Part III — Symplectic Geometry
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

The first part of the course will be an overview of the basic structures of symplectic geometry, including symplectic linear algebra, symplectic manifolds, symplectomorphisms, Darboux theorem, cotangent bundles, Lagrangian submanifolds, and Hamiltonian systems. The course will then go further into two topics. The first one is moment maps and toric symplectic manifolds, and the second one is capacities and symplectic embedding problems.

Pre-requisites

Some familiarity with basic notions from Differential Geometry and Algebraic Topology will be assumed. The material covered in the respective Michaelmas Term courses would be more than enough background.

Contents

1 Symplectic manifolds

1.1 Symplectic linear algebra

Definition (Symplectic vector space)**.** A *symplectic vector space* is a real vector space V together with a non-degenerate skew-symmetric bilinear map $\Omega : V \times V \to \mathbb{R}$.

Definition (Non-degenerate bilinear map)**.** We say a bilinear map Ω is non-degenerate if the induced map $\tilde{\Omega} : V \to V^*$ given by $v \mapsto \Omega(v, \cdot)$ is bijective.

Definition (Symplectic subspace)**.** If (V, Ω) is a symplectic vector space, a *symplectic subspace* is a subspace $W \subseteq V$ such that $\Omega|_{W \times W}$ is non-degenerate.

Definition (Isotropic subspace)**.** If (V, Ω) is a symplectic vector space, an *isotropic subspace* is a subspace $W \subseteq V$ such that $\Omega|_{W \times W} = 0$.

Definition (Lagrangian subspace)**.** If (V, Ω) is a symplectic vector space, an *Lagrangian subspace* is an isotropic subspace W with $\dim W = \frac{1}{2} \dim V$.

Definition (Symplectomorphism)**.** A *symplectomorphism* between symplectic vector spaces $(V, \Omega), (V', \Omega')$ is an isomorphism $\varphi : V \to V'$ such that $\varphi^* \Omega' = \Omega$.

1.2 Symplectic manifolds

Definition (Symplectic manifold)**.** A *symplectic manifold* is a manifold M of dimension $2n$ equipped with a 2-form ω that is closed (i.e. $\mathrm{d}\omega = 0$) and non-degenerate (i.e. $\omega^{\wedge n} \neq 0$). We call ω the *symplectic form*.

Definition (Symplectomorphism)**.** Let (X_1, ω_1) and (X_2, ω_2) be symplectic manifolds. A *symplectomorphism* is a diffeomorphism $f : X_1 \to X_2$ such that $f^* \omega_2 = \omega_1$.

Definition (Strongly isotopic)**.** Two symplectic structures on M are *strongly isotopic* if there is an isotopy taking one to the other.

Definition (Deformation equivalent)**.** Two symplectic structures ω_0, ω_1 on M are *deformation equivalent* if there is a family of symplectic forms ω_t that start and end at ω_0 and ω_1 respectively.

Definition (Isotopic)**.** Two symplectic structures ω_0, ω_1 on M are *isotopic* if there is a family of symplectic forms ω_t that start and end at ω_0 and ω_1 respectively, and further the cohomology class $[\omega_t]$ is independent of t.

1.3 Symplectomorphisms and Lagrangians

Definition (Lagrangian submanifold)**.** Let (M, ω) be a symplectic manifold, and $L \subseteq M$ a submanifold. Then L is a *Lagrangian submanifold* if for all $p \in L$, $T_p L$ is a Lagrangian subspace of $T_p M$. Equivalently, the restriction of ω to L vanishes and $\dim L = \frac{1}{2} \dim M$.

1.4 Periodic points of symplectomorphisms

Definition (Periodic point)**.** Let (M, ω) be a symplectic manifold, and $\varphi : M \to M$ a symplectomorphism. An *n-periodic point* of φ is an $x \in M$ such that $\varphi^n(x) = x$. A *periodic point* is an n-periodic point for some n.

1.5 Lagrangian submanifolds and fixed points

Definition (Exactly homotopic)**.** We say φ is *exactly homotopic* to the identity if there is isotopy $\rho_t : M \to M$ such that $\rho_0 = \mathrm{id}$ and $\rho_1 = \varphi$, and further there is some 1-periodic family of functions h_t such that ρ_t is generated by the vector field v_t defined by $\iota_{v_t}^* \omega = \mathrm{d}h_t$.

Definition (Non-degenerate function)**.** A endomorphism $\varphi : M \to M$ is non-degenerate iff all its fixed points are non-degenerate, i.e. if p is a fixed point, then $\det(\mathrm{id} - \mathrm{d}\varphi_p) \neq 0$.

2 Complex structures

2.1 Almost complex structures

Definition (Complex structure)**.** Let V be a vector space. A *complex structure* is a linear $J : V \to V$ with $J^2 = -1$.

Definition (Compatible complex structure)**.** Let (V, Ω) be a symplectic vector space, and J a complex structure on V. We say J is *compatible* with Ω if $G_J(u, v) = \Omega(u, Jv)$ is an inner product. In other words, we need

$$\Omega(Ju, Jv) = \Omega(u, v), \quad \Omega(v, Jv) \geq 0$$

with equality iff $v = 0$.

Notation. Let (V, Ω) be a symplectic vector space. We write $\mathcal{J}(V, \Omega)$ for the space of all compatible complex structures on (V, Ω).

Definition (Almost complex structure)**.** An *almost complex structure* J on a manifold is a smooth field of complex structures on the tangent space $J_p : T_pM \to T_pM$, $J_p^2 = -1$.

Definition (Integrable almost complex structure)**.** An almost complex structure on M is called *integrable* if it is induced by a complex structure.

Definition (Compatible almost complex structure)**.** An almost complex structure J on M is *compatible* with a symplectic structure ω if J_p is compatible with ω_p for all $p \in M$. In this case, (ω, g_J, J) is called a *compatible triple.*

Notation. Let (M, ω) be a symplectic manifold. We write $\mathcal{J}(V, \Omega)$ for the space of all compatible almost complex structures on (M, ω).

2.2 Dolbeault theory

Notation. We write $T_{1,0}$ for the $+i$ eigenspace of J and $T_{1,0}$ for the $-i$-eigenspace of J. These are called the *J-holomorphic tangent vectors* and the *J-anti-holomorphic tangent vectors* respectively.

Definition (J-holomorphic)**.** We say a function f is *J-holomorphic* if $\bar{\partial} f = 0$, and *J-anti-holomorphic* if $\partial f = 0$.

Definition (Dolbeault cohomology groups)**.** Let (M, J) be a manifold with an integrable almost complex structure. The *Dolbeault cohomology groups* are the cohomology groups of the cochain complex

$$\Omega^{p,q} \xrightarrow{\bar{\partial}} \Omega^{p,q+1} \xrightarrow{\bar{\partial}} \Omega^{p,q+2} \xrightarrow{\bar{\partial}} \cdots .$$

Explicitly,

$$H^{p,q}_{\mathrm{Dolb}}(M) = \frac{\ker(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\operatorname{im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}.$$

2.3 Kähler manifolds

Definition (Kähler manifold)**.** A *Kähler manifold* is a symplectic manifold equipped with a compatible integrable almost complex structure. Then ω is called a *Kähler form*.

Definition (Strictly plurisubharmonic (spsh))**.** A function $\rho \in C^\infty(M, \mathbb{R})$ is strictly plurisubharmonic (spsh) if locally, $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)$ is positive definite.

2.4 Hodge theory

Definition (Hodge star)**.** The Hodge $*$-operator $* : \Lambda^k \to \Lambda^{m-k}$ is defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, e_1 \wedge \cdots \wedge e_m.$$

Definition (Hodge star operator)**.** The *Hodge $*$-operator* on forms $* : \Omega^k(M) \to \Omega^{m-k}(M)$ is defined by the equation

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \text{ Vol}.$$

Definition (Codifferential operator)**.** We define the *codifferential operator* $\delta : \Omega^k \to \Omega^{k-1}$ to be the L^2-formal adjoint of d. In other words, we require

$$\langle \mathrm{d}\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2}$$

for all $\alpha \in \Omega^k$ and $\beta \in \Omega^{k+1}$.

Definition (Laplace–Beltrami operator)**.** We define the *Laplacian*, or the *Laplace–Beltrami operator* to be

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} : \Omega^k \to \Omega^k.$$

Definition (Harmonic form)**.** A form α is *harmonic* if $\Delta\alpha = 0$. We write

$$\mathcal{H}^k = \{\alpha \in \Omega^k(m) \mid \Delta\alpha = 0\}$$

for the space of harmonic forms.

3 Hamiltonian vector fields

3.1 Hamiltonian vector fields

Definition (Hamiltonian vector field). Let (M, ω) be a symplectic manifold. If $H \in C^\infty(M)$, then since $\tilde{\omega} : TM \to T^*M$ is an isomorphism, there is a unique vector field X_H on M such that

$$\iota_{X_H} \omega = \mathrm{d}H.$$

We call X_H the *Hamiltonian vector field* with *Hamiltonian function* H.

Definition (Symplectic vector field). A vector field X on (M, ω) is a *symplectic vector field* if $\mathcal{L}_X \omega = 0$.

Definition (Poisson bracket). Let $f, g \in C^\infty(M)$. We then define the *Poisson bracket* $\{f, g\}$ by

$$\{f, g\} = \omega(X_f, X_g).$$

3.2 Integrable systems

Definition (Hamiltonian system). A *Hamiltonian system* is a triple (M, ω, H) where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$, called the *Hamiltonian function*.

Definition (Integral of motion). A *integral of motion/first integral/constant of motion/conserved quantity* of a Hamiltonian system is a function $f \in C^\infty(M)$ such that $\{f, H\} = 0$.

Definition (Independent integrals of motion). We say $f_1, \ldots, f_n \in C^\infty(M)$ are *independent* if $(\mathrm{d}f_1)_p, \ldots, (\mathrm{d}f_n)_p$ are linearly independent at all points on some dense subset of M.

Definition (Commuting integrals of motion). We say $f_1, \ldots, f_n \in C^\infty$ *commute* if $\{f_i, f_j\} = 0$.

Definition (Completely integrable system). A Hamiltonian system (M, ω, H) of dimension $\dim M = 2n$ is *(completely) integrable* if it has n independent commuting integrals of motion $f_1 = H, f_2, \ldots, f_n$.

Definition (Liouville torus). A *Liouville torus* is a compact connected component of $f^{-1}(c)$.

3.3 Classical mechanics

Definition (Momentum). The *momentum* of a particle is

$$y = m \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Definition (Energy). The *energy* of a particle is

$$H(x, y) = \frac{1}{2m} |y|^2 + V(x).$$

Definition (Action). The *action* of a path $\gamma \in \mathcal{P}$ is

$$A_\gamma = \int_a^b \left(\sum_k \frac{m_k}{2} \left| \frac{\mathrm{d}\gamma_k}{\mathrm{d}t}(t) \right|^2 - V(\gamma(t)) \right) \mathrm{d}t.$$

3.4 Hamiltonian actions

Definition (Symplectic action). A *symplectic action* is a smooth group action $\psi : G \to \mathrm{Diff}(M)$ such that each ψ_g is a symplectomorphism. In other words, it is a map $G \to \mathrm{Symp}(M, \omega)$.

Definition (Hamiltonian action). An action of $\mathbb{R}$ or S^1 is *Hamiltonian* if the corresponding vector field is Hamiltonian.

Definition (Left-invariant vector field). A *left-invariant vector field* on a Lie group G is a vector field X such that

$$(L_g)_* X = X$$

for all $g \in G$.

We write $\mathfrak{g}$ for the space of all left-invariant vector fields on G, which comes with the Lie bracket on vector fields. This is called the *Lie algebra* of G.

Definition (Hamiltonian action). We say $\psi : G \to \mathrm{Symp}(M, \omega)$ is a *Hamiltonian action* if there exists a map $\mu : M \to \mathfrak{g}^*$ such that

(i) For all $X \in \mathfrak{g}$, $X^\#$ is the Hamiltonian vector field generated by μ^X, where $\mu^X : M \to \mathbb{R}$ is given by

$$\mu^X(p) = \langle \mu(p), X \rangle.$$

(ii) μ is G-equivariant, where G acts on $\mathfrak{g}^*$ by the coadjoint action. In other words,

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu \text{ for all } g \in G.$$

μ is then called a *moment map* for the action ψ.

3.5 Symplectic reduction

Definition (Orbit). If $p \in M$, the *orbit* of p under G is

$$\mathcal{O}_p = \{\psi_g(p) : g \in G\}.$$

Definition (Stabilizer). The *stabilizer* or *isotropy group* of $p \in M$ is the closed subgroup

$$G_p = \{g \in G : \psi_g(p) = p\}.$$

We write $\mathfrak{g}_p$ for the Lie algebra of G_p.

Definition (Transitive action). We say G acts *transitively* if M is one orbit.

Definition (Free action). We say G acts *freely* if $G_p = \{e\}$ for all p.

Definition (Locally free action). We say G acts *locally freely* if $\mathfrak{g}_p = \{0\}$, i.e. G_p is discrete.

Definition (Orbit space). The *orbit space* is M/G, and we write $\pi : M \to M/G$ for the orbit projection. We equip M/G with the quotient topology.

Definition (Symplectic quotient)**.** We call M_{red} the *symplectic quotient/reduced space/symplectic reduction* of (M, ω) with respect to the given G-action and moment map.

Definition (Ehresmann Connection)**.** An *(Ehresmann) connection* on P is a choice of subbundle $H \subseteq TP$ such that

(i) $P = V \oplus H$

(ii) H is G-invariant.

Such an H is called a *horizontal bundle*.

Definition (Flat connection)**.** A connection A is *flat* if $\operatorname{curv} A = 0$.

3.6 The convexity theorem

Definition (Morse(-Bott) function)**.** f is a *Morse function* if at each $p \in \operatorname{Crit}(f)$, $H_p f$ is non-degenerate.

f is a *Morse–Bott function* if the connected components of $\operatorname{Crit}(f)$ are submanifolds and for all $p \in \operatorname{Crit}(p)$, $T_p \operatorname{Crit}(f) = \ker(H_p f)$.

3.7 Toric manifolds

Definition (Effective action)**.** An action G on M is *effective* (or *faithful*) if every non-identity $g \in G$ moves at least one point of M.

Definition ((Symplectic) toric manifold)**.** A *(symplectic) toric manifold* is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective T^n action of an n-torus together with a choice of corresponding moment map μ.

Definition (Equivalent toric manifolds)**.** Fix a torus $T = \mathbb{R}^{2n}/(2\pi\mathbb{Z})^n$, and fix an identification $\mathfrak{t}^* \cong \mathfrak{t} \cong \mathbb{R}^n$. Given two toric manifolds $(M_i, \omega_i, T, \mu_i)$ for $i = 1, 2$, We say they are

(i) *equivalent* if there exists a symplectomorphism $\varphi : M_1 \to M_2$ such that $\varphi(x \cdot p) = x \cdot \varphi(p)$ and $\mu_2 \circ \varphi = \mu_1$.

(ii) *weakly equivalent* if there exists an automorphism $\lambda : T \to T$ and $\varphi : M_1 \to M_2$ symplectomorphism such that $\varphi(x, p) = \lambda(x) \cdot \varphi(p)$.

Definition.

$$\operatorname{AGL}(n, \mathbb{Z}) = \{x \mapsto Bx + c : B \in \operatorname{GL}(n, \mathbb{Z}), c \in \mathbb{R}^n\}.$$

Definition (Delzant polytope)**.** A *Delzant polytope* in $\mathbb{R}^n$ is a compact convex polytope satisfying

(i) *Simplicity*: There exists exactly n edges out meeting at each vertex.

(ii) *Rationality*: The edges meeting at each vertex P are of the form $P + tu_i$ for $t \geq 0$ and $u_i \in \mathbb{Z}^n$.

(iii) *Smoothness*: For each vertex, the corresponding u_i's can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

4 Symplectic embeddings

Definition (Symplectic embedding)**.** A symplectic embedding is an embedding $\varphi : M_1 \hookrightarrow M_2$ such that $\varphi^*\omega_2 = \omega_1$. The notation we use is $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$.

Definition (Symplectic capacity)**.** A *symplectic capacity* is a function c from the set of $2n$-dimensional manifolds to $[0, \infty]$ such that

(i) Monotonicity: if $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$, then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.

(ii) Conformality: $c(M, \lambda\omega) = \lambda c(M, \omega)$.

(iii) Non-triviality: $c(B^{2n}(1), \omega_0) > 0$ and $c(Z^{2n}(1), \omega_0) < \infty$.

If we only have (i) and (ii), this is called a *generalized capacity*.