathbb{C}^n \cong \mathbb{R}^{2n}$, we take

$$\rho(\mathbf{z}) = |\mathbf{z}|^2 = \sum z_k \bar{z}_k.$$

Then we have

$$h_{jk} = \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = \delta_{jk},$$

so this is strictly plurisubharmonic. Then

$$\begin{aligned} \omega &= \frac{i}{2} \sum dz_j \wedge d\bar{z}_k \\ &= \frac{i}{2} \sum d(x_j + iy_j) \wedge d(x_k - iy_k) \\ &= \sum dx_k \wedge dy_k, \end{aligned}$$

which is the standard symplectic form. So (\mathbb{C}^n, ω) is Kähler and $\rho = |\mathbf{z}|^2$ is a (global) Kähler potential for ω_0 .

There is a local converse to this result.

Proposition. Let M be a complex manifold, ω a closed real-valued $(1, 1)$ -form and $p \in M$, then there exists a neighbourhood U of p in M and a $\rho \in C^\infty(U, \mathbb{R})$ such that

$$\omega = i\partial\bar{\partial}\rho \text{ on } U.$$

Proof. This uses the holomorphic version of the Poincaré lemma. □

When ρ is Kähler, such a ρ is called a *local Kähler potential* for ω .

Note that it is not possible to have a global Kähler potential on a closed Kähler manifold, because if $\omega = \frac{i}{2}\partial\bar{\partial}\rho$, then

$$\omega = d\left(\frac{i}{2}\bar{\partial}\rho\right)$$

is exact, and we know symplectic forms cannot be exact.

Example. Let $M = \mathbb{C}^n$ and

$$\rho(z) = \log(|z|^2 + 1).$$

It is an exercise to check that ρ is strictly plurisubharmonic. Then

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}(\log(|z|^2 + 1))$$

is another Kähler form on \mathbb{C}^n , called the *Fubini–Study form*.

The reason this is interesting is that it allows us to put a Kähler structure on $\mathbb{C}\mathbb{P}^n$.

Example. Let $M = \mathbb{C}\mathbb{P}^n$. Using homogeneous coordinates, this is covered by the open sets

$$U_j = \{[z_0, \dots, z_n] \in \mathbb{C}\mathbb{P}^n \mid z_j \neq 0\}.$$

with the chart given by

$$\begin{aligned} \varphi_j : U_j &\rightarrow \mathbb{C}^n \\ [z_0, \dots, z_n] &\mapsto \left(\frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j}\right). \end{aligned}$$

One can check that $\varphi_j^*\omega_{FS} = \varphi_k^*\omega_{FS}$. Thus, these glue to give the *Fubini–Study form* on $\mathbb{C}\mathbb{P}^n$, making it a Kähler manifold.

2.4 Hodge theory

So what we have got so far is that if we have a complex manifold, then we can decompose

$$\Omega^k(M; \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q},$$

and using $\bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}$, we defined the Dolbeault cohomology groups

$$H_{\text{Dolb}}^{p,q}(M) = \frac{\ker \bar{\partial}}{\text{im } \bar{\partial}}.$$

It would be nice if we also had a decomposition

$$H_{\text{dR}}^k \cong \bigoplus_{p+q=k} H_{\text{Dolb}}^{p,q}(M).$$

This is not always true. However, it is true for compact Kähler manifolds:

Theorem (Hodge decomposition theorem). Let (M, ω) be a compact Kähler manifold. Then

$$H_{\text{dR}}^k \cong \bigoplus_{p+q=k} H_{\text{Dolb}}^{p,q}(M).$$

To prove the theorem, we will first need a “real” analogue of the theorem. This is an analytic theorem that lets us find canonical representatives for each cohomology class. We can develop the same theory for Dolbeault cohomology, and see that the canonical representatives for Dolbeault cohomology are the same as those for de Rham cohomology. We will not prove the analytic theorems, but just say how these things piece together.

Real Hodge theory

Let V be a real oriented vector space m with inner product G . Then this induces an inner product on each $\Lambda^k = \bigwedge^k(V)$, denoted $\langle \cdot, \cdot \rangle$, defined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(G(v_i, w_j))_{i,j}$$

Let e_1, \dots, e_n be an oriented orthonormal basis for V . Then

$$\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \leq j_1 < \cdots < j_k \leq m\}$$

is an orthonormal basis for Λ^k .

Definition (Hodge star). The Hodge $*$ -operator $*$: $\Lambda^k \rightarrow \Lambda^{m-k}$ is defined by the relation

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle e_1 \wedge \cdots \wedge e_m.$$

It is easy to see that the Hodge star is in fact an isomorphism. It is also not hard to verify the following properties:

Proposition.

- $*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_m$
- $*(e_{k+1} \wedge \cdots \wedge e_m) = (-1)^{k(m-k)} e_1 \wedge \cdots \wedge e_k$.
- $** = \alpha = (-1)^{k(m-k)} \alpha$ for $\alpha \in \Lambda^k$. □

In general, let (M, g) be a compact real oriented Riemannian manifold of dimension m . Then g gives an isomorphism $TM \cong T^*M$, and induces an inner product on each T_p^*M , which we will still denote g_p . This induces an inner product on each $\bigwedge^k T_p^*M$, which we will denote $\langle \cdot, \cdot \rangle$ again.

The Riemannian metric and orientation gives us a volume form $\text{Vol} \in \Omega^m(M)$, defined locally by

$$\text{Vol}_p(e_1 \wedge \cdots \wedge e_m),$$

where e_1, \dots, e_m is an oriented basis of T_p^*M . This induces an L^2 -inner product on $\Omega^k(M)$,

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle \text{Vol}.$$

Now apply Hodge $*$ -operator to each $(V, G) = (T_p^*M, g_p)$ and $p \in M$. We then get

Definition (Hodge star operator). The *Hodge *-operator* on forms $*$: $\Omega^k(M) \rightarrow \Omega^{m-k}(M)$ is defined by the equation

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \text{Vol.}$$

We again have some immediate properties.

Proposition.

(i) $**\alpha = (-1)^{k(m-k)}\alpha$ for $\alpha \in \Omega^k(M)$.

(ii) $*1 = \text{Vol}$ □

We now introduce the codifferential operator

Definition (Codifferential operator). We define the *codifferential operator* δ : $\Omega^k \rightarrow \Omega^{k-1}$ to be the L^2 -formal adjoint of d . In other words, we require

$$\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2}$$

for all $\alpha \in \Omega^k$ and $\beta \in \Omega^{k+1}$.

We immediately see that

Proposition. $\delta^2 = 0$.

Using the Hodge star, there is a very explicit formula for the codifferential (which in particular shows that it exists).

Proposition.

$$\delta = (-1)^{m(k+1)+1} * d * : \Omega^k \rightarrow \Omega^{k-1}.$$

Proof.

$$\begin{aligned} \langle d\alpha, \beta \rangle_{L^2} &= \int_M d\alpha \wedge *\beta \\ &= \int_M d(\alpha \wedge *\beta) - (-1)^k \int_M \alpha \wedge d(*\beta) \\ &= (-1)^{k+1} \int_M \alpha \wedge d(*\beta) \quad (\text{Stokes'}) \\ &= (-1)^{k+1} \int_M (-1)^{(m-k)k} \alpha \wedge **d(*\beta) \\ &= (-1)^{k+1+(m-k)k} \int_M \langle \alpha, *d*\beta \rangle. \quad \square \end{aligned}$$

We can now define the Laplace–Beltrami operator

Definition (Laplace–Beltrami operator). We define the *Laplacian*, or the *Laplace–Beltrami operator* to be

$$\Delta = d\delta + \delta d : \Omega^k \rightarrow \Omega^k.$$

Example. If $M = \mathbb{R}^m$ with the Euclidean inner product, and $f \in \Omega^0(M) = C^\infty(M)$, then

$$\Delta f = - \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

It is an exercise to prove the following

Proposition.

$$(i) \quad \Delta * = * \Delta : \Omega^k \rightarrow \Omega^{m-k}$$

$$(ii) \quad \Delta = (d + \delta)^2$$

$$(iii) \quad \langle \Delta \alpha, \beta \rangle_{L^2} = \langle \alpha, \Delta \beta \rangle_{L^2}.$$

$$(iv) \quad \Delta \alpha = 0 \text{ iff } d\alpha = \delta \alpha = 0. \quad \square$$

In particular, (iv) follows from the fact that

$$\langle \Delta \alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta \alpha, \delta \alpha \rangle = \|d\alpha\|_{L^2}^2 + \|\delta \alpha\|_{L^2}^2.$$

Similar to IA Vector Calculus, we can define

Definition (Harmonic form). A form α is *harmonic* if $\Delta \alpha = 0$. We write

$$\mathcal{H}^k = \{\alpha \in \Omega^k(m) \mid \Delta \alpha = 0\}$$

for the space of harmonic forms.

Observe there is a natural map $\mathcal{H}^k \rightarrow H_{\text{dR}}^k(M)$, sending α to $[\alpha]$. The main result is that

Theorem (Hodge decomposition theorem). Let (M, g) be a compact oriented Riemannian manifold. Then every cohomology class in $H_{\text{dR}}^k(M)$ has a unique harmonic representation, i.e. the natural map $\mathcal{H}^k \rightarrow H_{\text{dR}}^k(M)$ is an isomorphism.

We will not prove this.

Complex Hodge theory

We now see what we can do for complex Kähler manifolds. First check that

Proposition. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$ and (M, ω) Kähler. Then

$$(i) \quad * : \Omega^{p,q} \rightarrow \Omega^{n-p, n-q}.$$

$$(ii) \quad \Delta : \Omega^{p,q} \rightarrow \Omega^{p,q}. \quad \square$$

define the L^2 -adjoints $\bar{\partial}^* = \pm * \bar{\partial} *$ and $\partial^* = - - * \partial *$ with the appropriate signs as before, and then

$$d = \partial + \bar{\partial}, \quad \delta = \partial^* + \bar{\partial}^*.$$

We can then define

$$\Delta_{\partial} = \partial \partial^* + \partial^* \partial : \Omega^{p,q} \rightarrow \Omega^{p,q}$$

$$\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q}.$$

Proposition. If our manifold is Kähler, then

$$\Delta = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}. \quad \square$$

So if we have a harmonic form, then it is in fact ∂ and $\bar{\partial}$ -harmonic, and in particular it is ∂ and $\bar{\partial}$ -closed. This gives us a *Hodge decomposition*

$$\mathcal{H}_{\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q},$$

where

$$\mathcal{H}^{p,q} = \{\alpha \in \Omega^{p,q}(M) : \Delta\alpha = 0\}.$$

Theorem (Hodge decomposition theorem). Let (M, ω) be a compact Kähler manifold. The natural map $\mathcal{H}^{p,q} \rightarrow H_{\text{Dolb}}^{p,q}$ is an isomorphism. Hence

$$H_{\text{dR}}^k(M; \mathbb{C}) \cong \mathcal{H}_{\mathbb{C}}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong \bigoplus_{p+q=k} H_{\text{Dolb}}^{p,q}(M). \quad \square$$

What are some topological consequences of this? Recall the *Betti numbers* are defined by

$$b_k = \dim_{\mathbb{R}} H_{\text{dR}}^k(M) = \dim_{\mathbb{C}} H_{\text{dR}}^k(M; \mathbb{C}).$$

We can further define the *Hodge numbers*

$$h_{p,q} = \dim_{\mathbb{C}} H_{\text{Dolb}}^{p,q}(M).$$

Then the Hodge theorem says

$$b_k = \sum_{p+q=k} h_{p,q}.$$

Moreover, since

$$H_{\text{Dolb}}^{p,q}(M) = \overline{H_{\text{Dolb}}^{q,p}(M)}.$$

So we have *Hodge symmetry*,

$$h_{p,q} = h_{q,p}.$$

Moreover, the $*$ operator induces an isomorphism

$$H_{\text{Dolb}}^{p,q} \cong H_{\text{Dolb}}^{n-p, n-q}.$$

So we have

$$h_{p,q} = h_{n-p, n-q}.$$

There is called *central symmetry*, or *Serre duality*. Thus, we know that

Corollary. Odd Betti numbers are even.

Proof.

$$b_{2k+1} = \sum_{p+q=2k+1} h_{p,q} = 2 \left(\sum_{p=0}^k h_{p, 2k+1-p} \right). \quad \square$$

Corollary. $h_{1,0} = h_{0,1} = \frac{1}{2}b_1$ is a topological invariant.

We have also previously seen that for a general compact Kähler manifold, we have

Proposition. Even Betti numbers are positive.

Recall that we proved this by arguing that $[\omega^k] \neq 0 \in H^{2k}(M)$. In fact, $\omega^k \in H_{\text{Dolb}}^{k,k}(M)$, and so

Proposition. $h_{k,k} \neq 0$. □

We usually organize these $h_{p,q}$ in the form of a *Hodge diamond*, e.g.

$$\begin{array}{ccccc} & & h_{2,2} & & \\ & h_{2,1} & & h_{1,2} & \\ h_{2,0} & & h_{1,1} & & h_{0,2} \\ & h_{1,0} & & h_{0,1} & \\ & & h_{0,0} & & \end{array}$$

3 Hamiltonian vector fields

Symplectic geometry was first studied by physicists, who modeled their systems by a symplectic manifold. The *Hamiltonian function* $H \in C^\infty(M)$, which returns the energy of a particular configuration, generates a vector field on M which is the equation of motion of the system. In this chapter, we will discuss how this process works, and further study what happens when we have families of such Hamiltonian functions, giving rise to Lie group actions.

3.1 Hamiltonian vector fields

Definition (Hamiltonian vector field). Let (M, ω) be a symplectic manifold. If $H \in C^\infty(M)$, then since $\tilde{\omega} : TM \rightarrow T^*M$ is an isomorphism, there is a unique vector field X_H on M such that

$$\iota_{X_H}\omega = dH.$$

We call X_H the *Hamiltonian vector field* with *Hamiltonian function* H .

Suppose X_H is complete (e.g. when M is compact). This means we can integrate X_H , i.e. solve

$$\frac{\partial \rho_t}{\partial t}(p) = X_H(\rho_t(p)), \quad \rho_0(p) = p.$$

These flow have some nice properties.

Proposition. If X_H is a Hamiltonian vector field with flow ρ_t , then $\rho_t^*\omega = \omega$. In other words, each ρ_t is a symplectomorphism.

Proof. It suffices to show that $\frac{\partial}{\partial t}\rho_t^*\omega = 0$. We have

$$\frac{d}{dt}(\rho_t^*\omega) = \rho_t^*(\mathcal{L}_{X_H}\omega) = \rho_t^*(d\iota_{X_H}\omega + \iota_{X_H}d\omega) = \rho_t^*(ddH) = 0. \quad \square$$

Thus, every function H gives us a one-parameter subgroup of symplectomorphisms.

Proposition. ρ_t preserves H , i.e. $\rho_t^*H = H$.

Proof.

$$\frac{d}{dt}\rho_t^*H = \rho_t^*(\mathcal{L}_{X_H}H) = \rho_t^*(\iota_{X_H}dH) = \rho_t^*(\iota_{X_H}\iota_{X_H}\omega) = 0. \quad \square$$

So the flow lines of our vector field are contained in level sets of H .

Example. Take $(S^2, \omega = d\theta \wedge dh)$. Take

$$H(h, \theta) = h$$

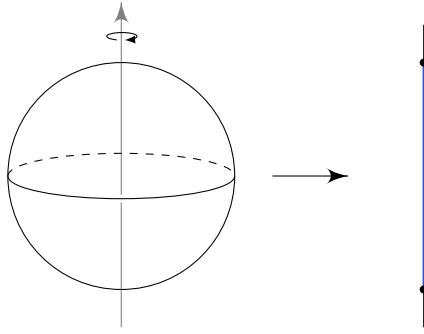
to be the height function. Then X_H solves

$$\iota_{X_H}(d\theta \wedge dh) = dh.$$

So

$$X_H = \frac{\partial}{\partial \theta}, \quad \rho_t(h, \theta) = (h, \theta + t).$$

As expected, the flow preserves height and the area form.



We have seen that Hamiltonian vector fields are symplectic:

Definition (Symplectic vector field). A vector field X on (M, ω) is a *symplectic vector field* if $\mathcal{L}_X \omega = 0$.

Observe that

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega = d\iota_X \omega.$$

So X is symplectic iff $\iota_X \omega$ is closed, and is Hamiltonian if it is exact. Thus, locally, every symplectic vector field is Hamiltonian and globally, the obstruction lies in $H_{\text{DR}}^1(M)$.

Example. Take $(T^2, \omega = d\theta_1 \wedge d\theta_2)$. Then $X_i = \frac{\partial}{\partial \theta_i}$ are symplectic but not Hamiltonian, since $\iota_{X_i} \omega = d\theta_{2-i}$ is closed but not exact.

Proposition. Let X, Y be symplectic vector fields on (M, ω) . Then $[X, Y]$ is Hamiltonian.

Recall that if X, Y are vector fields on M and $f \in C^\infty(M)$, then their *Lie bracket* is given by

$$[X, Y]f = (XY - YX)f.$$

This makes $\chi(M)$, the space of vector fields on M , a Lie algebra.

In order to prove the proposition, we need the following identity:

Exercise. $\iota_{[X, Y]} \alpha = [\mathcal{L}_X, \iota_Y] \alpha = [\iota_X, \mathcal{L}_Y] \alpha$.

Proof of proposition. We need to check that $\iota_{[X, Y]} \omega$ is exact. By the exercise, this is

$$\iota_{[X, Y]} \omega = \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega = d(\iota_X \iota_Y \omega) + \iota_X d\iota_Y \omega + \iota_Y d\iota_X \omega - \iota_Y \iota_X d\omega.$$

Since X, Y are symplectic, we know $d\iota_Y \omega = d\iota_X \omega = 0$, and the last term always vanishes. So this is exact, and $\omega(Y, X)$ is a Hamiltonian function for $[X, Y]$. \square

Definition (Poisson bracket). Let $f, g \in C^\infty(M)$. We then define the *Poisson bracket* $\{f, g\}$ by

$$\{f, g\} = \omega(X_f, X_g).$$

This is defined so that

$$X_{\{f, g\}} = -[X_f, X_g].$$

Exercise. The Poisson bracket satisfies the Jacobi identity, and also the Leibniz rule

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

Thus, if (M, ω) is symplectic, then $(C^\infty(M), \{\cdot, \cdot\})$ is a *Poisson algebra*. This means it is a commutative, associative algebra with a Lie bracket that satisfies the Leibniz rule.

Further, the map $C^\infty(M) \rightarrow \chi(M)$ sending $H \mapsto X_H$ is a Lie algebra (anti-)homomorphism.

Proposition. $\{f, g\} = 0$ iff f is constant along integral curves of X_g .

Proof.

$$\mathcal{L}_{X_g} f = \iota_{X_g} df = \iota_{X_g} \iota_{X_f} \omega = \omega(X_f, X_g) = \{f, g\} = 0. \quad \square$$

Example. If $M = \mathbb{R}^{2n}$ and $\omega = \omega_0 = \sum dx_j \wedge dy_j$, and $f \in C^\infty(\mathbb{R}^{2n})$, then

$$X_f = \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

If $\rho_0(p) = p$, then $\rho_t(p) = (x(t), y(t))$ is an integral curve for X_f iff

$$\frac{dx_i}{dt} = \frac{\partial f}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial f}{\partial x_i}.$$

In classical mechanics, these are known as *Hamilton equations*.

3.2 Integrable systems

In classical mechanics, we usually have a fixed H , corresponding to the energy.

Definition (Hamiltonian system). A *Hamiltonian system* is a triple (M, ω, H) where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, called the *Hamiltonian function*.

Definition (Integral of motion). A *integral of motion/first integral/constant of motion/conserved quantity* of a Hamiltonian system is a function $f \in C^\infty(M)$ such that $\{f, H\} = 0$.

For example, H is an integral of motion. Are there others? Of course, we can write down $2H, H^2, H^{12}, e^H$, etc., but these are more-or-less the same as H .

Definition (Independent integrals of motion). We say $f_1, \dots, f_n \in C^\infty(M)$ are *independent* if $(df_1)_p, \dots, (df_n)_p$ are linearly independent at all points on some dense subset of M .

Definition (Commuting integrals of motion). We say $f_1, \dots, f_n \in C^\infty$ *commute* if $\{f_i, f_j\} = 0$.

If we have n independent integrals of motion, then we clearly have $\dim M \geq n$. In fact, the commuting condition implies:

Exercise. Let f_1, \dots, f_n be independent commuting functions on (M, ω) . Then $\dim M \geq 2n$.

The idea is that the $(df_i)_p$ are not only independent, but span an isotropic subspace of TM .

If we have the maximum possible number of independent commuting first integrals, then we say we are integrable.

Definition (Completely integrable system). A Hamiltonian system (M, ω, H) of dimension $\dim M = 2n$ is (completely) integrable if it has n independent commuting integrals of motion $f_1 = H, f_2, \dots, f_n$.

Example. If $\dim M = 2$, then we only need one integral of motion, which we can take to be H . Then (M, ω, H) is integrable as long as the set of non-critical points of H is dense.

Example. The physics of a simple pendulum of length 1 and mass 1 can be modeled by the symplectic manifold $M = T^*S^1$, where the S^1 refers to the angular coordinate θ of the pendulum, and the cotangent vector is the momentum. The Hamiltonian function is

$$H = K + V = \text{kinetic energy} + \text{potential energy} = \frac{1}{2}\xi^2 + (1 - \cos \omega).$$

We can check that the critical points of H are $(\theta, \xi) = (0, 0)$ and $(\pi, 0)$. So (M, ω, H) is integrable.

Example. If $\dim M = 4$, then (M, ω, H) is integrable as long as there exists an integral motion independent of H . For example, on a spherical pendulum, we have $M = T^*S^2$, and H is the total energy. Then the angular momentum is an integral of motion.

What can we do with a completely integrable system? Suppose (M, ω, H) is completely integrable system with $\dim M = 2n$ and $f_1 = H, f_2, \dots, f_n$ are commuting. Let c be a regular value of $f = (f_1, \dots, f_n)$. Then $f^{-1}(c)$ is an n -dimensional submanifold of M . If $p \in f^{-1}(c)$, then

$$T_p(f^{-1}(c)) = \ker(df)_p.$$

Since

$$df_p = \begin{pmatrix} (df_1)_p \\ \vdots \\ (df_n)_p \end{pmatrix} = \begin{pmatrix} \iota_{X_{f_1}} \omega \\ \vdots \\ \iota_{X_{f_n}} \omega \end{pmatrix},$$

we know

$$T_p(f^{-1}(c)) = \ker(df_p) = \text{span}\{(X_{f_1})_p, \dots, (X_{f_n})_p\},$$

Moreover, since

$$\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0,$$

we know that $T_p(f^{-1}(c))$ is an isotropic subspace of (T_pM, ω_p) .

If X_{f_1}, \dots, X_{f_n} are complete, then following their flows, we obtain global coordinates of (the connected components of) $f^{-1}(c)$, where $q \in f^{-1}(c)$ has coordinates $(\varphi_1, \dots, \varphi_m)$ (angle coordinates) if q is achieved from the base point p by following the flow of X_{f_i} for φ_i seconds for each i . The fact that the f_i are Poisson commuting implies the vector fields commute, and so the order does not matter, and this gives a genuine coordinate system.

By the independence of X_{f_i} , the connected components look like $\mathbb{R}^{n-k} \times T^k$, where $T^k = (S^1)^k$ is the k torus. In the extreme case $k = n$, we simply get a torus, which is a compact connected component.

Definition (Liouville torus). A *Liouville torus* is a compact connected component of $f^{-1}(c)$.

It would be nice if the (φ_i) are part of a Darboux chart of M , and this is true.

Theorem (Arnold–Liouville theorem). Let (M, ω, H) be an integrable system with $\dim M = 2n$ and $f_1 = H, f_2, \dots, f_n$ integrals of motion, and $c \in \mathbb{R}$ a regular value of $f = (f_1, \dots, f_n)$.

- (i) If the flows of X_{f_i} are complete, then the connected components of $f^{-1}(\{c\})$ are homogeneous spaces for \mathbb{R}^n and admit affine coordinates $\varphi_1, \dots, \varphi_n$ (*angle coordinates*), in which the flows of X_{f_i} are linear.
- (ii) There exists coordinates ψ_1, \dots, ψ_n (*action coordinates*) such that the ψ_i 's are integrals of motion and $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n$ form a Darboux chart.

3.3 Classical mechanics

As mentioned at the beginning, symplectic geometry was first studied by physicists. In this section, we give a brief overview of how symplectic geometry arises in physics. Historically, there have been three “breakthroughs” in classical mechanics:

- (i) In ~ 1687 , Newton wrote down his three laws of physics, giving rise to Newtonian mechanics.
- (ii) In ~ 1788 , this was reformulated into the Lagrangian formalism.
- (iii) In ~ 1833 , these were further developed into the Hamiltonian formalism.

Newtonian mechanics

In Newtonian mechanics, we consider a particle of mass m moving in \mathbb{R}^3 under the potential $V(x)$. *Newton's second law* then says the trajectory of the particle obeys

$$m \frac{d^2x}{dt^2} = -\nabla V(x).$$

Hamiltonian mechanics.

To do Hamiltonian mechanics, a key concept to introduce is the momentum:

Definition (Momentum). The *momentum* of a particle is

$$y = m \frac{dx}{dt}.$$

We also need the energy function

Definition (Energy). The *energy* of a particle is

$$H(x, y) = \frac{1}{2m} |y|^2 + V(x).$$

We call \mathbb{R}^3 the *configuration space* and $T^*\mathbb{R}^3$ the *phase space*, parametrized by (x, y) . This has a canonical symplectic form $\omega = dx_i \wedge dy_i$.

Newton's second law can be written as

$$\frac{dy_i}{dt} = -\frac{\partial V}{\partial x_i}.$$

Combining with the definition of y , we find that (x, y) evolves under.

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i} \\ \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i}\end{aligned}$$

So physical motion is given by Hamiltonian flow under H . H is called the *Hamiltonian* of the system.

Lagrangian mechanics

Lagrangian mechanics is less relevant to our symplectic picture, but is nice to know about nevertheless. This is formulated via a variational principle.

In general, consider a system with N particles of masses m_1, \dots, m_N moving in \mathbb{R}^3 under a potential $V \in C^\infty(\mathbb{R}^{3N})$. The Hamiltonian function can be defined exactly as before:

$$H(x, y) = \sum_k \frac{1}{2m_k} |y_k|^2 + V(x),$$

where $x(t) = (x_1, \dots, x_n)$ and each x_i is a 3-vector; and similarly for y with $y_k = m_k \frac{dx_k}{dt}$. Then in Hamiltonian mechanics, we say (x, y) evolves under Hamiltonian flow.

Now fix $a, b \in \mathbb{R}$ and $p, q \in \mathbb{R}^{3N}$. Write \mathcal{P} for the space of all piecewise differentiable paths $\gamma = (\gamma_1, \dots, \gamma_n) : [a, b] \rightarrow \mathbb{R}^{3N}$.

Definition (Action). The *action* of a path $\gamma \in \mathcal{P}$ is

$$A_\gamma = \int_a^b \left(\sum_k \frac{m_k}{2} \left| \frac{d\gamma_k}{dt}(t) \right|^2 - V(\gamma(t)) \right) dt.$$

The integrand is known as the *Lagrangian function*. We will see that $\gamma(t) = x(t)$ is (locally) a stationary point of A_γ iff

$$m_k \frac{d^2 x_t}{dt} = -\frac{\partial V}{\partial x_k},$$

i.e. if and only if Newton's second law is satisfied.

The Lagrangian formulation works more generally if our particles are constrained to live in some submanifold $X \subseteq \mathbb{R}^{3n}$. For example, if we have a pendulum, then the particle is constrained to live in S^1 (or S^2). Then we set \mathcal{P} to be the maps $\gamma : [a, b] \rightarrow X$ that go from p to q . The Lagrangian formulation is then exactly the same, except the minimization problem is now performed within this \mathcal{P} .

More generally, suppose we have an n -dimensional manifold X , and $F : TX \rightarrow \mathbb{R}$ is a *Lagrangian function*. Given a curve $\gamma : [a, b] \rightarrow X$, there is a lift

$$\begin{aligned} \tilde{\gamma} : [a, b] &\rightarrow TX \\ t &\mapsto \left(\gamma(t), \frac{d\gamma}{dt}(t) \right). \end{aligned}$$

The *action* is then

$$A_\gamma = \int_a^b (\tilde{\gamma}^* F)(t) = \int_a^b F \left(\gamma(t), \frac{d\gamma}{dt}(t) \right) dt.$$

To find the critical points of the action, we use *calculus of variations*. Pick a chart (x_1, \dots, x_n) for X and naturally extend to a chart $(x_1, \dots, x_n, v_1, \dots, v_n)$ for TX . Consider a small perturbation of our path

$$\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon c_1(t), \dots, \gamma_n(t) + \varepsilon c_n(t))$$

for some functions $c_1, \dots, c_n \in C^\infty([a, b])$ such that $c_i(a) = c_i(b) = 0$. We think of this as an infinitesimal variation of γ . We then find that

$$\left. \frac{dA_{\gamma_\varepsilon}}{d\varepsilon} \right|_{\varepsilon=0} = 0 \Leftrightarrow \frac{\partial F}{\partial x_k} = \frac{d}{dt} \frac{\partial F}{\partial v_k} \text{ for } k = 1, \dots, n.$$

These are the *Euler–Lagrange equations*.

Example. In $X = \mathbb{R}^{3N}$ and

$$F(x_1, \dots, x_n, v_1, \dots, v_n) = \sum_k \frac{m_k}{2} |v_k|^2 - V(x_1, \dots, x_n).$$

Then the Euler–Lagrange equations are

$$-\frac{\partial V}{\partial x_{k_i}} = m_k \frac{d^2 x_{k_i}}{dt^2}.$$

Example. On a Riemannian manifold, if we set $F : TX \rightarrow \mathbb{R}$ be $(x, v) \mapsto |v|^2$, then we obtain the Christoffel equations for a geodesic.

In general, there need not be solutions to the Euler–Lagrange equation. However, if we satisfy the *Legendre condition*

$$\det \left(\frac{\partial^2 F}{\partial v_i \partial v_j} \right) \neq 0,$$

then the Euler–Lagrange equations become second order ODEs, and there is a unique solution given $\gamma(0)$ and $\dot{\gamma}(0)$. If furthermore this is positive definite, then the solution is actually a locally minimum.

3.4 Hamiltonian actions

In the remainder of the course, we are largely interested in how Lie groups can act on symplectic manifolds via Hamiltonian vector fields. These are known as *Hamiltonian actions*. We first begin with the notion of symplectic actions.

Let (M, ω) be a symplectic manifold, and G a Lie group.

Definition (Symplectic action). A *symplectic action* is a smooth group action $\psi : G \rightarrow \text{Diff}(M)$ such that each ψ_g is a symplectomorphism. In other words, it is a map $G \rightarrow \text{Symp}(M, \omega)$.

Example. Let $G = \mathbb{R}$. Then a map $\psi : G \rightarrow \text{Diff}(M)$ is a one-parameter group of transformations $\{\psi_t : t \in \mathbb{R}\}$. Given such a group action, we obtain a complete vector field

$$X_p = \left. \frac{d\psi_t}{dt}(p) \right|_{t=0}.$$

Conversely, given a complete vector field X , we can define

$$\psi_t = \exp tX,$$

and this gives a group action by \mathbb{R} .

Under this correspondence, symplectic actions correspond to complete symplectic vector fields.

Example. If $G = S^1$, then a symplectic action of S^1 is a symplectic action of \mathbb{R} which is periodic.

In the case where G is \mathbb{R} or S^1 , it is easy to define what it means for an action to be Hamiltonian:

Definition (Hamiltonian action). An action of \mathbb{R} or S^1 is *Hamiltonian* if the corresponding vector field is Hamiltonian.

Example. Take $(S^2, \omega = d\theta \wedge dh)$. Then we have a rotation action

$$\psi_t(\theta, h) = (\theta + t, h)$$

generated by the vector field $\frac{\partial}{\partial \theta}$. Since $\iota_{\frac{\partial}{\partial \theta}} \omega = dh$ is exact, this is in fact a Hamiltonian S^1 action.

Example. Take $(T^2, d\theta_1 \wedge d\theta_2)$. Consider the action

$$\psi_t(\theta_1, \theta_2) = (\theta_1 + t, \theta_2).$$

This is generated by the vector field $\frac{\partial}{\partial \theta_1}$. But $\iota_{\frac{\partial}{\partial \theta_1}} \omega = d\theta_2$, which is closed but not exact. So this is a symplectic action that is not Hamiltonian.

How should we define Hamiltonian group actions for groups that are not \mathbb{R} or S^1 ? The simplest possible next case is the torus $G = T^n = S^1 \times \cdots \times S^1$. If we have a map $\psi : T^n \rightarrow \text{Symp}(M, \omega)$, then for this to be Hamiltonian, it should definitely be the case that the restriction to each S^1 is Hamiltonian in the previous sense. Moreover, for these to be compatible, we would expect each Hamiltonian function to be preserved by the other factors as well.

For the general case, we need to talk about the Lie group of G . Let G be a Lie group. For each $g \in GG$, there is a left multiplication map

$$\begin{aligned} L_g : G &\rightarrow G \\ a &\mapsto ga. \end{aligned}$$

Definition (Left-invariant vector field). A *left-invariant vector field* on a Lie group G is a vector field X such that

$$(L_g)_*X = X$$

for all $g \in G$.

We write \mathfrak{g} for the space of all left-invariant vector fields on G , which comes with the Lie bracket on vector fields. This is called the *Lie algebra* of G .

If X is left-invariant, then knowing X_e tells us what X is everywhere, and specifying X_e produces a left-invariant vector field. Thus, we have an isomorphism $\mathfrak{g} \cong T_eG$.

The Lie algebra admits a natural action of G , called the *adjoint action*. To construct this, note that G acts on itself by conjugation,

$$\varphi_g(a) = gag^{-1}.$$

This fixes the identity, and taking the derivative gives $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$, or equivalently, Ad is a map $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$. The dual \mathfrak{g}^* admits the dual action of G , called the *coadjoint action*. Explicitly, this is given by

$$\langle \text{Ad}_g^*(\xi), x \rangle = \langle \xi, \text{Ad}_g x \rangle.$$

An important case is when G is abelian, i.e. a product of S^1 's and \mathbb{R} 's, in which case the conjugation action is trivial, hence the (co)adjoint action is trivial.

Returning to group actions, the correspondence between complete vector fields and \mathbb{R}/S^1 actions can be described as follows: Given a smooth action $\psi : G \rightarrow \text{Diff}(M)$ and a point $p \in M$, there is a map

$$\begin{aligned} G &\rightarrow M \\ g &\mapsto \psi_g(p). \end{aligned}$$

Differentiating this at e gives

$$\begin{aligned} \mathfrak{g} &\cong T_eG \rightarrow T_pM \\ X &\mapsto X_p^\#. \end{aligned}$$

We call $X^\#$ the vector field on M generated by $X \in \mathfrak{g}$. In the case where $G = S^1$ or \mathbb{R} , we have $\mathfrak{g} \cong \mathbb{R}$, and the complete vector field corresponding to the action is the image of 1 under this map.

We are now ready to define

Definition (Hamiltonian action). We say $\psi : G \rightarrow \text{Symp}(M, \omega)$ is a *Hamiltonian action* if there exists a map $\mu : M \rightarrow \mathfrak{g}^*$ such that

- (i) For all $X \in \mathfrak{g}$, $X^\#$ is the Hamiltonian vector field generated by μ^X , where $\mu^X : M \rightarrow \mathbb{R}$ is given by

$$\mu^X(p) = \langle \mu(p), X \rangle.$$

- (ii) μ is G -equivariant, where G acts on \mathfrak{g}^* by the coadjoint action. In other words,

$$\mu \circ \psi_g = \text{Ad}_g^* \circ \mu \text{ for all } g \in G.$$

μ is then called a *moment map* for the action ψ .

In the case where G is abelian, condition (ii) just says μ is G -invariant.

Example. Let $M = \mathbb{C}^n$, and

$$\omega = \frac{1}{2} \sum_j dz_j \wedge d\bar{z}_j = \sum_i r_j dr_j \wedge d\theta_j.$$

We let

$$T^n = \{(t_1, \dots, t_n) \in \mathbb{C}^n : |t_k| = 1 \text{ for all } k\},$$

acting by

$$\psi_{(t_1, \dots, t_n)}(z_1, \dots, z_n) = (t_1^{k_1} z_1, \dots, t_n^{k_n} z_n)$$

where $k_1, \dots, k_n \in \mathbb{Z}$.

We claim this action has moment map

$$\begin{aligned} \mu : \mathbb{C}^n &\rightarrow (\mathfrak{t}^n)^* \cong \mathbb{R}^n \\ (x_1, \dots, z_n) &\mapsto -\frac{1}{2}(k_1|z_1|^2, \dots, k_n|z_n|^2). \end{aligned}$$

It is clear that this is invariant, and if $X = (a_1, \dots, a_n) \in \mathfrak{t}^n \in \mathbb{R}^n$, then

$$X^\# = k_1 a_1 \frac{\partial}{\partial \theta_1} + \dots + k_n a_n \frac{\partial}{\partial \theta_n}.$$

Then we have

$$d\mu^X = d\left(-\frac{1}{2} \sum k_j a_j r_j^2\right) = -\sum k_j a_j r_j dr_j = \iota_{X^\#} \omega.$$

Example. A nice example of a non-abelian Hamiltonian action is coadjoint orbits. Let G be a Lie group, and \mathfrak{g} the Lie algebra. If $X \in \mathfrak{g}$, then we get a vector field ${}^{\mathfrak{g}}X^\#$ generated by X via the adjoint action, and also a vector field ${}^{\mathfrak{g}}X$ on \mathfrak{g}^* generated by the co-adjoint action.

If $\xi \in \mathfrak{g}^*$, then we can define the *coadjoint orbit* through ξ

$$\mathcal{O}_\xi = \{\text{Ad}_g^*(\xi) : g \in G\}.$$

What is interesting about this is that this coadjoint orbit is actually a symplectic manifold, with symplectic form given by

$$\omega_\xi(X_\xi^\#, Y_\xi^\#) = \langle \xi, [X, Y] \rangle.$$

Then the coadjoint action of G on \mathcal{O}_ξ has moment map $\mathcal{O}_\xi \rightarrow \mathfrak{g}^*$ given by the inclusion.

3.5 Symplectic reduction

Given a Lie group action of G on M , it is natural to ask what the “quotient” of M looks like. What we will study is not quite the quotient, but a *symplectic reduction*, which is a natural, well-behaved subspace of the quotient that is in particular symplectic.

We first introduce some words. Let $\psi : G \rightarrow \text{Diff}(M)$ be a smooth action.

Definition (Orbit). If $p \in M$, the *orbit* of p under G is

$$\mathcal{O}_p = \{\psi_g(p) : g \in G\}.$$

Definition (Stabilizer). The *stabilizer* or *isotropy group* of $p \in M$ is the closed subgroup

$$G_p = \{g \in G : \psi_g(p) = p\}.$$

We write \mathfrak{g}_p for the Lie algebra of G_p .

Definition (Transitive action). We say G acts *transitively* if M is one orbit.

Definition (Free action). We say G acts *freely* if $G_p = \{e\}$ for all p .

Definition (Locally free action). We say G acts *locally freely* if $\mathfrak{g}_p = \{0\}$, i.e. G_p is discrete.

Definition (Orbit space). The *orbit space* is M/G , and we write $\pi : M \rightarrow M/G$ for the orbit projection. We equip M/G with the quotient topology.

The main theorem is the following:

Theorem (Marsden–Weinstein, Meyer). Let G be a compact Lie group, and (M, ω) a symplectic manifold with a Hamiltonian G -action with moment map $\mu : M \rightarrow \mathfrak{g}^*$. Write $i : \mu^{-1}(0) \hookrightarrow M$ for the inclusion. Suppose G acts freely on $\mu^{-1}(0)$. Then

- (i) $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold;
- (ii) $\pi : \mu^{-1}(0) \rightarrow M_{\text{red}}$ is a principal G -bundle; and
- (iii) There exists a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{\text{red}}$.

Definition (Symplectic quotient). We call M_{red} the *symplectic quotient/reduced space/symplectic reduction* of (M, ω) with respect to the given G -action and moment map.

What happens if we do reduction at other levels? In other words, what can we say about $\mu^{-1}(\xi)/G$ for other $\xi \in \mathfrak{g}^*$?

If we want to make sense of this, we need ξ to be preserved under the coadjoint action of G . This is automatically satisfied if G is abelian, and in this case, we simply have $\mu^{-1}(\xi) = \varphi^{-1}(0)$, where $\varphi = \mu - \xi$ is another moment map. So this is not more general.

If ξ is not preserved by G , then we can instead consider $\mu^{-1}(\xi)/G_\xi$, or equivalently take $\mu^{-1}(\mathcal{O}_\xi)/G$. We check that

$$\mu^{-1}(\xi)/G_\xi \cong \mu^{-1}(\mathcal{O}_\xi)/G \cong \varphi^{-1}(0)/G,$$

where

$$\begin{aligned} \varphi : M \times \mathcal{O}_\xi &\rightarrow \mathfrak{g}^* \\ (\rho, \eta) &\mapsto \mu(\rho) - \eta \end{aligned}$$

is a moment map for the product action of G on $(M \times \mathcal{O}_\xi, \omega \times \omega_\xi)$.

So in fact there is no loss in generality for considering just $\mu^{-1}(0)$.

Proof. We first show that $\mu^{-1}(0)$ is a manifold. This follows from the following claim:

Claim. G acts locally freely at p iff p is a regular point of μ .

We compute the dimension of $\text{im } d\mu_p$ using the rank-nullity theorem. We know $d\mu_p v = 0$ iff $\langle d\mu_p(v), X \rangle = 0$ for all $X \in \mathfrak{g}$. We can compute

$$\langle d\mu_p(v), X \rangle = (d\mu^X)_p(v) = (\iota_{X_p^\#} \omega)(v) = \omega_p(X_p^\#, v).$$

Moreover, the span of the $X_p^\#$ is exactly $T_p \mathcal{O}_p$. So

$$\ker d\mu_p = (T_p \mathcal{O}_p)^\omega.$$

Thus,

$$\dim(\text{im } d\mu_p) = \dim \mathcal{O}_p = \dim G - \dim G_p.$$

In particular, $d\mu_p$ is surjective iff $G_p = 0$.

Then (i) and (ii) follow from the following theorem:

Theorem. Let G be a compact Lie group and Z a manifold, and G acts freely on Z . Then Z/G is a manifold and $Z \rightarrow Z/G$ is a principal G -bundle.

Note that if G does not act freely on $\mu^{-1}(0)$, then by Sard's theorem, generically, ξ is a regular value of μ , and so $\mu^{-1}(\xi)$ is a manifold, and G acts locally freely on $\mu^{-1}(\xi)$. If $\mu^{-1}(\xi)$ is preserved by G , then $\mu^{-1}(\xi)/G$ is a symplectic orbifold.

It now remains to construct the symplectic structure. Observe that if $p \in \mu^{-1}(0)$, then

$$T_p \mathcal{O}_p \subseteq T_p \mu^{-1}(0) = \ker d\mu_p = (T_p \mathcal{O}_p)^\omega.$$

So $T_p \mathcal{O}_p$ is an isotropic subspace of $(T_p M, \omega)$. We then observe the following straightforward linear algebraic result:

Lemma. Let (V, Ω) be a symplectic vector space and I an isotropic subspace. Then Ω induces a canonical symplectic structure Ω_{red} on I^Ω/I , given by $\Omega_{\text{red}}([u], [v]) = \Omega(u, v)$.

Applying this, we get a canonical symplectic structure on

$$\frac{(T_p \mathcal{O}_p)^\omega}{T_p \mathcal{O}_p} = \frac{T_p \mu^{-1}(0)}{T_p \mathcal{O}_p} = T_{[p]} M_{\text{red}}.$$

This defines ω_{red} on M_{red} , which is well-defined because ω is G -invariant, and is smooth by local triviality and canonicity.

It remains to show that $d\omega_{\text{red}} = 0$. By construction, $i^* \omega = \pi^* \omega_{\text{red}}$. So

$$\pi^*(d\omega_{\text{red}}) = d\pi^* \omega_{\text{red}} = di^* \omega = i^* d\omega = 0$$

Since π^* is injective, we are done. □

Example. Take

$$(M, \omega) = \left(\mathbb{C}^n, \omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k = \sum r_k dr_k \wedge d\theta_k \right).$$

We let $G = S^1$ act by multiplication

$$e^{it} \cdot (z_1, \dots, z_n) = (e^{it}z_1, \dots, e^{it}z_n).$$

This action is Hamiltonian with moment map

$$\begin{aligned} \mu : \mathbb{C}^n &\rightarrow \mathbb{R} \\ z &\mapsto -\frac{|z|^2}{2} + \frac{1}{2}. \end{aligned}$$

The $+\frac{1}{2}$ is useful to make the inverse image of 0 non-degenerate. To check this is a moment map, we compute

$$d\left(\frac{1}{2}|z|^2\right) = d\left(-\frac{1}{2}\sum r_k^2\right) = -\sum r_k dr_k.$$

On the other hand, if

$$X = a \in \mathfrak{g} \cong \mathbb{R},$$

then we have

$$X^\# = a \left(\frac{\partial}{\partial \theta_1} + \dots + \frac{\partial}{\partial \theta_n} \right).$$

So we have

$$\iota_{X^\#}\omega = -a \sum r_k dr_k = d\mu^X.$$

It is also clear that μ is S^1 -invariant. We then have

$$\mu^{-1}(0) = \{z \in \mathbb{C}^n : |z|^2 = 1\} = S^{2n-1}.$$

Then we have

$$M_{\text{red}} = \mu^{-1}(0)/S^1 = \mathbb{C}\mathbb{P}^{n-1}.$$

One can check that this is in fact the Fubini–Study form. So $(\mathbb{C}\mathbb{P}^{n-1}, \omega_{FS})$ is the symplectic quotient of (\mathbb{C}^n, ω_0) with respect to the diagonal S^1 action and the moment map μ .

Example. Fix $k, \ell \in \mathbb{Z}$ relatively prime. Then S^1 acts on \mathbb{C}^2 by

$$e^{it}(z_1, z_2) = (e^{ikt}z_1, e^{i\ell t}z_2).$$

This action is Hamiltonian with moment map

$$\begin{aligned} \mu : \mathbb{C}^2 &\rightarrow \mathbb{R} \\ (z_1, z_2) &\mapsto -\frac{1}{2}(k|z_1|^2 + \ell|z_2|^2). \end{aligned}$$

There is no level set of μ where the action is free, since

- $(z, 0)$ has stabilizer $\mathbb{Z}/k\mathbb{Z}$
- $(0, z)$ has stabilizer $\mathbb{Z}/\ell\mathbb{Z}$
- (z_1, z_2) has trivial stabilizer if $z_1, z_2 \neq 0$.

On the other hand, the action is still locally free on $\mathbb{C}^2 \setminus \{(0,0)\}$ since the stabilizers are discrete.

The reduced spaces $\mu^{-1}(\xi)/S^1$ for $\xi \neq 0$ are *orbifolds*, called *weighted* or *twisted projective spaces*.

The final example is an infinite dimensional one by Atiyah and Bott. We will not be able to prove the result in full detail, or any detail at all, but we will build up to the statement. The summary of the result is that performing symplectic reduction on the space of all connections gives the moduli space of flat connections.

Let $G \rightarrow P \xrightarrow{\pi} B$ be a principal G -bundle, and $\psi : G \rightarrow \text{Diff}(P)$ the associated free action. Let

$$\begin{aligned} d\psi : \mathfrak{g} &\rightarrow \chi(P) \\ X &\mapsto X^\# \end{aligned}$$

be the associated infinitesimal action. Let X_1, \dots, X_k be a basis of the Lie algebra \mathfrak{g} . Then since ψ is a free action, $X_1^\#, \dots, X_k^\#$ are all linearly independent at each $p \in P$.

Define the *vertical tangent space*

$$V_p = \text{span}\{(X_1^\#)_p, \dots, (X_k^\#)_p\} \subseteq T_p P = \ker(d\pi_p)$$

We can then put these together to get $V \subseteq TP$, the *vertical tangent bundle*.

Definition (Ehresmann Connection). An (*Ehresmann*) *connection* on P is a choice of subbundle $H \subseteq TP$ such that

- (i) $P = V \oplus H$
- (ii) H is G -invariant.

Such an H is called a *horizontal bundle*.

There is another way of describing a connection. A *connection form* on P is a \mathfrak{g} -valued 1-form $A \in \Omega^1(P) \otimes \mathfrak{g}$ such that

- (i) $\iota_{X^\#} A = X$ for all $X \in \mathfrak{g}$
- (ii) A is G -invariant for the action

$$g \cdot (A_i \otimes X_i) = (\psi_{g^{-1}})^* A_i \otimes \text{Ad}_g X_i.$$

Lemma. Giving an Ehresmann connection is the same as giving a connection 1-form.

Proof. Given an Ehresmann connection H , we define

$$A_p(v) = X,$$

where $v = X_p^\# + h_p \in V \oplus H$.

Conversely, given an A , we define

$$H_p = \ker A_p = \{v \in T_p P : i_v A_p = 0\}.$$

□

We next want to define the notion of curvature. We will be interested in *flat* connections, i.e. connections with zero curvature.

To understand curvature, if we have a connection $TP = V \oplus H$, then we get further decompositions

$$T^*P = V^* \oplus H^*, \quad \Lambda^2(T^*P) = (\Lambda^2V^*) \oplus (V^* \wedge H^*) \oplus (\Lambda^2H^*).$$

So we end up having

$$\begin{aligned} \Omega^1(P) &= \Omega_{\text{vert}}^1(P) \oplus \Omega_{\text{hor}}^1(P) \\ \Omega^2(P) &= \Omega_{\text{vert}}^2 \oplus \Omega_{\text{mixed}} \oplus \Omega_{\text{hor}}^2 \end{aligned}$$

If $A = \sum_{i=1}^k A_i \otimes X_i \in \Omega^1 \otimes \mathfrak{g}$, then

$$dA \in \Omega^2 \otimes \mathfrak{g}.$$

We can then decompose this as

$$dA = (dA)_{\text{vert}} + (dA)_{\text{mix}} + (dA)_{\text{hor}}.$$

The first part is uninteresting, in the sense that it is always given by

$$(dA)_{\text{vert}}(X, Y) = [X, Y],$$

the second part always vanishes, and the last is the *curvature form*

$$\text{curv } A \in \Omega_{\text{hor}}^2 \otimes \mathfrak{g}.$$

Definition (Flat connection). A connection A is *flat* if $\text{curv } A = 0$.

We write \mathcal{A} for the space of all connections on P . Observe that if $A_1, A_0 \in \mathcal{A}$, then $A_1 - A_0$ is G -invariant and

$$\iota_{X\#}(A_1 - A_0) = X - X = 0$$

for all $X \in \mathfrak{g}$. So $A_1 - A_0 \in (\Omega_{\text{hor}}^1 \otimes \mathfrak{g})^G$. So

$$\mathcal{A} = A_0 + (\Omega_{\text{hor}}^1 \otimes \mathfrak{g})^G,$$

and $T_{A_0}\mathcal{A} = (\Omega_{\text{hor}}^1 \otimes \mathfrak{g})^G$.

Suppose B is a compact Riemann surface, and G a compact or semisimple Lie group. Then there exists an Ad-invariant inner product on \mathfrak{g} . We can then define

$$\omega : (\Omega_{\text{hor}}^1 \otimes \mathfrak{g})^G \times (\Omega_{\text{hor}}^1 \otimes \mathfrak{g})^G \rightarrow \mathbb{R},$$

sending

$$\left(\sum a_i X_i, \sum b_i X_i \right) \mapsto \int_B \sum_{i,j} a_i \wedge b_j (X_i, X_j).$$

This is easily seen to be bilinear, anti-symmetric and non-degenerate. It is also closed, if suitably interpreted, since it is effectively constant across the affine space \mathcal{A} . Thus, \mathcal{A} is an “infinite-dimensional symplectic manifold”.

To perform symplectic reduction, let \mathcal{G} be the *gauge group*, i.e. the group of G -equivariant diffeomorphisms $f : P \rightarrow P$ covering the identity. \mathcal{G} acts on \mathcal{A} by

$$V \oplus H \mapsto V \oplus H_f,$$

where H_f is the image of H by df . Atiyah and Bott proved that the action of \mathcal{G} on (\mathcal{A}, ω) is Hamiltonian with moment map

$$\begin{aligned} \mu : \mathcal{A} &\mapsto \text{Lie}(\mathcal{G})^* = (\Omega_{hor}^2 \otimes \mathfrak{g})^G \\ A &\mapsto \text{curv } A \end{aligned}$$

Performing symplectic reduction, we get

$$\mathcal{M} = \mu^{-1}(0)/\mathcal{G},$$

the moduli space of flat connections, which has a symplectic structure. It turns out this is in fact a finite-dimensional symplectic orbifold.

3.6 The convexity theorem

We focus on the case $G = T^n$. It turns out the image of the moment map μ has a very rigid structure.

Theorem (Convexity theorem (Atiyah, Guillemin–Sternberg)). Let (M, ω) be a compact connected symplectic manifold, and

$$\mu : M \rightarrow \mathbb{R}^n$$

a moment map for a Hamiltonian torus action. Then

- (i) The levels $\mu^{-1}(c)$ are connected for all c
- (ii) The image $\mu(M)$ is convex.
- (iii) The image $\mu(M)$ is in fact the convex hull of $\mu(M^G)$.

We call $\mu(M)$ the *moment polytope*.

Here we identify $G \cong T^n \simeq \mathbb{R}^n/\mathbb{Z}^n$, which gives us an identification $\mathfrak{g} \cong \mathbb{R}^n$ and $\mathfrak{g}^* \cong (\mathbb{R}^n)^* \cong \mathbb{R}^n$.

Example. Consider $(M = \mathbb{C}\mathbb{P}^n, \omega_{FS})$. T^n acts by letting $t = (t_1, \dots, t_n) \in T^n \cong \text{U}(1)^n$ send

$$\psi_t([z_0 : \dots : z_n]) = [z_0 : t_1 z_1 : \dots : t_n z_n].$$

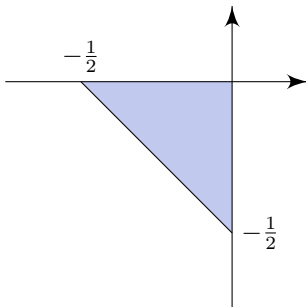
This has moment map

$$\mu([z_0 : \dots : z_n]) = -\frac{1}{2} \frac{(|z_1|^2, \dots, |z_n|^2)}{|z_0|^2 + \dots + |z_n|^2}.$$

The image of this map is

$$\mu(M) = \left\{ x \in \mathbb{R}^n : x_k \leq 0, x_1 + \dots + x_n \geq -\frac{1}{2} \right\}.$$

For example, in the $\mathbb{C}\mathbb{P}^2$ case, the moment image lives in \mathbb{R}^2 , and is just



The three vertices are $\mu([0 : 0 : 1])$, $\mu([0 : 1 : 0])$ and $\mu([1 : 0 : 0])$.

We now want to prove the convexity theorem. We first look at (iii). While it seems like a very strong claim, it actually follows formally from (ii).

Lemma. (ii) implies (iii).

Proof. Suppose the fixed point set of the action has k connected components $Z = Z_1 \cup \dots \cup Z_k$. Then μ is constant on each Z_j , since $X^\#|_{Z_j} = 0$ for all X . Let $\mu(Z_j) = \eta_j \in \mathbb{R}^n$. By (ii), we know the convex hull of $\{\eta_1, \dots, \eta_k\}$ is contained in $\mu(M)$.

To see that $\mu(M)$ is exactly the convex hull, observe that if $X \in \mathbb{R}^n$ has rationally independent components, so that X topologically generates T , then p is fixed by T iff $X_p^\# = 0$, iff $d\mu_p^X = 0$. Thus, μ^X attains its maximum on one of the Z_j .

Now if ξ is not in the convex hull of $\{\eta_j\}$, then we can pick an $X \in \mathbb{R}^n$ with rationally independent components such that $\langle \xi, X \rangle > \langle \eta_j, X \rangle$ for all j , since the space of such X is open and non-empty. Then

$$\langle \xi, X \rangle > \sup_{p \in \bigcup Z_j} \langle \mu(p), X \rangle = \sup_{p \in M} \langle \mu(p), X \rangle.$$

So $\xi \notin \mu(M)$. □

With a bit more work, (i) also implies (ii).

Lemma. (i) implies (ii).

Proof. The case $n = 1$ is immediate, since $\mu(M)$ is compact and connected, hence a closed interval.

In general, to show that $\mu(M)$ is convex, we want to show that the intersection of $\mu(M)$ with any line is connected. In other words, if $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is any projection and $\nu = \pi \circ \mu$, then

$$\pi^{-1}(c) \cap \mu(M) = \mu(\nu^{-1}(c))$$

is connected. This would follow if we knew $\nu^{-1}(c)$ were connected, which would follow from (i) if ν were a moment map of an T^n action. Unfortunately, most of the time, it is just the moment map of an \mathbb{R}^n action. For it to come from a T^n action, we need π to be represented by an *integer* matrix. Then

$$T = \{\pi^T t : t \in T^n = \mathbb{R}^n / \mathbb{Z}^n\} \subseteq T^{n+1}$$

is a subtorus, and one readily checks that ν is the moment map for the T action.

Now for any $p_0, p_1 \in M$, we can find p'_0, p'_1 arbitrarily close to p_0, p_1 and a line of the form $\pi^{-1}(c)$ with π integral. Then the line between p'_0 and p'_1 is contained in $\mu(M)$ by the above argument, and we are done since $\mu(M)$ is compact, hence closed. \square

It thus remains to prove (i), where we have to put in some genuine work. This requires *Morse–Bott theory*.

Let M be a manifold, $\dim M = m$, and $f : M \rightarrow \mathbb{R}$ a smooth map. Let

$$\text{Crit}(f) = \{p \in M : df_p = 0\}$$

be the set of critical points. For $p \in \text{Crit}(f)$ and (U, x_1, \dots, x_m) a coordinate chart around p , we have a *Hessian matrix*

$$H_p f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

Definition (Morse(-Bott) function). f is a *Morse function* if at each $p \in \text{Crit}(f)$, $H_p f$ is non-degenerate.

f is a *Morse–Bott function* if the connected components of $\text{Crit}(f)$ are submanifolds and for all $p \in \text{Crit}(p)$, $T_p \text{Crit}(f) = \ker(H_p f)$.

If f is Morse, then the critical points are isolated. If f is Morse–Bott, then the Hessian is non-degenerate in the normal bundle to $\text{Crit}(f)$.

If M is compact, then there is a finite number of connected components of $\text{Crit}(f)$. So we have

$$\text{Crit}(f) = Z_1 \cup \dots \cup Z_k,$$

and the Z_i are called the *critical submanifolds*.

For $p \in Z_i$, the Hessian $H_p f$ is a quadratic form and we can write

$$T_p M = E_p^- \oplus T_p Z_i \oplus E_p^+,$$

where E_p^\pm are the positive and negative eigenspaces of $H_p f$ respectively. Note that $\dim E_p^\pm$ are locally constant, hence constant along Z_i . So we can define the *index* of Z_i to be $\dim E_p^-$, and the *coindex* to be $\dim E_p^+$.

We can then define a vector bundle $E^- \rightarrow Z_i$, called the *negative normal bundle*.

Morse theory tells us how the topology of $M_c^- = \{p \in M : f(p) \leq c\}$ changes with $c \in \mathbb{R}$.

Theorem (Morse theory).

- (i) If $f^{-1}([c_1, c_2])$ does not contain any critical point. Then $f^{-1}(c_1) \cong f^{-1}(c_2)$ and $M_{c_1} \cong M_{c_2}$ (where \cong means diffeomorphic).
- (ii) If $f^{-1}([c_1, c_2])$ contains one critical manifold Z , then $M_{c_2}^- \simeq M_{c_1}^- \cup D(E^-)$, where $D(E^-)$ is the disk bundle of E^- .

In particular, if Z is an isolated point, $M_{c_2}^-$ is, up to homotopy equivalence, obtained by adding a $\dim E_p^-$ -cell to $M_{c_1}^-$. \square

The key lemma in this proof is the following result:

Lemma. Let M be a compact connected manifold, and $f : M \rightarrow \mathbb{R}$ a Morse–Bott function with no critical submanifold of index or coindex 1. Then

- (i) f has a unique local maximum and local minimum
- (ii) All level sets of $f^{-1}(c)$ are connected.

Proof sketch. There is always a global minimum since f is compact. If there is another local minimum at c , then the disk bundle is trivial, and so in

$$M_{c+\varepsilon}^- \simeq M_{c-\varepsilon}^- \cup D(E^-)$$

for ε small enough, the union is a disjoint union. So $M_{c+\varepsilon}$ has two components. Different connected components can only merge by crossing a level of index 1, so this cannot happen. To handle the maxima, consider $-f$.

More generally, the same argument shows that a change in connectedness must happen by passing through a index or coindex 1 critical submanifold. \square

To apply this to prove the convexity theorem, we will show that for any X , μ^X is a Morse–Bott function where all the critical submanifolds are symplectic. In particular, they are of even index and coindex.

Lemma. For any $X \in \mathbb{R}^n$, μ^X is a Morse–Bott function where all critical submanifolds are symplectic.

Proof sketch. Note that p is a fixed point iff $X_p^\# = 0$ iff $d\mu_p^X = 0$ iff p is a critical point. So the critical points are exactly the fixed points of X .

$(T_p M, \omega_p)$ models (M, ω) in a neighbourhood of p by Darboux theorem. Near a fixed point T^n , an equivariant version of the Darboux theorem tells us there is a coordinate chart U where (M, ω, μ) looks like

$$\begin{aligned} \omega|_U &= \sum dx_i \wedge dy_i \\ \mu|_U &= \mu(p) - \frac{1}{2} \sum_i (x_i^2 + y_i^2) \alpha_i, \end{aligned}$$

where $\alpha_i \in \mathbb{Z}$ are *weights*.

Then the critical submanifolds of μ are given by

$$\{x_i = y_i = 0 : \alpha_i \neq 0\},$$

which is locally a symplectic manifold and has even index and coindex. \square

Finally, we can prove the theorem.

Lemma. (i) holds.

Proof. The $n = 1$ case follows from the previous lemmas. We then induct on n .

Suppose the theorem holds for n , and let $\mu = (\mu_1, \dots, \mu_n) : M \rightarrow \mathbb{R}^{n+1}$ be a moment map for a Hamiltonian T^{n+1} -action. We want to show that for all $c = (c_1, \dots, c_n) \in \mathbb{R}^{n+1}$, the set

$$\mu^{-1}(c) = \mu_1^{-1}(c_1) \cap \dots \cap \mu_{n+1}^{-1}(c_{n+1})$$

is connected.

The idea is to set

$$N = \mu_1^{-1}(c_1) \cap \cdots \cap \mu_1^{-1}(c_n),$$

and then show that $\mu_{n+1}|_N : N \rightarrow \mathbb{R}$ is a Morse–Bott function with no critical submanifolds of index or coindex 1.

We may assume that $d\mu_1, \dots, d\mu_n$ are linearly independent, or equivalently, $d\mu^X \neq 0$ for all $X \in \mathbb{R}^n$. Otherwise, this reduces to the case of an n -torus.

To make sense of N , we must pick c to be a regular value. Density arguments imply that

$$\mathcal{C} = \bigcup_{X \neq 0} \text{Crit}(\mu^X) = \bigcup_{X \in \mathbb{Z}^{n+1} \setminus \{0\}} \text{Crit} \mu^X.$$

Since $\text{Crit} \mu^X$ is a union of codimension ≥ 2 submanifolds, its complement is dense. Hence by the Baire category theorem, \mathcal{C} has dense complement. Then a continuity argument shows that we only have to consider the case when c is a regular value of μ , hence N is a genuine submanifold of codimension n .

By the induction hypothesis, N is connected. We now show that $\mu_{n+1}|_N : N \rightarrow \mathbb{R}$ is Morse–Bott with no critical submanifolds of index or coindex 1.

Let x be a critical point. Then the theory of Lagrange multipliers tells us there are some $\lambda_i \in \mathbb{R}$ such that

$$\left[d\mu_{n+1} + \sum_{i=1}^n \lambda_i d\mu_i \right]_x = 0$$

Thus, μ is critical in M for the function

$$\mu^Y = \mu_{n+1} + \sum_{i=1}^n \lambda_i \mu_i,$$

where $Y = (\lambda_1, \dots, \lambda_n, 1) \in \mathbb{R}^{n+1}$. So by the claim, μ^Y is Morse–Bott with only even indices and coindices. Let W be a critical submanifold of μ^Y containing x .

Claim. W intersects N transversely at x .

If this were true, then $\mu^Y|_N$ has $W \cap N$ as a non-degenerate critical submanifold of even index and coindex, since the coindex doesn't change and W is even-dimensional. Moreover, when restricted to N , $\sum \lambda_i \mu_i$ is a constant. So $\mu_{n+1}|_N$ satisfies the same properties.

To prove the claim, note that

$$T_x N = \ker d\mu_1|_x \cap \cdots \cap \ker d\mu_n|_x.$$

With a moment's thought, we see that it suffices to show that $d\mu_1, \dots, d\mu_n$ remain linearly independent when restricted to $T_x W$. Now observe that the Hamiltonian vector fields $X_1^\#|_x, \dots, X_n^\#|_x$ are independent since $d\mu_1|_x, \dots, d\mu_n|_x$ are, and they live in $T_x W$ since their flows preserve W .

Since W is symplectic (by the claim), for all $k = (k_1, \dots, k_n)$, there exists $v \in T_x W$ such that

$$\omega \left(\sum k_i X_i^\#|_x, v \right) \neq 0.$$

In other words,

$$\left(\sum k_i d\mu_i \right) (v) \neq 0. \quad \square$$

It is natural to seek a non-abelian generalization of this, and it indeed exists. Let (M, ω) be a symplectic manifold, and G a compact Lie group with a Hamiltonian action on M with moment map $\mu : M \rightarrow \mathfrak{g}^*$. From Lie group theory, there is a maximal torus $T \subseteq G$ with Lie algebra \mathfrak{t} , and the *Weyl group* $W = N(T)/T$ is finite (where $N(T)$ is the normalizer of T).

Then under the coadjoint action, we have

$$\mathfrak{g}^*/G \cong \mathfrak{t}^*/W,$$

Pick a *Weyl chamber* \mathfrak{t}_+^* of \mathfrak{t}^* , i.e. a fundamental domain of \mathfrak{t}^* under W . Then μ induces a moment map $\mu_+ : M \rightarrow \mathfrak{t}_+^*$, and the non-abelian convexity theorem says

Theorem (Kirwan, 1984). $\mu_+(M) \subseteq \mathfrak{t}_+^*$ is a convex polytope.

We shall end with an application of the convexity theorem to linear algebra. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\lambda_1 \geq \lambda_2$, and \mathcal{H}_λ^2 the set of all 2×2 Hermitian matrices with eigenvalues (λ_1, λ_2) . What can the diagonal entries of $A \in \mathcal{H}_\lambda^2$ be?

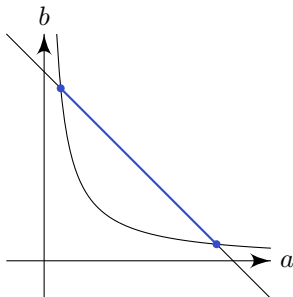
We can definitely solve this by brute force, since any entry in \mathcal{H} looks like

$$A = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$. We know

$$\begin{aligned} \operatorname{tr} a &= a + b = \lambda_1 + \lambda_2 \\ \det a &= ab - |z|^2 = \lambda_1 \lambda_2. \end{aligned}$$

The first implies $b = \lambda_1 + \lambda_2 - a$, and all the second condition gives is that $ab > \lambda_1 \lambda_2$.



This completely determines the geometry of \mathcal{H}_λ^2 , since for each allowed value of a , there is a unique value of b , which in turn determines the norm of z . Topologically, this is a sphere, since there is a S^1 's worth of choices of z except at the two end points $ab = \lambda_1 \lambda_2$.

What has this got to do with Hamiltonian actions? Observe that $U(2)$ acts transitively on \mathcal{H}_λ^2 by conjugation, and

$$T^2 = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \right\} \subseteq U(2).$$

This contains a copy of S^1 given by

$$S^1 = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \right\} \subseteq T^2.$$

We can check that

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}^{-1} = \begin{pmatrix} a & e^{i\theta} z \\ e^{i\theta} z & b \end{pmatrix}$$

Thus, if $\varphi : \mathcal{H}_\lambda^2 \rightarrow \mathbb{R}^2$ is the map that selects the diagonal elements, then

$$\varphi^{-1}(a, b) = \begin{pmatrix} a & |z|e^{i\theta} \\ |z|e^{-i\theta} & b \end{pmatrix}$$

is one S^1 -orbit. This is reminiscent of the S^1 action of S^2 quotienting out to a line segment.

We can think more generally about \mathcal{H}_λ^n , the $n \times n$ Hermitian matrices with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$, and ask what the diagonal elements can be.

Take $\varphi : \mathcal{H}_\lambda^n \rightarrow \mathbb{R}^n$ be the map that selects the diagonal entries. Then the image lies on the plane $\text{tr } A = \sum \lambda_i$. This certainly contains the $n!$ points whose coordinates are all possible permutation of the λ_i , given by the diagonal matrices.

Theorem (Schur–Horn theorem). $\varphi(\mathcal{H}_\lambda^n)$ is the convex hull of the $n!$ points from the diagonal matrices.

To view this from a symplectic perspective, let $M = \mathcal{H}_\lambda^n$, and $U(n)$ acts transitively by conjugation. For $A \subseteq M$, let G_A be the stabilizer of A . Then

$$\mathcal{H}_\lambda^n = U(n)/G_A.$$

Thus,

$$T_A \mathcal{H}_\lambda^n \cong \frac{i\mathcal{H}^n}{\mathfrak{g}_A}$$

where \mathcal{H}^n is the Hermitian matrices. The point of this is to define a symplectic form. We define

$$\begin{aligned} \omega_A : i\mathcal{H}^n \times i\mathcal{H}^n &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto i \text{tr}([X, Y]A) = i \text{tr}(X(YA - AY)) \end{aligned}$$

So

$$\ker \omega_A = \{Y : [A, Y] = 0\} = \mathfrak{g}_A.$$

So ω_A induces a non-degenerate form on $T_A \mathcal{H}_\lambda^n$. In fact, this gives a symplectic form on \mathcal{H}_λ^n .

Let $T^n \subseteq U(n)$ be the maximal torus consisting of diagonal entries. We can show that the T^n action is Hamiltonian with moment map φ . Since T^n fixes exactly the diagonal matrices, we are done.

3.7 Toric manifolds

What the convexity theorem tells us is that if we have a manifold M with a torus action, then the image of the moment map is a convex polytope. How much information is retained by a polytope?

Of course, if we take a torus that acts trivially on M , then no information is retained.

Definition (Effective action). An action G on M is *effective* (or *faithful*) if every non-identity $g \in G$ moves at least one point of M .

But we can still take the trivial torus T^0 that acts trivially, and it will still be effective. Of course, no information is retained in the polytope as well. Thus, we want to have as large of a torus action as we can. The following proposition puts a bound on “how much” torus action we can have:

Proposition. Let (M, ω) be a compact, connected symplectic manifold with moment map $\mu : M \rightarrow \mathbb{R}^n$ for a Hamiltonian T^n action. If the T^n action is effective, then

- (i) There are at least $n + 1$ fixed points.
- (ii) $\dim M \geq 2n$.

We first state without proof a result that is just about smooth actions.

Fact. An effective action of T^n has orbits of dimension n . □

This doesn't mean all orbits are of dimension n . It just means *some* orbit has dimension n .

Proof.

- (i) If $\mu = (\mu_1, \dots, \mu_n) : M \rightarrow \mathbb{R}^n$ and p is a point in an n -dimensional orbit, then $\{(d\mu_i)_p\}$ are linearly independent. So $\mu(p)$ is an interior point (if p is not in the interior, then there exists a direction X pointing out of $\mu(M)$. So $(d\mu^X)_p = 0$, and $d\mu^X$ gives a non-trivial linear combination of the $d\mu_i$'s that vanishes).

So if there is an interior point, we know $\mu(M)$ is a non-degenerate polytope in \mathbb{R}^n . This mean it has at least $n + 1$ vertices. So there are at least $n + 1$ fixed points.

- (ii) Let \mathcal{O} be an orbit of p in M . Then μ is constant on \mathcal{O} by invariance of μ . So

$$T_p\mathcal{O} \subseteq \ker(d\mu_p) = (T_p\mathcal{O})^\omega.$$

So all orbits of a Hamiltonian torus action are isotropic submanifolds. So $\dim \mathcal{O} \leq \frac{1}{2} \dim M$. So we are done. □

In the “optimal” case, we have $\dim M = 2n$.

Definition ((Symplectic) toric manifold). A *(symplectic) toric manifold* is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective T^n action of an n -torus together with a choice of corresponding moment map μ .

Example. Take $(\mathbb{C}\mathbb{P}^n, \omega_{FS})$, where the moment map is given by

$$\mu([z_0 : z_1 : \dots : z_n]) = -\frac{1}{2} \frac{(|z_1|^2, \dots, |z_n|^2)}{|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}.$$

Then this is a symplectic toric manifold.

Note that if $(M, \omega, T\mu)$ is a toric manifold and $\mu = (\mu_0, \dots, \mu_n) : M \rightarrow \mathbb{R}^n$, then μ_1, \dots, μ_n are commuting integrals of motion

$$\{\mu_i, \mu_j\} = \omega(X_i^\#, X_j^\#) = 0.$$

So we get an integrable system.

The punch line of the section is that there is a correspondence between toric manifolds and polytopes of a suitable kind. First, we need a suitable notion of equivalence of toric manifolds.

Definition (Equivalent toric manifolds). Fix a torus $T = \mathbb{R}^{2n}/(2\pi\mathbb{Z})^n$, and fix an identification $\mathfrak{t}^* \cong \mathfrak{t} \cong \mathbb{R}^n$. Given two toric manifolds $(M_i, \omega_i, T, \mu_i)$ for $i = 1, 2$, We say they are

- (i) *equivalent* if there exists a symplectomorphism $\varphi : M_1 \rightarrow M_2$ such that $\varphi(x \cdot p) = x \cdot \varphi(p)$ and $\mu_2 \circ \varphi = \mu_1$.
- (ii) *weakly equivalent* if there exists an automorphism $\lambda : T \rightarrow T$ and $\varphi : M_1 \rightarrow M_2$ symplectomorphism such that $\varphi(x, p) = \lambda(x) \cdot \varphi(p)$.

We also need a notion of equivalence of polytopes. Recall that $\text{Aut}(T) = \text{GL}(n, \mathbb{Z})$, and we can define

Definition.

$$\text{AGL}(n, \mathbb{Z}) = \{x \mapsto Bx + c : B \in \text{GL}(n, \mathbb{Z}), c \in \mathbb{R}^n\}.$$

Finally, not all polytopes can arise from the image of a moment map. It is not hard to see that the following are some necessary properties:

Definition (Delzant polytope). A *Delzant polytope* in \mathbb{R}^n is a compact convex polytope satisfying

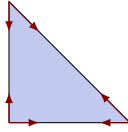
- (i) *Simplicity*: There exists exactly n edges out meeting at each vertex.
- (ii) *Rationality*: The edges meeting at each vertex P are of the form $P + tu_i$ for $t \geq 0$ and $u_i \in \mathbb{Z}^n$.
- (iii) *Smoothness*: For each vertex, the corresponding u_i 's can be chosen to be a \mathbb{Z} -basis of \mathbb{Z}^n .

Observe that all polytopes arising as $\mu(M)$ satisfy these properties.

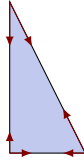
We can equivalently define rationality and smoothness as being the exact same conditions on the outward-pointing normals to the facets (co-dimension 1 faces) meeting at P .

Example. In \mathbb{R} , there is any Delzant polytope is a line segment. This corresponds to the toric manifold $S^2 = \mathbb{C}\mathbb{P}^1$ as before, and the length of the polytope corresponds to the volume of $\mathbb{C}\mathbb{P}^1$ under ω .

Example. In \mathbb{R}^2 , this is Delzant polytope:



On the other hand, this doesn't work:



since on the bottom right vertex, we have

$$\det \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} = -2 \neq \pm 1.$$

To fix this, we can do



Of course, we can also do boring things like rectangles.

There is in fact a classification of all Delzant polytopes in \mathbb{R}^2 , but we shall not discuss this.

Example. The rectangular pyramid in \mathbb{R}^3 is not Delzant because it is not simple. The tetrahedron is.

Theorem (Delzant). There are correspondences

$$\left\{ \begin{array}{l} \text{symplectic toric manifolds} \\ \text{up to equivalence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Delzant polytopes} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{symplectic toric manifolds} \\ \text{up to weak equivalence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Delzant polytopes} \\ \text{modulo } \text{AGL}(n, \mathbb{Z}) \end{array} \right\}$$

Proof sketch. Given a Delzant polytope Δ in $(\mathbb{R}^n)^*$ with d facets, we want to construct $(M_\Delta, \omega_\Delta, T_\Delta, \mu_\Delta)$ with $\mu_\Delta(M_\Delta) = \Delta$. The idea is to perform the construction for the “universal” Delzant polytope with d facets, and then obtain the desired M_Δ as a symplectic reduction of this universal example. As usual, the universal example will be “too big” to be a genuine symplectic toric manifold. Instead, it will be non-compact.

If Δ has d facets with primitive outward-point normal vectors v_1, \dots, v_d (i.e. they cannot be written as a \mathbb{Z} -multiple of some other \mathbb{Z} -vector), then we can write Δ as

$$\Delta = \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle \leq \lambda_i \text{ for } i = 1, \dots, d\}$$

for some λ_i .

There is a natural (surjective) map $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^n$ that sends the basis vector e_d of \mathbb{R}^d to v_d . If $\lambda = (\lambda_1, \dots, \lambda_d)$, and we have a pullback diagram

$$\begin{array}{ccc} \Delta & \longrightarrow & \mathbb{R}_\lambda^d \\ \downarrow & & \downarrow \\ (\mathbb{R}^n)^* & \xleftarrow{\pi^*} & (\mathbb{R}^d)^* \end{array}$$

where

$$\mathbb{R}_\lambda^d = \{X \in (\mathbb{R}^d)^* : \langle X, e_i \rangle \leq \lambda_i \text{ for all } i\}.$$

In more down-to-earth language, this says

$$\pi^*(x) \in \mathbb{R}_\lambda^d \iff x \in \Delta,$$

which is evident from definition.

Now there is a universal “toric manifold” with $\mu(M) = \mathbb{R}_\lambda^d$, namely (\mathbb{C}^d, ω_0) with the diagonal action

$$(t_1, \dots, t_d) \cdot (z_1, \dots, z_d) = (e^{it_1} z_1, \dots, e^{it_n} z_n),$$

using the moment map

$$\phi(z_1, \dots, z_d) = -\frac{1}{2}(|z_1|^2, \dots, |z_d|^2) + (\lambda_1, \dots, \lambda_d).$$

We now want to pull this back along π^* . To this extent, note that π sends \mathbb{Z}^d to \mathbb{Z}^n , hence induces a map $T^d \rightarrow T^n$ with kernel N . If \mathfrak{n} is the Lie algebra of N , then we have a short exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0.$$

Since $\text{im } \pi^* = \ker i^*$, the pullback of \mathbb{C}^d along π^* is exactly

$$Z = (i^* \circ \phi)^{-1}(0).$$

It is easy to see that this is compact.

Observe that $i^* \circ \phi$ is exactly the the moment map of the induced action by N . So Z/N is the symplectic reduction of \mathbb{C}^d by N , and in particular has a natural symplectic structure. It is natural to consider Z/N instead of Z itself, since Z carries a T^d action, but we only want to be left with a T^n action. Thus, after quotienting out by N , the T^d action becomes a $T^d/N \cong T^n$ action, with moment map given by the unique factoring of

$$Z \hookrightarrow \mathbb{C}^d \rightarrow (\mathbb{R}^d)^*$$

through $(\mathbb{R}^n)^*$. The image is exactly Δ . □

4 Symplectic embeddings

We end with a tiny chapter on symplectic embeddings, as promised in the course description.

Definition (Symplectic embedding). A symplectic embedding is an embedding $\varphi : M_1 \hookrightarrow M_2$ such that $\varphi^*\omega_2 = \omega_1$. The notation we use is $(M_1, \omega_1) \hookrightarrow^s (M_2, \omega_2)$.

A natural question to ask is, if we have two symplectic manifolds, is there a symplectic embedding between them?

For concreteness, take $(\mathbb{C}^n, \omega_0) \cong (\mathbb{R}^{2n}, \omega_0)$, and consider the subsets $B^{2n}(r)$ and $Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$ (where the product is one of symplectic manifolds). If $r \leq R$, then there is a natural inclusion of $B^{2n}(r)$ into $Z^{2n}(R)$? If we only ask for volume-preserving embeddings, then we can always embed $B^{2n}(r)$ into $Z^{2n}(R)$, since $Z^{2n}(R)$ has infinite volume. It turns out, if we require the embedding to be symplectic, we have

Theorem (Non-squeezing theorem, Gromov, 1985). There is an embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r < R$.

When studying symplectic embeddings, it is natural to consider the following:

Definition (Symplectic capacity). A *symplectic capacity* is a function c from the set of $2n$ -dimensional manifolds to $[0, \infty]$ such that

- (i) Monotonicity: if $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$, then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.
- (ii) Conformality: $c(M, \lambda\omega) = \lambda c(M, \omega)$.
- (iii) Non-triviality: $c(B^{2n}(1), \omega_0) > 0$ and $c(Z^{2n}(1), \omega_0) < \infty$.

If we only have (i) and (ii), this is called a *generalized capacity*.

Note that the volume is a generalized capacity, but not a symplectic capacity.

Proposition. The existence of a symplectic capacity is equivalent to Gromov's non-squeezing theorem.

Proof. The \Rightarrow direction is clear by monotonicity and conformality. Conversely, if we know Gromov's non-squeezing theorem, we can define the *Gromov width*

$$W_G(M, \omega) = \sup\{\pi r^2 \mid (B^{2n}(r), \omega_0) \hookrightarrow (M, \omega)\}.$$

This clearly satisfies (i) and (ii), and (iii) follows from Gromov non-squeezing. Note that Darboux's theorem says there is always an embedding of $B^{2n}(r)$ into any symplectic manifold as long as r is small enough. \square

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