Part III — Symplectic Geometry
Theorems

Based on lectures by A. R. Pires
Notes taken by Dexter Chua

Lent 2018

The first part of the course will be an overview of the basic structures of symplectic geometry, including symplectic linear algebra, symplectic manifolds, symplectomorphisms, Darboux theorem, cotangent bundles, Lagrangian submanifolds, and Hamiltonian systems. The course will then go further into two topics. The first one is moment maps and toric symplectic manifolds, and the second one is capacities and symplectic embedding problems.

Pre-requisites

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

Contents

1 Symplectic manifolds

1.1 Symplectic linear algebra

Theorem (Standard form theorem)**.** Let V be a real vector space and Ω a skew-symmetric bilinear form. Then there is a basis $\{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that

(i) $\Omega(u_i, v) = 0$ for all $v \in V$

(ii) $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$.

(iii) $\Omega(e_i, f_j) = \delta_{ij}$.

1.2 Symplectic manifolds

Proposition. If a compact manifold M^{2n} is such that $H^{2k}_{\mathrm{dR}}(M) = 0$ for some $k < n$, then M does not admit a symplectic structure.

Theorem (Moser)**.** If M is compact with a family ω_t of symplectic forms with $[\omega_t]$ constant, then there is an isotopy $\rho_t : M \to M$ with $\rho_t^* \omega_t = \omega_0$.

Theorem (Relative Moser)**.** Let $X \subseteq M$ be a compact manifold of a manifold M, and ω_0, ω_1 symplectic forms on M agreeing on X. Then there are neighbourhoods U_0, U_1 of X and a diffeomorphism $\varphi : U_0 \to U_1$ fixing X such that $\varphi^* \omega_1 = \omega_0$.

Theorem (Darboux theorem)**.** If (M, ω) is a symplectic manifold, and $p \in M$, then there is a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ about p on which

$$\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Proposition. Let $\pi : M = T^* X \to X$ be the projection map, and $\pi^* : T^* X \to T^* M$ the pullback map. Then for $\xi \in M$, we have $\alpha_\xi = \pi^* \xi$.

Proposition. Let μ be a one-form of X, i.e. a section $s_\mu : X \to T^* X$. Then $s_\mu^* \alpha = \mu$.

1.3 Symplectomorphisms and Lagrangians

Proposition. Let $L = N^* S$ and $M = T^* X$. Then $L \hookrightarrow M$ is a Lagrangian submanifold.

Proposition. f is a symplectomorphism iff T_f is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.

1.4 Periodic points of symplectomorphisms

Proposition. The fixed point of φ are in one-to-one correspondence with the critical points of ψ.

Proposition. The n-periodic points of φ are in one-to-one correspondence with the critical points of

$$\psi_n(x_1, \ldots, x_n) = f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_n, x_1).$$

Theorem (Poincaré's last geometric theorem (Birkhoff, 1925))**.** Let $\varphi : A \to A$ be an area-preserving diffeomorphism such that φ preserves the boundary components, and twists them in opposite directions. Then φ has at least two fixed points.

1.5 Lagrangian submanifolds and fixed points

Theorem (Lagrangian neighbourhood theorem)**.** Let (M, ω) be a symplectic manifold, X a compact Lagrangian submanifold, and ω_0 the canonical symplectic form on T^*X. Then there exists neighbourhoods U_0 of X in T^*X and U of X in M and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ sending X to X.

Theorem (Weinstein)**.** Let M be a $2n$-dimensional manifold, X n-dimensional compact submanifold, and $i : X \hookrightarrow M$ the inclusion, and symplectic forms ω_0, ω_1 on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e. X is Lagrangian with respect to both symplectic structures. Then there exists neighbourhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M such that $\rho|_X = \mathrm{id}_X$ and $\rho^*\omega_1 = \omega_0$.

Lemma. Let V be a $2n$-dimensional vector space, Ω_0, Ω_1 symplectic structures on V. Suppose U is a subspace of V Lagrangian with respect to both Ω_0 and Ω_1, and W is any complement of V. Then we can construct canonically a linear isomorphism $H : V \to V$ such that $H|_U = \mathrm{id}_U$ and $H^*\Omega_1 = \Omega_2$.

Note that the statement of the theorem doesn't mention W, but the construction of H requires a complement of V, so it is canonical only after we pick a W.

Theorem (Whitney extension theorem)**.** Let X be a submanifold of M, $H_p : T_pM \to T_pM$ smooth family of isomorphisms such that $H_p|_{T_pX} = \mathrm{id}_{T_pX}$. Then there exists an neighbourhood $\mathcal{N}$ of X in M and an embedding $h : \mathcal{N} \to M$ such that $h|_X = \mathrm{id}_X$ and for all $p \in X$, $\mathrm{d}h_p = H_p$.

Theorem. Let (M, ω) be a compact symplectic manifold such that $H^1_{\mathrm{dR}}(M) = 0$. Then any symplectomorphism $\varphi : M \to M$ sufficiently close to the identity has at least two fixed points.

Theorem (Arnold conjecture)**.** Let (M, ω) be a compact symplectic manifold of dimension $2n$, and $\varphi : M \to M$ a symplectomorphism. Suppose φ is *exactly homotopic* to the identity and *non-degenerate*. Then the number of fixed points of φ is at least $\sum_{i=0}^{2n} \dim H^i(M, \mathbb{R})$.

2 Complex structures

2.1 Almost complex structures

Lemma. There is a correspondence between real vector spaces with a complex structure and complex vector spaces, where J acts as multiplication by i. □

Proposition (Polar decomposition)**.** Let (V, Ω) be a symplectic vector space, and G an inner product on V. Then from G, we can *canonically* construct a compatible complex structure J on (V, Ω). If $G = G_J$ for some J, then this process returns J.

Proposition. $\mathcal{J}(V, \Omega)$ is path-connected.

Proposition. Let (M, ω) be a symplectic manifold, and g a metric on M. Then from g we can canonically construct a compatible almost complex structure J. □

Corollary. Any symplectic manifold has a compatible almost complex structure. □

Proposition. $J(M, \omega)$ is contractible. □

Proposition. Let J be an almost complex structure on M that is compatible with ω_0 and ω_1. Then ω_0 and ω_1 are deformation equivalent.

Proposition. Let (M, ω) be a symplectic manifold, J a compatible almost complex structure. If X is an almost complex submanifold of (M, J), i.e. $J(TX) = TX$, then X is a symplectic submanifold of (M, ω).

Theorem (Gromov)**.** Let (M, J) be an almost complex manifold with M open, i.e. M has no closed connected components. Then there exists a symplectic form ω in any even 2-cohomology class and such that J is homotopic to an almost complex structure compatible with ω. □

2.2 Dolbeault theory

Theorem (Newlander–Nirenberg)**.** The following are equivalent:

- $\bar{\partial}^2 = 0$
- $\partial^2 = 0$
- $\mathrm{d} = \partial + \bar{\partial}$
- J is integrable
- $\mathcal{N} = 0$

where $\mathcal{N}$ is the *Nijenhuis torsion*

$$\mathcal{N}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$ □

2.3 Kähler manifolds

Lemma. $\omega \in \Omega^{1,1}$.

Theorem. A Kähler form ω on a complex manifold M is a ∂- and $\bar{\partial}$-closed form of type $(1,1)$ which on a local chart is given by

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

where at each point, the matrix (h_{jk}) is Hermitian and positive definite.

Proposition. Let (M, ω) be a complex Kähler manifold. If $X \subseteq M$ is a complex submanifold, then $(X, i^*\omega)$ is Kähler, and this is called a *Kähler submanifold*. □

Proposition. Let M be a complex manifold, and $\rho \in C^\infty(M; \mathbb{R})$ strictly plurisubharmonic. Then

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

is a Kähler form.

Proposition. Let M be a complex manifold, ω a closed real-valued $(1,1)$-form and $p \in M$, then there exists a neighbourhood U of p in M and a $\rho \in C^\infty(U, \mathbb{R})$ such that

$$\omega = i\partial\bar{\partial}\rho \text{ on } U.$$

2.4 Hodge theory

Theorem (Hodge decomposition theorem)**.** Let (M, ω) be a compact Kḧaler manifold. Then

$$H^k_{\mathrm{dR}} \cong \bigoplus_{p+q=k} H^{p,q}_{\mathrm{Dolb}}(M).$$

Proposition.

- $*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_m$
- $*(e_{k+1} \wedge \cdots \wedge e_m) = (-1)^{k(m-k)} e_1 \wedge \cdots \wedge e_k$.
- $** = \alpha = (-1)^{k(m-k)}\alpha$ for $\alpha \in \Lambda^k$. □

Proposition.

(i) $**\alpha = (-1)^{k(m-k)}\alpha$ for $\alpha \in \Omega^k(M)$.

(ii) $*1 = \mathrm{Vol}$ □

Proposition. $\delta^2 = 0$.

Proposition.

$$\delta = (-1)^{m(k+1)+1} * \mathrm{d} * : \Omega^k \to \Omega^{k-1}.$$

Proposition.

(i) $\Delta * = * \Delta : \Omega^k \to \Omega^{m-k}$

(ii) $\Delta = (\mathrm{d} + \delta)^2$

(iii) $\langle \Delta\alpha, \beta\rangle_{L^2} = \langle \alpha, \Delta\beta\rangle_{L^2}$.

(iv) $\Delta\alpha = 0$ iff $\mathrm{d}\alpha = \delta\alpha = 0$. □

Theorem (Hodge decomposition theorem)**.** Let (M, g) be a compact oriented Riemannian manifold. Then every cohomology class in $H^k_{\mathrm{dR}}(M)$ has a unique harmonic representation, i.e. the natural map $\mathcal{H}^k \to H^k_{\mathrm{dR}}(M)$ is an isomorphism.

Proposition. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$ and (M, ω) Kähler. Then

(i) $* : \Omega^{p,q} \to \Omega^{n-p,n-q}$.

(ii) $\Delta : \Omega^{p,q} \to \Omega^{p,q}$. □

Proposition. If our manifold is Kähler, then

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$ □

Theorem (Hodge decomposition theorem)**.** Let (M, ω) be a compact Kähler manifold. The natural map $\mathcal{H}^{p,q} \to H^{p,q}_{\mathrm{Dolb}}$ is an isomorphism. Hence

$$H^k_{\mathrm{dR}}(M; \mathbb{C}) \cong \mathcal{H}^k_{\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \cong \bigoplus_{p+q=k} H^{p,q}_{\mathrm{Dolb}}(M).$$ □

Corollary. Odd Betti numbers are even.

Corollary. $h_{1,0} = h_{0,1} = \frac{1}{2} b_1$ is a topological invariant.

Proposition. Even Betti numbers are positive.

Proposition. $h_{k,k} \neq 0$. □

3 Hamiltonian vector fields

3.1 Hamiltonian vector fields

Proposition. If X_H is a Hamiltonian vector field with flow ρ_t, then $\rho_t^* \omega = \omega$. In other words, each ρ_t is a symplectomorphism.

Proposition. ρ_t preserves H, i.e. $\rho_t^* H = H$.

Proposition. Let X, Y be symplectic vector fields on (M, ω). Then $[X, Y]$ is Hamiltonian.

Proposition. $\{f, g\} = 0$ iff f is constant along integral curves of X_g.

3.2 Integrable systems

Theorem (Arnold–Liouville thoerem)**.** Let (M, ω, H) be an integrable system with $\dim M = 2n$ and $f_1 = H, f_2, \ldots, f_n$ integrals of motion, and $c \in \mathbb{R}$ a regular value of $f = (f_1, \ldots, f_n)$.

(i) If the flows of X_{f_i} are complete, then the connected components of $f^{-1}(\{c\})$ are homogeneous spaces for $\mathbb{R}^n$ and admit affine coordinates $\varphi_1, \ldots, \varphi_n$ (*angle coordinates*), in which the flows of X_{f_i} are linear.

(ii) There exists coordinates $\psi_1, \ldots, \psi_n$ (*action coordinates*) such that the ψ_i's are integrals of motion and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ form a Darboux chart.

3.3 Classical mechanics

3.4 Hamiltonian actions

3.5 Symplectic reduction

Theorem (Marsden–Weinstein, Meyer)**.** Let G be a compact Lie group, and (M, ω) a symplectic manifold with a Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$. Write $i : \mu^{-1}(0) \hookrightarrow M$ for the inclusion. Suppose G acts freely on $\mu^{-1}(0)$. Then

(i) $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold;

(ii) $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ is a principal G-bundle; and

(iii) There exists a symplectic form ω_{red} on M_{red} such that $i^* \omega = \pi^* \omega_{\mathrm{red}}$.

Theorem. Let G be a compact Lie group and Z a manifold, and G acts freely on Z. Then Z/G is a manifold and $Z \to Z/G$ is a principal G-bundle.

Lemma. Let (V, Ω) be a symplectic vector space and I an isotropic subspace. Then Ω induces a canonical symplectic structure Ω_{red} on I^Ω / I, given by $\Omega_{\mathrm{red}}([u], [v]) = \Omega(u, v)$.

Lemma. Giving an Ehresmann connection is the same as giving a connection 1-form.

3.6 The convexity theorem

Theorem (Convexity theorem (Atiyah, Guillemin–Sternberg))**.** Let (M, ω) be a compact connected symplectic manifold, and

$$\mu : M \to \mathbb{R}^n$$

a moment map for a Hamiltonian torus action. Then

(i) The levels $\mu^{-1}(c)$ are connected for all c

(ii) The image $\mu(M)$ is convex.

(iii) The image $\mu(M)$ is in fact the convex hull of $\mu(M^G)$.

We call $\mu(M)$ the *moment polytope*.

Lemma. (ii) implies (iii).

Lemma. (i) implies (ii).

Theorem (Morse theory)**.**

(i) If $f^{-1}([c_1, c_2])$ does not contain any critical point. Then $f^{-1}(c_1) \cong f^{-1}(c_2)$ and $M_{c_1} \cong M_{c_2}$ (where $\cong$ means diffeomorphic).

(ii) If $f^{-1}([c_1, c_2])$ contains one critical manifold Z, then $M^-_{c_2} \simeq M^-_{c_1} \cup D(E^-)$, where $D(E^-)$ is the disk bundle of E^-.

In particular, if Z is an isolated point, $M^-_{c_2}$ is, up to homotopy equivalence, obtained by adding a $\dim E^-_p$-cell to $M^-_{c_1}$. □

Lemma. Let M be a compact connected manifold, and $f : M \to \mathbb{R}$ a Morse–Bott function with no critical submanifold of index or coindex 1. Then

(i) f has a unique local maximum and local minimum

(ii) All level sets of $f^{-1}(c)$ are connected.

Lemma. For any $X \in \mathbb{R}^n$, μ^X is a Morse–Bott function where all critical submanifolds are symplectic.

Lemma. (i) holds.

Theorem (Kirwan, 1984)**.** $\mu_+(M) \subseteq \mathfrak{t}^*_+$ is a convex polytope.

Theorem (Schur–Horn theorem)**.** $\varphi(\mathcal{H}^n_\lambda)$ is the convex hull of the $n!$ points from the diagonal matrices.

3.7 Toric manifolds

Proposition. Let (M, ω) be a compact, connected symplectic manifold with moment map $\mu : M \to \mathbb{R}^n$ for a Hamiltonian T^n action. If the T^n action is effective, then

(i) There are at least $n + 1$ fixed points.

(ii) $\dim M \geq 2n$.

Theorem (Delzant)**.** There are correspondences

$$\left\{\begin{array}{c}\text{symplectic toric manifolds}\\ \text{up to equivalence}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Delzant polytopes}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{symplectic toric manifolds}\\ \text{up to weak equivalence}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Delzant polytopes}\\ \text{modulo } \mathrm{AGL}(n, \mathbb{Z})\end{array}\right\}$$

4 Symplectic embeddings

Theorem (Non-squeezing theorem, Gromov, 1985)**.** There is an embedding $B^2(n) \hookrightarrow Z^{2n}(R)$ iff $r < R$.

Proposition. The existence of a symplectic capacity is equivalent to Gromov's non-squeezing theorem.