Part III — Symplectic Geometry
Theorems with proof

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The first part of the course will be an overview of the basic structures of symplectic geometry, including symplectic linear algebra, symplectic manifolds, symplectomorphisms, Darboux theorem, cotangent bundles, Lagrangian submanifolds, and Hamiltonian systems. The course will then go further into two topics. The first one is moment maps and toric symplectic manifolds, and the second one is capacities and symplectic embedding problems.

Pre-requisites

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

Contents

1 Symplectic manifolds

1.1 Symplectic linear algebra

Theorem (Standard form theorem)**.** Let V be a real vector space and Ω a skew-symmetric bilinear form. Then there is a basis $\{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that

(i) $\Omega(u_i, v) = 0$ for all $v \in V$

(ii) $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$.

(iii) $\Omega(e_i, f_j) = \delta_{ij}$.

Proof. Let
$$U = \{u \in V : \Omega(u, v) = 0 \text{ for all } v \in V\},$$
and pick a basis $u_1, \ldots, u_k$ of this. Choose any W complementary to U.

First pick $e_1 \in W \setminus \{0\}$ arbitrarily. Since $e_1 \notin U$, we can pick f_1 such that $\Omega(e_1, f_1) = 1$. Then define $W_1 = \mathrm{span}\{e_i, f_i\}$, and
$$W_1^\Omega = \{w \in W : \Omega(w, v) = 0 \text{ for all } v \in W_1\}.$$
It is clear that $W_1 \cap W_1^\Omega = \{0\}$. Moreover, $W = W_1 \oplus W_1^\Omega$. Indeed, if $v \in W$, then
$$v = (\Omega(v, f_1)e_1 - \Omega(v, e_1)f_1) + (v - (\Omega(v, f_1)e_1 - \Omega(v, e_1)f_1)),$$
Then we are done by induction on the dimension. □

1.2 Symplectic manifolds

Proposition. If a compact manifold M^{2n} is such that $H^{2k}_{\mathrm{dR}}(M) = 0$ for some $k < n$, then M does not admit a symplectic structure.

Theorem (Moser)**.** If M is compact with a family ω_t of symplectic forms with $[\omega_t]$ constant, then there is an isotopy $\rho_t : M \to M$ with $\rho_t^* \omega_t = \omega_0$.

Proof. We set ρ_0 to be the identity. Then the equation $\rho_t^*\omega_t = \omega_0$ is equivalent to $\rho_t^*\omega_t$ being constant. If v_t is the associated vector field to ρ_t, then we need
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^*\omega_t) = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right).$$
So we want to solve
$$\mathcal{L}_{v_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t} = 0.$$
To solve for this, since $[\frac{\mathrm{d}\omega_t}{\mathrm{d}t}] = 0$, it follows that there is a family μ_t of 1-forms such that $\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \mathrm{d}\mu_t$. Then our equation becomes
$$\mathcal{L}_{v_t}\omega_t + \mathrm{d}\mu_t = 0.$$
By assumption, $\mathrm{d}\omega_t = 0$. So by Cartan's magic formula, we get
$$\mathrm{d}\iota_{v_t}\omega_t + \mathrm{d}\mu_t = 0.$$

To solve this equation, it suffices to solve *Moser's equation*,

$$\iota_{v_t}\omega_t + \mu_t = 0,$$

which can be solved since ω_t is non-degenerate. $\square$

Theorem (Relative Moser)**.** Let $X \subseteq M$ be a compact manifold of a manifold M, and ω_0, ω_1 symplectic forms on M agreeing on X. Then there are neighbourhoods U_0, U_1 of X and a diffeomorphism $\varphi : U_0 \to U_1$ fixing X such that $\varphi^*\omega_1 = \omega_0$.

Proof. We set

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Then this is certainly closed, and since this is constantly ω_0 on X, it is non-degenerate on X, hence, by compactness, non-degenerate on a small tubular neighbourhood U_0 of X (by compactness). Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \omega_1 - \omega_0,$$

and we know this vanishes on X. Since the inclusion of X into a tubular neighbourhood is a homotopy equivalence, we know $[\omega_1 - \omega_0] = 0 \in H^1_{\mathrm{dR}}(U_0)$. Thus, we can find some μ such that $\omega_1 - \omega_0 = \mathrm{d}\mu$, and by translation by a constant, we may suppose μ vanishes on X. We then solve Moser's equation, and the resulting ρ will be constant on X since μ vanishes. $\square$

Theorem (Darboux theorem)**.** If (M, ω) is a symplectic manifold, and $p \in M$, then there is a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ about p on which

$$\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Proof. ω can certainly be written in this form *at* p. Then relative Moser with $X = \{p\}$ promotes this to hold in a neighbourhood of p. $\square$

Proposition. Let $\pi : M = T^*X \to X$ be the projection map, and $\pi^* : T^*X \to T^*M$ the pullback map. Then for $\xi \in M$, we have $\alpha_\xi = \pi^*\xi$.

Proof. On a chart, we have

$$\pi^*\xi\left(\frac{\partial}{\partial x_j}\right) = \xi\left(\frac{\partial}{\partial x_j}\right) = \xi_j = \alpha_\xi\left(\frac{\partial}{\partial x_j}\right),$$

and similarly both vanish on $\frac{\partial}{\partial \xi_j}$. $\square$

Proposition. Let μ be a one-form of X, i.e. a section $s_\mu : X \to T^*X$. Then $s_\mu^*\alpha = \mu$.

Proof. By definition,

$$\alpha_\xi = \xi \circ \mathrm{d}\pi.$$

So for $x \in X$, we have

$$s_\mu^*\alpha_{\mu(x)} = \mu(x) \circ \mathrm{d}\pi \circ \mathrm{d}s_\mu = \mu(x) \circ \mathrm{d}(\pi \circ s_\mu) = \mu(x) \circ \mathrm{d}(\mathrm{id}_X) = \mu(x). \quad \square$$

1.3 Symplectomorphisms and Lagrangians

Proposition. Let $L = N^*S$ and $M = T^*X$. Then $L \hookrightarrow M$ is a Lagrangian submanifold.

Proof. If L is k-dimensional, then $S \hookrightarrow X$ locally looks like $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$, and it is clear in this case. □

Proposition. f is a symplectomorphism iff T_f is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.

Proof. The first condition is $f^*\omega_2 = \omega_1$, and the second condition is $\gamma_f^*\tilde{\omega} = 0$, and

$$\gamma_f^*\tilde{\omega} = \gamma_f^*\mathrm{pr}_1^*\omega_1 = \gamma_f^*\mathrm{pr}_2^*\omega_2 = \omega_1 - f^*\omega_2.$$ □

1.4 Periodic points of symplectomorphisms

Proposition. The fixed point of φ are in one-to-one correspondence with the critical points of ψ.

Proposition. The n-periodic points of φ are in one-to-one correspondence with the critical points of

$$\psi_n(x_1, \ldots, x_n) = f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_n, x_1).$$

Theorem (Poincaré's last geometric theorem (Birkhoff, 1925))**.** Let $\varphi : A \to A$ be an area-preserving diffeomorphism such that φ preserves the boundary components, and twists them in opposite directions. Then φ has at least two fixed points.

1.5 Lagrangian submanifolds and fixed points

Theorem (Lagrangian neighbourhood theorem)**.** Let (M, ω) be a symplectic manifold, X a compact Lagrangian submanifold, and ω_0 the canonical symplectic form on T^*X. Then there exists neighbourhoods U_0 of X in T^*X and U of X in M and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ sending X to X.

Theorem (Weinstein)**.** Let M be a $2n$-dimensional manifold, X n-dimensional compact submanifold, and $i : X \hookrightarrow M$ the inclusion, and symplectic forms ω_0, ω_1 on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e. X is Lagrangian with respect to both symplectic structures. Then there exists neighbourhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M such that $\rho|_X = \mathrm{id}_X$ and $\rho^*\omega_1 = \omega_0$.

Proof of equivalence. If (V, Ω) is a symplectic vector space, L a Lagrangian subspace, the bilinear form

$$\begin{aligned} \Omega : V/L \times L &\to \mathbb{R} \\ ([v], u) &\to \Omega(v, u). \end{aligned}$$

is non-degenerate and gives a natural isomorphism $V/L \cong L^*$. Taking $V = T_pM$ and $L = T_pX$, we get an isomorphism

$$NX = TM|_X/TX \cong T^*X.$$

Thus, by the standard tubular neighbourhood theorem, there is a neighbourhood $\mathcal{N}_0$ of X in NX and a neighbourhood $\mathcal{N}$ of X in M, and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$. We now have two symplectic forms on $\mathcal{N}_0$ — the one from the cotangent bundle and the pullback of that from M. Then applying the second theorem gives the first.

Conversely, if we know the first theorem, then applying this twice gives us symplectomorphisms between neighbourhoods of X in M under each symplectic structure with a neighbourhood in the cotangent bundle. □

Proof of second theorem. For $p \in X$, we define $V = T_pM$ and $U = T_pX$, and W any complement of U. By assumption, U is a Lagrangian subspace of both $(V, \omega_0|_p = \Omega_0)$ and $(V, \omega_1|_p = \Omega_1)$. We apply the following linear-algebraic lemma:

Lemma. Let V be a $2n$-dimensional vector space, Ω_0, Ω_1 symplectic structures on V. Suppose U is a subspace of V Lagrangian with respect to both Ω_0 and Ω_1, and W is any complement of V. Then we can construct canonically a linear isomorphism $H : V \to V$ such that $H|_U = \mathrm{id}_U$ and $H^*\Omega_1 = \Omega_2$.

Note that the statement of the theorem doesn't mention W, but the construction of H requires a complement of V, so it is canonical only after we pick a W.

By this lemma, we get canonically an isomorphism $H_p : T_pM \to T_pM$ such that $H_p|_{T_pX} = \mathrm{id}_{T_pX}$ and $H_p^*\omega_1|_p = \omega_0|_p$. The canonicity implies H_p varies smoothly with p. We now apply the Whitney extension theorem

Theorem (Whitney extension theorem)**.** Let X be a submanifold of M, $H_p : T_pM \to T_pM$ smooth family of isomorphisms such that $H_p|_{T_pX} = \mathrm{id}_{T_pX}$. Then there exists an neighbourhood $\mathcal{N}$ of X in M and an embedding $h : \mathcal{N} \to M$ such that $h|_X = \mathrm{id}_X$ and for all $p \in X$, $\mathrm{d}h_p = H_p$.

So at $p \in X$, we have $h^*\omega_1|_p = (\mathrm{d}h_p)^*\omega_1|_P = h_p^*\omega_1|_p = \omega_0|_p$. So we are done by relative Moser. □

Theorem. Let (M, ω) be a compact symplectic manifold such that $H^1_{\mathrm{dR}}(M) = 0$. Then any symplectomorphism $\varphi : M \to M$ sufficiently close to the identity has at least two fixed points.

Proof. The graph of φ corresponds to a closed 1-form on M. Since μ is closed and $H^1_{\mathrm{dR}}(M) = 0$, we know $\mu = \mathrm{d}h$ for some $h \in C^\infty$. Since M is compact, h has at least two critical points (the global maximum and minimum). Since the fixed points corresponding to the points where μ vanish (so that Γ_φ intersects Δ), we are done. □

Theorem (Arnold conjecture)**.** Let (M, ω) be a compact symplectic manifold of dimension $2n$, and $\varphi : M \to M$ a symplectomorphism. Suppose φ is *exactly homotopic* to the identity and *non-degenerate.* Then the number of fixed points of φ is at least $\sum_{i=0}^{2n} \dim H^i(M, \mathbb{R})$.

2 Complex structures

2.1 Almost complex structures

Lemma. There is a correspondence between real vector spaces with a complex structure and complex vector spaces, where J acts as multiplication by i. □

Proposition (Polar decomposition)**.** Let (V, Ω) be a symplectic vector space, and G an inner product on V. Then from G, we can *canonically* construct a compatible complex structure J on (V, Ω). If $G = G_J$ for some J, then this process returns J.

Proof. Since Ω, G are non-degenerate, we know

$$\Omega(u, v) = G(u, Av)$$

for some $A : V \to V$. If $A^2 = -1$, then we are done, and set $J = A$. In general, we also know that A is skew-symmetric, i.e. $G(Au, v) = G(u, -Av)$, which is clear since Ω is anti-symmetric. Since AA^t is symmetric and positive definite, it makes sense to write down $\sqrt{AA^T}$ (e.g. by diagonalizing), and we take

$$J = \sqrt{AA^t}^{-1} A = \sqrt{-A^2}^{-1} A.$$

It is clear that $J^2 = -1$, since A commutes with $\sqrt{AA^t}$, so this is a complex structure. We can write this as $A = \sqrt{AA^T} J$, and this is called the *(left) polar decomposition* of A.

We now check that J is a compatible, i.e. $G_J(u, v) = \Omega(u, Jv)$ is symmetric and positive definite. But

$$G_J(u, v) = G(u, \sqrt{AA^t} v),$$

and we are done since $\sqrt{AA^t}$ is positive and symmetric. □

Proposition. $\mathcal{J}(V, \Omega)$ is path-connected.

Proof. Let $J_0, J_1 \in \mathcal{J}(V, \Omega)$. Then this induces inner products G_{J_0}, G_{J_1}. Let $G_t = (1 - t)G_{J_0} + tG_{J_1}$ be a smooth family of inner products on V. Then apply polar decomposition to get a family of complex structures that start from J_0 to J_1. □

Proposition. Let (M, ω) be a symplectic manifold, and g a metric on M. Then from g we can canonically construct a compatible almost complex structure J. □

Corollary. Any symplectic manifold has a compatible almost complex structure. □

Proposition. $J(M, \omega)$ is contractible. □

Proposition. Let J be an almost complex structure on M that is compatible with ω_0 and ω_1. Then ω_0 and ω_1 are deformation equivalent.

Proof. Check that $\omega_t = (1 - t)\omega_0 + t\omega_1$ works, which is non-degenerate since $\omega_t(\,\cdot\,, J\,\cdot\,)$ is a positive linear combination of inner products, hence is non-degenerate. □

Proposition. Let (M, ω) be a symplectic manifold, J a compatible almost complex structure. If X is an almost complex submanifold of (M, J), i.e. $J(TX) = TX$, then X is a symplectic submanifold of (M, ω).

Proof. We only have to check $\omega|_{TX}$ is non-degenerate, but $\Omega(\,\cdot\,, J\,\cdot\,)$ is a metric, so is in particular non-degenerate. □

Theorem (Gromov)**.** Let (M, J) be an almost complex manifold with M open, i.e. M has no closed connected components. Then there exists a symplectic form ω in any even 2-cohomology class and such that J is homotopic to an almost complex structure compatible with ω. □

2.2 Dolbeault theory

Theorem (Newlander–Nirenberg)**.** The following are equivalent:

- $\bar{\partial}^2 = 0$
- $\partial^2 = 0$
- $\mathrm{d} = \partial + \bar{\partial}$
- J is integrable
- $\mathcal{N} = 0$

where $\mathcal{N}$ is the *Nijenhuis torsion*

$$\mathcal{N}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]. \qquad \square$$

2.3 Kähler manifolds

Lemma. $\omega \in \Omega^{1,1}$.

Proof. Since $\omega(\,\cdot\,, J\,\cdot\,)$ is symmetric, we have

$$J^*\omega(u, v) = \omega(Ju, Jv) = \omega(v, JJu) = -\omega(v, -u) = \omega(u, v).$$

So $J^*\omega = \omega$.

On the other hand, J^* acts on holomorphic forms as multiplication by i and anti-holomorphic forms by multiplication by -1 (by definition). So it acts on $\Omega^{2,0}$ and $\Omega^{0,2}$ by multiplication by -1 (locally $\Omega^{2,0}$ is spanned by $\mathrm{d}z_i \wedge \mathrm{d}z_j$, etc.), while it fixes $\Omega^{1,1}$. So ω must lie in $\Omega^{1,1}$. □

Theorem. A Kähler form ω on a complex manifold M is a ∂- and $\bar{\partial}$-closed form of type $(1, 1)$ which on a local chart is given by

$$\omega = \frac{i}{2} \sum_{j,k} h_{jk}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

where at each point, the matrix (h_{jk}) is Hermitian and positive definite.

Proposition. Let (M, ω) be a complex Kähler manifold. If $X \subseteq M$ is a complex submanifold, then $(X, i^*\omega)$ is Kähler, and this is called a *Kähler submanifold*. □

Proposition. Let M be a complex manifold, and $\rho \in C^\infty(M; \mathbb{R})$ strictly plurisubharmonic. Then

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

is a Kähler form.

Proof. ω is indeed a 2-form of type (1, 1). Since $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$, we know $\partial\omega = \bar{\partial}\omega = 0$. We also have

$$\omega = \frac{i}{2}\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

and the matrix is Hermitian positive definite by assumption. □

Proposition. Let M be a complex manifold, ω a closed real-valued $(1,1)$-form and $p \in M$, then there exists a neighbourhood U of p in M and a $\rho \in C^\infty(U, \mathbb{R})$ such that

$$\omega = i\partial\bar{\partial}\rho \text{ on } U.$$

Proof. This uses the holomorphic version of the Poincaré lemma. □

2.4 Hodge theory

Theorem (Hodge decomposition theorem)**.** Let (M, ω) be a compact Kḧaler manifold. Then

$$H^k_{\mathrm{dR}} \cong \bigoplus_{p+q=k} H^{p,q}_{\mathrm{Dolb}}(M).$$

Proposition.

- $*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_m$
- $*(e_{k+1} \wedge \cdots \wedge e_m) = (-1)^{k(m-k)} e_1 \wedge \cdots \wedge e_k$.
- $** = \alpha = (-1)^{k(m-k)}\alpha$ for $\alpha \in \Lambda^k$. □

Proposition.

(i) $* * \alpha = (-1)^{k(m-k)}\alpha$ for $\alpha \in \Omega^k(M)$.

(ii) $*1 = \mathrm{Vol}$ □

Proposition. $\delta^2 = 0$.

Proposition.

$$\delta = (-1)^{m(k+1)+1} * \mathrm{d} * : \Omega^k \to \Omega^{k-1}.$$

Proof.

$$\begin{aligned}
\langle \mathrm{d}\alpha, \beta\rangle_{L^2} &= \int_M \mathrm{d}\alpha \wedge *\beta \\
&= \int_M \mathrm{d}(\alpha \wedge *\beta) - (-1)^k \int_M \alpha \wedge \mathrm{d}(*\beta) \\
&= (-1)^{k+1} \int_M \alpha \wedge \mathrm{d}(*\beta) && \text{(Stokes')} \\
&= (-1)^{k+1} \int_M (-1)^{(m-k)k} \alpha \wedge * * \mathrm{d}(*\beta) \\
&= (-1)^{k+1+(m-k)k} \int_M \langle \alpha, * \,\mathrm{d} *\beta\rangle.
\end{aligned}$$
□

Proposition.

(i) $\Delta * = * \Delta : \Omega^k \to \Omega^{m-k}$

(ii) $\Delta = (\mathrm{d} + \delta)^2$

(iii) $\langle \Delta\alpha, \beta\rangle_{L^2} = \langle \alpha, \Delta\beta\rangle_{L^2}$.

(iv) $\Delta\alpha = 0$ iff $\mathrm{d}\alpha = \delta\alpha = 0$. □

Theorem (Hodge decomposition theorem)**.** Let (M, g) be a compact oriented Riemannian manifold. Then every cohomology class in $H^k_{\mathrm{dR}}(M)$ has a unique harmonic representation, i.e. the natural map $\mathcal{H}^k \to H^k_{\mathrm{dR}}(M)$ is an isomorphism.

Proposition. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$ and (M, ω) Kähler. Then

(i) $* : \Omega^{p,q} \to \Omega^{n-p,n-q}$.

(ii) $\Delta : \Omega^{p,q} \to \Omega^{p,q}$. □

Proposition. If our manifold is Kähler, then

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}. \qquad \square$$

Theorem (Hodge decomposition theorem)**.** Let (M, ω) be a compact Kähler manifold. The natural map $\mathcal{H}^{p,q} \to H^{p,q}_{\mathrm{Dolb}}$ is an isomorphism. Hence

$$H^k_{\mathrm{dR}}(M; \mathbb{C}) \cong \mathcal{H}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong \bigoplus_{p+q=k} H^{p,q}_{\mathrm{Dolb}}(M). \qquad \square$$

Corollary. Odd Betti numbers are even.

Proof.

$$b_{2k+1} = \sum_{p+q=2k+1} h_{p,q} = 2\left(\sum_{p=0}^{k} h_{p,2k+1-p}\right). \qquad \square$$

Corollary. $h_{1,0} = h_{0,1} = \frac{1}{2} b_1$ is a topological invariant.

Proposition. Even Betti numbers are positive.

Proposition. $h_{k,k} \neq 0$. □

3 Hamiltonian vector fields

3.1 Hamiltonian vector fields

Proposition. If X_H is a Hamiltonian vector field with flow ρ_t, then $\rho_t^*\omega = \omega$. In other words, each ρ_t is a symplectomorphism.

Proof. It suffices to show that $\frac{\partial}{\partial t}\rho_t^*\omega = 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^*\omega) = \rho_t^*(\mathcal{L}_{X_H}\omega) = \rho_t^*(\mathrm{d}\iota_{X_H}\omega + \iota_{X_H}\mathrm{d}\omega) = \rho_t^*(\mathrm{dd}H) = 0. \qquad \square$$

Proposition. ρ_t preserves H, i.e. $\rho_t^* H = H$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^* H = \rho_t^*(\mathcal{L}_{X_H} H) = \rho_t^*(\iota_{X_H}\mathrm{d}H) = \rho_t^*(\iota_{X_H}\iota_{X_H}\omega) = 0. \qquad \square$$

Proposition. Let X, Y be symplectic vector fields on (M, ω). Then $[X, Y]$ is Hamiltonian.

Proof of proposition. We need to check that $\iota_{[X,Y]}\omega$ is exact. By the exercise, this is

$$\iota_{[X,Y]}\omega = \mathcal{L}_X \iota_Y \omega - \iota_Y \mathcal{L}_X \omega = \mathrm{d}(\iota_X\iota_Y\omega) + \iota_X \mathrm{d}\iota_Y\omega + \iota_Y\mathrm{d}\iota_X\omega - \iota_Y\iota_Y\mathrm{d}\omega.$$

Since X, Y are symplectic, we know $\mathrm{d}\iota_Y\omega = \mathrm{d}\iota_X\omega = 0$, and the last term always vanishes. So this is exact, and $\omega(Y, X)$ is a Hamiltonian function for $[X, Y]$. $\square$

Proposition. $\{f, g\} = 0$ iff f is constant along integral curves of X_g.

Proof.

$$\mathcal{L}_{X_g} f = \iota_{X_g}\,\mathrm{d}f = \iota_{X_g}\iota_{X_f}\omega = \omega(X_f, X_g) = \{f, g\} = 0. \qquad \square$$

3.2 Integrable systems

Theorem (Arnold–Liouville thoerem)**.** Let (M, ω, H) be an integrable system with $\dim M = 2n$ and $f_1 = H, f_2, \ldots, f_n$ integrals of motion, and $c \in \mathbb{R}$ a regular value of $f = (f_1, \ldots, f_n)$.

(i) If the flows of X_{f_i} are complete, then the connected components of $f^{-1}(\{c\})$ are homogeneous spaces for $\mathbb{R}^n$ and admit affine coordinates $\varphi_1, \ldots, \varphi_n$ (*angle coordinates*), in which the flows of X_{f_i} are linear.

(ii) There exists coordinates $\psi_1, \ldots, \psi_n$ (*action coordinates*) such that the ψ_i's are integrals of motion and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ form a Darboux chart.

3.3 Classical mechanics

3.4 Hamiltonian actions

3.5 Symplectic reduction

Theorem (Marsden–Weinstein, Meyer)**.** Let G be a compact Lie group, and (M, ω) a symplectic manifold with a Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$. Write $i : \mu^{-1}(0) \hookrightarrow M$ for the inclusion. Suppose G acts freely on $\mu^{-1}(0)$. Then

(i) $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold;

(ii) $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle; and

(iii) There exists a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{\text{red}}$.

Proof. We first show that $\mu^{-1}(0)$ is a manifold. This follows from the following claim:

Claim. G acts locally freely at p iff p is a regular point of μ.

We compute the dimension of $\operatorname{im} \mathrm{d}\mu_p$ using the rank-nullity theorem. We know $\mathrm{d}\mu_p v = 0$ iff $\langle \mathrm{d}\mu_p(v), X\rangle = 0$ for all $X \in \mathfrak{g}$. We can compute

$$\langle \mathrm{d}\mu_p(v), X\rangle = (\mathrm{d}\mu^X)_p(v) = (\iota_{X_p^\#}\omega)(v) = \omega_p(X_p^\#, v).$$

Moreover, the span of the $X_p^\#$ is exactly $T_p\mathcal{O}_p$. So

$$\ker \mathrm{d}\mu_p = (T_p\mathcal{O}_p)^\omega.$$

Thus,

$$\dim(\operatorname{im} \mathrm{d}\mu_p) = \dim \mathcal{O}_p = \dim G - \dim G_p.$$

In particular, $\mathrm{d}\mu_p$ is surjective iff $G_p = 0$.

Then (i) and (ii) follow from the following theorem:

Theorem. Let G be a compact Lie group and Z a manifold, and G acts freely on Z. Then Z/G is a manifold and $Z \to Z/G$ is a principal G-bundle.

Note that if G does not act freely on $\mu^{-1}(0)$, then by Sard's theorem, generically, ξ is a regular value of μ, and so $\mu^{-1}(\xi)$ is a manifold, and G acts locally freely on $\mu^{-1}(\xi)$. If $\mu^{-1}(\xi)$ is preserved by G, then $\mu^{-1}(\xi)/G$ is a symplectic orbifold.

It now remains to construct the symplectic structure. Observe that if $p \in \mu^{-1}(0)$, then

$$T_p\mathcal{O}_p \subseteq T_p\mu^{-1}(0) = \ker \mathrm{d}\mu_p = (T_p\mathcal{O}_p)^\omega.$$

So $T_p\mathcal{O}_p$ is an isotropic subspace of (T_pM, ω). We then observe the following straightforward linear algebraic result:

Lemma. Let (V, Ω) be a symplectic vector space and I an isotropic subspace. Then Ω induces a canonical symplectic structure Ω_{red} on I^Ω/I, given by $\Omega_{\text{red}}([u], [v]) = \Omega(u, v)$.

Applying this, we get a canonical symplectic structure on

$$\frac{(T_p\mathcal{O}_p)^\omega}{T_p\mathcal{O}_p} = \frac{T_p\mu^{-1}(0)}{T_p\mathcal{O}_p} = T_{[p]}M_{\text{red}}.$$

This defines ω_{red} on M_{red}, which is well-defined because ω is G-invariant, and is smooth by local triviality and canonicity.

It remains to show that $\mathrm{d}\omega_{\text{red}} = 0$. By construction, $i^*\omega = \pi^*\omega_{\text{red}}$. So

$$\pi^*(\mathrm{d}\omega_{\text{red}}) = \mathrm{d}\pi^*\omega_{\text{red}} = \mathrm{d}i^*\omega = i^*\mathrm{d}\omega = 0$$

Since π^* is injective, we are done. □

Lemma. Giving an Ehresmann connection is the same as giving a connection 1-form.

Proof. Given an Ehresmann connection H, we define

$$A_p(v) = X,$$

where $v = X_p^\# + h_p \in V \oplus H$.

Conversely, given an A, we define

$$H_p = \ker A_p = \{v \in T_pP : i_v A_p = 0\}. \qquad \square$$

3.6 The convexity theorem

Theorem (Convexity theorem (Atiyah, Guillemin–Sternberg))**.** Let (M, ω) be a compact connected symplectic manifold, and

$$\mu : M \to \mathbb{R}^n$$

a moment map for a Hamiltonian torus action. Then

(i) The levels $\mu^{-1}(c)$ are connected for all c

(ii) The image $\mu(M)$ is convex.

(iii) The image $\mu(M)$ is in fact the convex hull of $\mu(M^G)$.

We call $\mu(M)$ the *moment polytope*.

Lemma. (ii) implies (iii).

Proof. Suppose the fixed point set of the action has k connected components $Z = Z_1 \cup \cdots \cup Z_k$. Then μ is constant on each Z_j, since $X^\#|_{Z_j} = 0$ for all X. Let $\mu(Z_j) = \eta_j \in \mathbb{R}^n$. By (ii), we know the convex hull of $\{\eta_1, \ldots \eta_k\}$ is contained in $\mu(M)$.

To see that $\mu(M)$ is exactly the convex hull, observe that if $X \in \mathbb{R}^n$ has rationally independent components, so that X topologically generates T, then p is fixed by T iff $X_p^\# = 0$, iff $\mathrm{d}\mu_p^X = 0$. Thus, μ^X attains its maximum on one of the Z_j.

Now if ξ is not in the convex hull of $\{\eta_j\}$, then we can pick an $X \in \mathbb{R}^n$ with rationally independent components such that $\langle \xi, X\rangle > \langle \eta_j, X\rangle$ for all j, since the space of such X is open and non-empty. Then

$$\langle \xi, X\rangle > \sup_{p \in \bigcup Z_j} \langle \mu(p), X\rangle = \sup_{p \in M} \langle \mu(p), X\rangle.$$

So $\xi \notin \mu(M)$. $\square$

Lemma. (i) implies (ii).

Proof. The case $n = 1$ is immediate, since $\mu(M)$ is compact and connected, hence a closed interval.

In general, to show that $\mu(M)$ is convex, we want to show that the intersection of $\mu(M)$ with any line is connected. In other words, if $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is any projection and $\nu = \pi \circ \mu$, then

$$\pi^{-1}(c) \cap \mu(M) = \mu(\nu^{-1}(c))$$

is connected. This would follow if we knew $\nu^{-1}(c)$ were connected, which would follow from (i) if ν were a moment map of an T^n action. Unfortunately, most of the time, it is just the moment map of an $\mathbb{R}^n$ action. For it to come from a T^n action, we need π to be represented by an *integer* matrix. Then

$$T = \{\pi^T t : t \in T^n = \mathbb{R}^n/\mathbb{Z}^n\} \subseteq T^{n+1}$$

is a subtorus, and one readily checks that ν is the moment map for the T action.

Now for any $p_0, p_1 \in M$, we can find p_0', p_1' arbitrarily close to p_0, p_1 and a line of the form $\pi^{-1}(c)$ with π integral. Then the line between p_0' and p_1' is contained in $\mu(M)$ by the above argument, and we are done since $\mu(M)$ is compact, hence closed. □

Theorem (Morse theory)**.**

(i) If $f^{-1}([c_1, c_2])$ does not contain any critical point. Then $f^{-1}(c_1) \cong f^{-1}(c_2)$ and $M_{c_1} \cong M_{c_2}$ (where $\cong$ means diffeomorphic).

(ii) If $f^{-1}([c_1, c_2])$ contains one critical manifold Z, then $M^-_{c_2} \simeq M^-_{c_1} \cup D(E^-)$, where $D(E^-)$ is the disk bundle of E^-.

In particular, if Z is an isolated point, $M^-_{c_2}$ is, up to homotopy equivalence, obtained by adding a $\dim E^-_p$-cell to $M^-_{c_1}$. □

Lemma. Let M be a compact connected manifold, and $f : M \to \mathbb{R}$ a Morse–Bott function with no critical submanifold of index or coindex 1. Then

(i) f has a unique local maximum and local minimum

(ii) All level sets of $f^{-1}(c)$ are connected.

Proof sketch. There is always a global minimum since f is compact. If there is another local minimum at c, then the disk bundle is trivial, and so in

$$M^-_{c+\varepsilon} \simeq M^-_{c-\varepsilon} \cup D(E^-)$$

for ε small enough, the union is a disjoint union. So $M_{c+\varepsilon}$ has two components. Different connected components can only merge by crossing a level of index 1, so this cannot happen. To handle the maxima, consider $-f$.

More generally, the same argument shows that a change in connectedness must happen by passing through a index or coindex 1 critical submanifold. □

Lemma. For any $X \in \mathbb{R}^n$, μ^X is a Morse–Bott function where all critical submanifolds are symplectic.

Proof sketch. Note that p is a fixed point iff $X^\#_p = 0$ iff $\mathrm{d}\mu^X_p = 0$ iff p is a critical point. So the critical points are exactly the fixed points of X.

(T_pM, ω_p) models (M, ω) in a neighbourhood of p by Darboux theorem. Near a fixed point T^n, an equivariant version of the Darboux theorem tells us there is a coordinate chart U where (M, ω, μ) looks like

$$\begin{aligned} \omega|_U &= \sum \mathrm{d}x_i \wedge \mathrm{d}y_i \\ \mu|_U &= \mu(p) - \frac{1}{2}\sum_i (x_i^2 + y_i^2)\alpha_i, \end{aligned}$$

where $\alpha_i \in \mathbb{Z}$ are *weights*.

Then the critical submanifolds of μ are given by

$$\{x_i = y_i = 0 : \alpha_i \neq 0\},$$

which is locally a symplectic manifold and has even index and coindex. □

Lemma. (i) holds.

Proof. The $n = 1$ case follows from the previous lemmas. We then induct on n.

Suppose the theorem holds for n, and let $\mu = (\mu_1, \ldots, \mu_n) : M \to \mathbb{R}^{n+1}$ be a moment map for a Hamiltonian T^{n+1}-action. We want to show that for all $c = (c_1, \ldots, c_n) \in \mathbb{R}^{n+1}$, the set

$$\mu^{-1}(c) = \mu_1^{-1}(c_1) \cap \cdots \cap \mu_{n+1}^{-1}(c_{n+1})$$

is connected.

The idea is to set

$$N = \mu_1^{-1}(c_1) \cap \cdots \mu_1^{-1}(c_n),$$

and then show that $\mu_{n+1}|_N : N \to \mathbb{R}$ is a Morse–Bott function with no critical submanifolds of index or coindex 1.

We may assume that $\mathrm{d}\mu_1, \ldots, \mathrm{d}\mu_n$ are linearly independent, or equivalently, $\mathrm{d}\mu^X \neq 0$ for all $X \in \mathbb{R}^n$. Otherwise, this reduces to the case of an n-torus.

To make sense of N, we must pick c to be a regular value. Density arguments imply that

$$\mathcal{C} = \bigcup_{X \neq 0} \mathrm{Crit}(\mu^X) = \bigcup_{X \in \mathbb{Z}^{n+1} \setminus \{0\}} \mathrm{Crit}\, \mu^X.$$

Since $\mathrm{Crit}\, \mu^X$ is a union of codimension ≥ 2 submanifolds, its complement is dense. Hence by the Baire category theorem, $\mathcal{C}$ has dense complement. Then a continuity argument shows that we only have to consider the case when c is a regular value of μ, hence N is a genuine submanifold of codimension n.

By the induction hypothesis, N is connected. We now show that $\mu_{n+1}|_N : N \to \mathbb{R}$ is Morse–Bott with no critical submanifolds of index or coindex 1.

Let x be a critical point. Then the theory of Lagrange multipliers tells us there are some $\lambda_i \in \mathbb{R}$ such that

$$\left[\mathrm{d}\mu_{n+1} + \sum_{n=1}^{n} \lambda_i \mathrm{d}\mu_i\right]_x = 0$$

Thus, μ is critical in M for the function

$$\mu^Y = \mu_{n+1} + \sum_{i=1}^{n} \lambda_i \mu_i,$$

where $Y = (\lambda_1, \ldots, \lambda_n, 1) \in \mathbb{R}^{n+1}$. So by the claim, μ^Y is Morse–Bott with only even indices and coindices. Let W be a critical submanifold of μ^Y containing x.

Claim. W intersects N transversely at x.

If this were true, then $\mu^Y|_N$ has $W \cap N$ as a non-degenerate critical submanifold of even index and coindex, since the coindex doesn't change and W is even-dimensional. Moreover, when restricted to N, $\sum \lambda_i \mu_i$ is a constant. So $\mu_{n+1}|_N$ satisfies the same properties.

To prove the claim, note that

$$T_x N = \ker \mathrm{d}\mu_1|_x \cap \cdots \cap \ker \mathrm{d}\mu_n|_x.$$

With a moments thought, we see that it suffices to show that $\mathrm{d}\mu_1, \ldots, \mathrm{d}\mu_n$ remain linearly independent when restricted to $T_x W$. Now observe that the Hamiltonian vector fields $X_1^\#|_x, \ldots, X_n^\#|_x$ are independent since $\mathrm{d}\mu_1|_x, \ldots \mathrm{d}\mu_n|_x$ are, and they live in $T_x W$ since their flows preserve W.

Since W is symplectic (by the claim), for all $k = (k_1, \ldots k_n)$, there exists $v \in T_x W$ such that

$$\omega\left(\sum k_i X_i^\#|_x, v\right) \neq 0.$$

In other words,

$$\left(\sum k_i \mathrm{d}\mu_i\right)(v) \neq 0.$$

□

Theorem (Kirwan, 1984). $\mu_+(M) \subseteq \mathfrak{t}_+^*$ is a convex polytope.

Theorem (Schur–Horn theorem). $\varphi(\mathcal{H}_\lambda^n)$ is the convex hull of the $n!$ points from the diagonal matrices.

3.7 Toric manifolds

Proposition. Let (M, ω) be a compact, connected symplectic manifold with moment map $\mu : M \to \mathbb{R}^n$ for a Hamiltonian T^n action. If the T^n action is effective, then

(i) There are at least $n + 1$ fixed points.

(ii) $\dim M \geq 2n$.

Proof.

(i) If $\mu = (\mu_1, \ldots, \mu_n) : M \to \mathbb{R}^n$ and p is a point in an n-dimensional orbit, then $\{(\mathrm{d}\mu_i)_p\}$ are linearly independent. So $\mu(p)$ is an interior point (if p is not in the interior, then there exists a direction X pointing out of $\mu(M)$. So $(\mathrm{d}\mu^X)_p = 0$, and $\mathrm{d}\mu^X$ gives a non-trivial linear combination of the $\mathrm{d}\mu_i$'s that vanishes).

So if there is an interior point, we know $\mu(M)$ is a non-degenerate polytope in $\mathbb{R}^n$. This mean it has at least $n + 1$ vertices. So there are at least $n + 1$ fixed points.

(ii) Let $\mathcal{O}$ be an orbit of p in M. Then μ is constant on $\mathcal{O}$ by invariance of μ. So

$$T_p \mathcal{O} \subseteq \ker(\mathrm{d}\mu_p) = (T_p \mathcal{O})^\omega.$$

So all orbits of a Hamiltonian torus action are isotropic submanifolds. So $\dim \mathcal{O} \leq \frac{1}{2} \dim M$. So we are done. □

Theorem (Delzant)**.** There are correspondences

$$\begin{Bmatrix}\text{symplectic toric manifolds}\\ \text{up to equivalence}\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{Delzant polytopes}\end{Bmatrix}$$

$$\begin{Bmatrix}\text{symplectic toric manifolds}\\ \text{up to weak equivalence}\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{Delzant polytopes}\\ \text{modulo } \mathrm{AGL}(n, \mathbb{Z})\end{Bmatrix}$$

Proof sketch. Given a Delzant polytope Δ in $(\mathbb{R}^n)^*$ with d facets, we want to construct $(M_\Delta, \omega_\Delta, T_\Delta, \mu_\Delta)$ with $\mu_\Delta(M_\Delta) = \Delta$. The idea is to perform the construction for the "universal" Delzant polytope with d facets, and then obtain the desired M_Δ as a symplectic reduction of this universal example. As usual, the universal example will be "too big" to be a genuine symplectic toric manifold. Instead, it will be non-compact.

If Δ has d facets with primitive outward-point normal vectors $v_1, \ldots, v_d$ (i.e. they cannot be written as a $\mathbb{Z}$-multiple of some other $\mathbb{Z}$-vector), then we can write Δ as

$$\Delta = \{x \in (\mathbb{R}^n)^* : \langle x, v_i\rangle \leq \lambda_i \text{ for } i = 1, \ldots, d\}$$

for some λ_i.

There is a natural (surjective) map $\pi : \mathbb{R}^d \to \mathbb{R}^n$ that sends the basis vector e_d of $\mathbb{R}^d$ to v_d. If $\lambda = (\lambda_1, \ldots, \lambda_d)$, and we have a pullback diagram

$$\begin{array}{ccc} \Delta & \longrightarrow & \mathbb{R}^d_\lambda \\ \big\downarrow & & \big\downarrow \\ (\mathbb{R}^n)^* & \overset{\pi^*}{\hookrightarrow} & (\mathbb{R}^d)^* \end{array}$$

where

$$\mathbb{R}^d_\lambda = \{X \in (\mathbb{R}^d)^* : \langle X, e_i\rangle \leq \lambda_i \text{ for all } i\}.$$

In more down-to-earth language, this says

$$\pi^*(x) \in \mathbb{R}^d_\lambda \iff x \in \Delta,$$

which is evident from definition.

Now there is a universal "toric manifold" with $\mu(M) = \mathbb{R}^d_\lambda$, namely $(\mathbb{C}^d, \omega_0)$ with the diagonal action

$$(t_1, \ldots, t_d) \cdot (z_1, \ldots, z_d) = (e^{it_1} z_1, \ldots, e^{it_n} z_n),$$

using the moment map

$$\phi(z_1, \ldots, z_d) = -\frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (\lambda_1, \ldots, \lambda_d).$$

We now want to pull this back along π^*. To this extent, note that π sends $\mathbb{Z}^d$ to $\mathbb{Z}^n$, hence induces a map $T^d \to T^n$ with kernel N. If $\mathfrak{n}$ is the Lie algebra of N, then we have a short exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \overset{\pi^*}{\longrightarrow} (\mathbb{R}^d)^* \overset{i^*}{\longrightarrow} \mathfrak{n}^* \longrightarrow 0.$$

Since $\operatorname{im} \pi^* = \ker i^*$, the pullback of $\mathbb{C}^d$ along π^* is exactly

$$Z = (i^* \circ \phi)^{-1}(0).$$

It is easy to see that this is compact.

Observe that $i^* \circ \phi$ is exactly the the moment map of the induced action by N. So Z/N is the symplectic reduction of $\mathbb{C}^d$ by N, and in particular has a natural symplectic structure. It is natural to consider Z/N instead of Z itself, since Z carries a T^d action, but we only want to be left with a T^n action. Thus, after quotienting out by N, the T^d action becomes a $T^d/N \cong T^n$ action, with moment map given by the unique factoring of

$$Z \hookrightarrow \mathbb{C}^d \to (\mathbb{R}^d)^*$$

through $(\mathbb{R}^n)^*$. The image is exactly Δ. □

4 Symplectic embeddings

Theorem (Non-squeezing theorem, Gromov, 1985)**.** There is an embedding $B^2(n) \hookrightarrow Z^{2n}(R)$ iff $r < R$.

Proposition. The existence of a symplectic capacity is equivalent to Gromov's non-squeezing theorem.

Proof. The $\Rightarrow$ direction is clear by monotonicity and conformality. Conversely, if we know Gromov's non-squeezing theorem, we can define the *Gromov width*

$$W_G(M, \omega) = \sup\{\pi r^2 \mid (B^{2n}(r), \omega_0) \hookrightarrow (M, \omega)\}.$$

This clearly satisfies (i) and (ii), and (iii) follows from Gromov non-squeezing. Note that Darboux's theorem says there is always an embedding of $B^{2n}(r)$ into any symplectic manifold as long as r is small enough. $\square$