Part III — Stochastic Calculus and Applications

Theorems with proof

Based on lectures by R. Bauerschmidt

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- *Brownian motion.* Existence and sample path properties.
- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita–Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins–Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, Yamada–Watanabe theorem, strong Markov property, and relation to second order partial differential equations.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.

Contents

0 Introduction **3**

1 The Lebesgue–Stieltjes integral **4**

2 Semi-martingales **5**
2.1 Finite variation processes 5
2.2 Local martingale 6
2.3 Square integrable martingales 8
2.4 Quadratic variation 9
2.5 Covariation 13
2.6 Semi-martingale 15

3 The stochastic integral **16**
3.1 Simple processes 16
3.2 Itô isometry 16
3.3 Extension to local martingales 18
3.4 Extension to semi-martingales 20
3.5 Itô formula 21
3.6 The Lévy characterization 23
3.7 Girsanov's theorem 25

4 Stochastic differential equations **28**
4.1 Existence and uniqueness of solutions 28
4.2 Examples of stochastic differential equations 30
4.3 Representations of solutions to PDEs 30

0 Introduction

Proposition. Let H be any separable Hilbert space. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Gaussian subspace $S \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ and an isometry $I : H \to S$. In other words, for any $f \in H$, there is a corresponding random variable $I(f) \sim N(0, (f, f)_H)$. Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ and $(f, g)_H = \mathbb{E}[I(f)I(g)]$.

Proof. By separability, we can pick a Hilbert space basis $(e_i)_{i=1}^\infty$ of H. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space that carries an infinite independent sequence of standard Gaussian random variables $X_i \sim N(0, 1)$. Then send e_i to X_i, extend by linearity and continuity, and take S to be the image. □

Proposition.

- For $A \subseteq \mathbb{R}_+$ with $|A| < \infty$, $WN(A) \sim N(0, |A|)$.
- For disjoint $A, B \subseteq \mathbb{R}_+$, the variables $WN(A)$ and $WN(B)$ are independent.
- If $A = \bigcup_{i=1}^\infty A_i$ for disjoint sets $A_i \subseteq \mathbb{R}_+$, with $|A| < \infty, |A_i| < \infty$, then

$$WN(A) = \sum_{i=1}^\infty WN(A_i) \text{ in } L^2 \text{ and a.s.}$$

Proof. Only the last point requires proof. Observe that the partial sum

$$M_n = \sum_{i=1}^n WN(A)$$

is a martingale, and is bounded in L^2 as well, since

$$\mathbb{E}M_n^2 = \sum_{i=1}^n \mathbb{E}WN(A_i)^2 = \sum_{i=1}^n |A_i| \leq |A|.$$

So we are done by the martingale convergence theorem. The limit is indeed $WN(A)$ because $\mathbf{1}_A = \sum_{n=1}^\infty \mathbf{1}_{A_i}$. □

1 The Lebesgue–Stieltjes integral

Theorem. For any two finite measures μ_1, μ_2, there is a signed measure μ with $\mu(A) = \mu_1(A) - \mu_2(A)$.

Proof. Let $\nu = \mu_1 + \mu_2$. By Radon–Nikodym, there are positive functions f_1, f_2 such that $\mu_i(\mathrm{d}t) = f_i(t)\nu(\mathrm{d}t)$. Then

$$(\mu_1 - \mu_2)(\mathrm{d}t) = (f_1 - f_2)^+(t) \cdot \nu(\mathrm{d}t) + (f_1 - f_2)^-(t) \cdot \nu(\mathrm{d}t). \qquad \square$$

Theorem. There is a bijection

$$\left\{ \text{ signed measures on } [0, T] \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{càdlàg functions of bounded} \\ \text{variation } a : [0, T] \to \mathbb{R} \end{array} \right\}$$

that sends a signed measure μ to $a(t) = \mu([0, t])$. To construct the inverse, given a, we define

$$a_\pm = \frac{1}{2}(V_a \pm a).$$

Then $a_\pm$ are both positive, and $a = a_+ - a_-$. We can then define $\mu_\pm$ by

$$\begin{aligned} \mu_\pm[0, t] &= a_\pm(t) - a_\pm(0) \\ \mu &= \mu_+ - \mu_- \end{aligned}$$

Moreover, $V_{\mu[0,t]} = |\mu|[0, t]$.

Proposition. Let a be càdlàg and BV on $[0, t]$, and h bounded and left-continuous. Then

$$\begin{aligned} \int_0^t h(s)\, \mathrm{d}a(s) &= \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \Big(a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \Big) \\ \int_0^t h(s)\, |\mathrm{d}a(s)| &= \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \Big| a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \Big| \end{aligned}$$

for any sequence of subdivisions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ of $[0, t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$.

Proof. We approximate h by h_m defined by

$$h_m(0) = 0, \quad h_m(s) = h(t_{i-1}^{(m)}) \text{ for } s \in (t_{i-1}^{(m)}, t_i^{(m)}].$$

Then by left continuity, we have

$$h(s) = \lim_{n\to\infty} h_m(s)$$

by left continuity, and moreover

$$\lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})(a(t_i^{(m)}) - a(t_{i-1}^{(m)})) = \lim_{m\to\infty} \int_{(0,t]} h_m(s)\mu(\mathrm{~d}s) = \int_{(0,t]} h(s)\mu(\mathrm{d}s)$$

by dominated convergence theorem. The statement about $|\mathrm{d}a(s)|$ is left as an exercise. $\square$

2 Semi-martingales

2.1 Finite variation processes

Proposition. The total variation process V of a càdlàg adapted process A is also càdlàg, finite variation and adapted, and it is also increasing.

Proof. We only have to check that it is adapted. But that follows directly from our previous expression of the integral as the limit of a sum. Indeed, let $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m} = t$ be a (nested) sequence of subdivisions of $[0, t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$. We have seen

$$V_t = \lim_{m\to\infty} \sum_{i=1}^{n_m} |A_{t_i^{(m)}} - A_{t_{i-1}^{(m)}}| + |A(0)| \in \mathcal{F}_t.$$

□

Proof. Since X is càdlàg adapted, it is clear that H is left-continuous and adapted. Since H is left-continuous, it is approximated by simple processes. Indeed, let

$$H_t^n = \sum_{i=1}^{2^n} H_{(i-1)2^{-n}} \mathbf{1}_{((i-1)2^{-n}, i2^{-n}]}(t) \wedge n \in \mathcal{E}.$$

Then $H_t^n \to H$ for all t by left continuity, and previsibility follows. □

Proposition. Let A be a finite variation process, and H previsible such that

$$\int_0^t |H(\omega, s)|\, |\mathrm{d}A(\omega, s)| < \infty \text{ for all } (\omega, t) \in \Omega \times [0, \infty).$$

Then $H \cdot A$ is a finite variation process.

Proof. The finite variation and càdlàg parts follow directly from the deterministic versions. We only have to check that $H \cdot A$ is adapted, i.e. $(H \cdot A)(\,\cdot\,, t) \in \mathcal{F}_t$ for all $t \geq 0$.

First, $H \cdot A$ is adapted if $H(\omega, s) = 1_{(u,v]}(s) 1_E(\omega)$ for some $u < v$ and $E \in \mathcal{F}_u$, since

$$(H \cdot A)(\omega, t) = 1_E(\omega)(A(\omega, t \wedge v) - A(\omega, t \wedge u)) \in \mathcal{F}_t.$$

Thus, $H \cdot A$ is adapted for $H = \mathbf{1}_F$ when $F \in \Pi$. Clearly, Π is a π system, i.e. it is closed under intersections and non-empty, and by definition it generates the previsible σ-algebra $\mathcal{P}$. So to extend the adaptedness of $H \cdot A$ to all previsible H, we use the monotone class theorem.

We let

$$\mathcal{V} = \{H : \Omega \times [0, \infty) \to \mathbb{R} : H \cdot A \text{ is adapted}\}.$$

Then

(i) $1 \in \mathcal{V}$

(ii) $1_F \in \mathcal{V}$ for all $F \in \Pi$.

(iii) $\mathcal{V}$ is closed under monotone limits.

So $\mathcal{V}$ contains all bounded $\mathcal{P}$-measurable functions. □

2.2 Local martingale

Theorem (Optional stopping theorem)**.** Let X be a càdlàg adapted integrable process. Then the following are equivalent:

(i) X is a martingale, i.e. $X_t \in L^1$ for every t, and
$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \text{ for all } t > s.$$

(ii) The *stopped process* $X^T = (X^T_t) = (X_{T \wedge t})$ is a martingale for all stopping times T.

(iii) For all stopping times T, S with T bounded, $X_T \in L^1$ and $\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{T \wedge S}$ almost surely.

(iv) For all bounded stopping times T, $X_T \in L^1$ and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

For X uniformly integrable, (iii) and (iv) hold for all stopping times.

Proposition. Let X be a local martingale and $X_t \geq 0$ for all t. Then X is a supermartingale.

Proof. Let (T_n) be a reducing sequence for X. Then
$$\begin{aligned}\mathbb{E}(X_t \mid \mathcal{F}_s) &= \mathbb{E}\left(\liminf_{n\to\infty} X_{t \wedge T_n} \mid \mathcal{F}_s\right)\\ &\leq \lim_{n\to\infty} \mathbb{E}(X_{t\wedge T_n} \mid \mathcal{F}_s)\\ &= \liminf_{T_n \to \infty} X_{s \wedge T_n}\\ &= X_s.\end{aligned}$$
□

Proposition. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then the set
$$\chi = \{\mathbb{E}(X \mid \mathcal{G}) : G \subseteq \mathcal{F} \text{ a sub-}\sigma\text{-algebra}\}$$
is uniformly integrable, i.e.
$$\sup_{Y \in \chi} \mathbb{E}(|Y| \mathbf{1}_{|Y| > \lambda}) \to 0 \text{ as } \lambda \to \infty.$$

Theorem (Vitali theorem)**.** $X_n \to X$ in L^1 iff (X_n) is uniformly integrable and $X_n \to X$ in probability.

Proposition. The following are equivalent:

(i) X is a martingale.

(ii) X is a local martingale, and for all $t \geq 0$, the set
$$\chi_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$
is uniformly integrable.

Proof.

– (a) ⇒ (b): Let X be a martingale. Then by the optional stopping theorem, $X_T = \mathbb{E}(X_t \mid \mathcal{F}_T)$ for any bounded stopping time $T \leq t$. So χ_t is uniformly integrable.

– (b) ⇒ (a): Let X be a local martingale with reducing sequence (T_n), and assume that the sets χ_t are uniformly integrable for all $t \geq 0$. By the optional stopping theorem, it suffices to show that $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for any bounded stopping time T.

  So let T be a bounded stopping time, say $T \leq t$. Then

$$\mathbb{E}(X_0) = \mathbb{E}(X_0^{T_n}) = \mathbb{E}(X_T^{T_n}) = \mathbb{E}(X_{T \wedge T_n})$$

  for all n. Now $T \wedge T_n$ is a stopping time $\leq t$, so $\{X_{T \wedge T_n}\}$ is uniformly integrable by assumption. Moreover, $T_n \wedge T \to T$ almost surely as $n \to \infty$, hence $X_{T \wedge T_n} \to X_T$ in probability. Hence by Vitali, this converges in L^1. So

$$\mathbb{E}(X_T) = \mathbb{E}(X_0). \qquad \square$$

Corollary. If $Z \in L^1$ is such that $|X_t| \leq Z$ for all t, then X is a martingale. In particular, every bounded local martingale is a martingale.

Proposition. Let X be a *continuous* local martingale with $X_0 = 0$. Define

$$S_n = \inf\{t \geq 0 : |X_t| = n\}.$$

Then S_n is a stopping time, $S_n \to \infty$ and X^{S_n} is a bounded martingale. In particular, (S_n) reduces X.

Proof. It is clear that S_n is a stopping time, since (if it is not clear)

$$\{S_n \leq t\} = \bigcap_{k \in \mathbb{N}} \left\{\sup_{s \leq t} |X_s| > n - \frac{1}{k}\right\} = \bigcap_{k \in \mathbb{N}} \bigcup_{s < t, s \in \mathbb{Q}} \left\{|X_s| > n - \frac{1}{k}\right\} \in \mathcal{F}_t.$$

It is also clear that $S_n \to \infty$, since

$$\sup_{s \leq t} |X_s| \leq n \leftrightarrow S_n \geq t,$$

and by continuity and compactness, $\sup_{s \leq t} |X_s|$ is finite for every (ω, t).

Finally, we show that X^{S_n} is a martingale. By the optional stopping theorem, $X^{T_n \wedge S_n}$ is a martingale, so X^{S_n} is a local martingale. But it is also bounded by n. So it is a martingale. $\square$

Theorem. Let X be a continuous local martingale with $X_0 = 0$. If X is also a finite variation process, then $X_t = 0$ for all t.

Proof. Let X be a finite-variation continuous local martingale with $X_0 = 0$. Since X is finite variation, we can define the total variation process (V_t) corresponding to X, and let

$$S_n = \inf\{t \geq 0 : V_t \geq n\} = \inf\left\{t \geq 0 : \int_0^1 |\mathrm{d}X_s| \geq n\right\}.$$

Then S_n is a stopping time, and $S_n \to \infty$ since X is assumed to be finite variation. Moreover, by optional stopping, X^{S_n} is a local martingale, and is also bounded, since

$$X_t^{S_n} \leq \int_0^{t \wedge S_n} |\mathrm{d}X_s| \leq n.$$

So X^{S_n} is in fact a martingale.

We claim its L^2-norm vanishes. Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0, t]$. Using the fact that X^{S_n} is a martingale and has orthogonal increments, we can write

$$\mathbb{E}((X_t^{S_n})^2) = \sum_{i=1}^k \mathbb{E}((X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n})^2).$$

Observe that X^{S_n} is finite variation, but the right-hand side is summing the *square* of the variation, which ought to vanish when we take the limit $\max |t_i - t_{i-1}| \to 0$. Indeed, we can compute

$$\begin{aligned}
\mathbb{E}((X_t^{S_n})^2) &= \sum_{i=1}^k \mathbb{E}((X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n})^2)\\
&\leq \mathbb{E}\left(\max_{1 \leq i \leq k} |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}| \sum_{i=1}^k |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|\right)\\
&\leq \mathbb{E}\left(\max_{1 \leq i \leq k} |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}| V_{t \wedge S_n}\right)\\
&\leq \mathbb{E}\left(\max_{1 \leq i \leq k} |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}| n\right).
\end{aligned}$$

Of course, the first term is also bounded by the total variation. Moreover, we can make further subdivisions so that the mesh size tends to zero, and then the first term vanishes in the limit by continuity. So by dominated convergence, we must have $\mathbb{E}((X_t^{S_n})^2) = 0$. So $X_t^{S_n} = 0$ almost surely for all n. So $X_t = 0$ for all t almost surely. □

2.3 Square integrable martingales

Theorem (Doob's inequality)**.** Let $X \in \mathcal{M}^2$. Then

$$\mathbb{E}\left(\sup_{t \geq 0} X_t^2\right) \leq 4\mathbb{E}(X_\infty^2).$$

Theorem. $\mathcal{M}^2$ is a Hilbert space and $\mathcal{M}_c^2$ is a closed subspace.

Proof. We need to check that $\mathcal{M}^2$ is complete. Thus let $(X^n) \subseteq \mathcal{M}^2$ be a Cauchy sequence, i.e.

$$\mathbb{E}((X_\infty^n - X_\infty^m)^2) \to 0 \text{ as } n, m \to \infty.$$

By passing to a subsequence, we may assume that

$$\mathbb{E}((X_\infty^n - X_\infty^{n-1})^2) \leq 2^{-n}.$$

First note that

$$\begin{aligned}
\mathbb{E}\left(\sum_n \sup_{t\geq 0} |X_t^n - X_t^{n-1}|\right) &\leq \sum_n \mathbb{E}\left(\sup_{t\geq 0}|X_t^n - X_t^{n-1}|^2\right)^{1/2} && \text{(CS)}\\
&\leq \sum_n 2\mathbb{E}\left(|X_\infty^n - X_\infty^{n-1}|^2\right)^{1/2} && \text{(Doob's)}\\
&\leq 2\sum_n 2^{-n/2} < \infty.
\end{aligned}$$

So

$$\sum_{n=1}^\infty \sup_{t\geq 0}|X_t^n - X_t^{n-1}| < \infty \text{ a.s.} \qquad (*)$$

So on this event, (X^n) is a Cauchy sequence in the space $(D[0,\infty), \|\cdot\|_\infty)$ of cádlág sequences. So there is some $X(\omega, \cdot) \in D[0,\infty)$ such that

$$\|X^n(\omega, \cdot) - X(\omega, \cdot)\|_\infty \to 0 \text{ for almost all } \omega.$$

and we set $X = 0$ outside this almost sure event $(*)$. We now claim that

$$\mathbb{E}\left(\sup_{t\geq 0}|X^n - X|^2\right) \to 0 \text{ as } n \to \infty.$$

We can just compute

$$\begin{aligned}
\mathbb{E}\left(\sup_t |X^n - X|^2\right) &= \mathbb{E}\left(\lim_{m\to\infty}\sup_t |X^n - X^m|^2\right)\\
&\leq \liminf_{m\to\infty}\mathbb{E}\left(\sup_t |X^n - X^m|^2\right) && \text{(Fatou)}\\
&\leq \liminf_{m\to\infty} 4\mathbb{E}(X_\infty^n - X_m^\infty)^2 && \text{(Doob's)}
\end{aligned}$$

and this goes to 0 in the limit $n \to \infty$ as well.

We finally have to check that X is indeed a martingale. We use the triangle inequality to write

$$\begin{aligned}
\|E(X_t \mid \mathcal{F}_s) - X_s\|_{L^2} &\leq \|\mathbb{E}(X_t - X_t^n \mid \mathcal{F}_s)\|_{L^2} + \|X_s^n - X_s\|_{L^2}\\
&\leq \mathbb{E}(\mathbb{E}((X_t - X_t^n)^2 \mid \mathcal{F}_s))^{1/2} + \|X_s^n - X_s\|_{L^2}\\
&= \|X_t - X_t^n\|_{L^2} + \|X_s^n - X_s\|_{L^2}\\
&\leq 2\mathbb{E}\left(\sup_t |X_t - X_t^n|^2\right)^{1/2} \to 0
\end{aligned}$$

as $n \to \infty$. But the left-hand side does not depend on n. So it must vanish. So $X \in \mathcal{M}^2$.

We could have done exactly the same with continuous martingales, so the second part follows. □

2.4 Quadratic variation

Theorem. Let M be a continuous local martingale with $M_0 = 0$. Then there exists a unique (up to indistinguishability) continuous adapted increasing process

$(\langle M\rangle_t)_{t\geq 0}$ such that $\langle M\rangle_0 = 0$ and $M_t^2 - \langle M\rangle_t$ is a continuous local martingale. Moreover,

$$\langle M\rangle_t = \lim_{n\to\infty} \langle M\rangle_t^{(n)}, \quad \langle M\rangle_t^{(n)} = \sum_{i=1}^{\lceil 2^n t\rceil} (M_{t2^{-n}} - M_{(i-1)2^{-n}})^2,$$

where the limit u.c.p.

Proof. To show uniqueness, we use that finite variation and local martingale are incompatible. Suppose (A_t) and $(\tilde{A}_t)$ obey the conditions for $\langle M\rangle$. Then $A_t - \tilde{A}_t = (M_t^2 - \tilde{A}_t) - (M_t^2 - A_t)$ is a continuous adapted local martingale starting at 0. Moreover, both A_t and $\tilde{A}_t$ are increasing, hence have finite variation. So $A - \tilde{A} = 0$ almost surely.

To show existence, we need to show that the limit exists and has the right property. We do this in steps.

Claim. The result holds if M is in fact bounded.

Suppose $|M(\omega, t)| \leq C$ for all (ω, t). Then $M \in \mathcal{M}_c^2$. Fix $T > 0$ deterministic. Let

$$X_t^n = \sum_{i=1}^{\lceil 2^n T\rceil} M_{(i-1)2^{-n}}(M_{i2^{-n}\wedge t} - M_{(i-1)2^{-n}\wedge t}).$$

This is defined so that

$$\langle M\rangle_{k2^{-n}}^{(n)} = M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n.$$

This reduces the study of $\langle M\rangle^{(n)}$ to that of $X_{k2^{-n}}^n$.

We check that (X_t^n) is a Cauchy sequence in $\mathcal{M}_c^2$. The fact that it is a martingale is an immediate computation. To show it is Cauchy, for $n \geq m$, we calculate

$$X_\infty^n - X_\infty^m = \sum_{i=1}^{\lceil 2^n T\rceil} (M_{(i-1)2^{-n}} - M_{\lfloor (i-1)2^{m-n}\rfloor 2^{-m}})(M_{i2^{-n}} - M_{(i-1)2^{-n}}).$$

We now take the expectation of the square to get

$$\begin{aligned}
\mathbb{E}(X_\infty^n - X_\infty^m)^2 &= \mathbb{E}\left(\sum_{i=1}^{\lceil 2^n T\rceil} (M_{(i-1)2^{-n}} - M_{\lfloor (i-1)2^{m-n}\rfloor 2^{-m}})^2 (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\right)\\
&\leq \mathbb{E}\left(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^2 \sum_{i=1}^{\lceil 2^n T\rceil} (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\right)\\
&= \mathbb{E}\left(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^2 \langle M\rangle_T^{(n)}\right)\\
&\leq \mathbb{E}\left(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^4\right)^{1/2} \mathbb{E}\left((\langle M\rangle_T^{(n)})^2\right)^{1/2}\\
&\qquad\qquad\text{(Cauchy–Schwarz)}
\end{aligned}$$

We shall show that the second factor is bounded, while the first factor tends to zero as $m \to \infty$. These are both not surprising — the first term vanishing in the limit corresponds to M being continuous, and the second term is bounded since M itself is bounded.

To show that the first term tends to zero, we note that we have

$$|M_t - M_s|^4 \leq 16C^4,$$

and moreover

$$\sup_{|s-t|\leq 2^{-m}} |M_t - M_s| \to 0 \text{ as } m \to \infty \text{ by uniform continuity.}$$

So we are done by the dominated convergence theorem.

To show the second term is bounded, we do (writing $N = \lceil 2^n T \rceil$)

$$\begin{aligned}
\mathbb{E}\left((\langle M\rangle_T^{(n)})^2\right) &= \mathbb{E}\left(\left(\sum_{i=1}^N (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\right)^2\right) \\
&= \sum_{i=1}^N \mathbb{E}\left((M_{i2^{-n}} - M_{(i-1)2^{-n}})^4\right) \\
&\quad + 2\sum_{i=1}^N \mathbb{E}\left((M_{i2^{-n}} - M_{(i-1)2^{-n}})^2 \sum_{k=i+1}^N (M_{k2^{-n}} - M_{(k-1)2^{-n}})^2\right)
\end{aligned}$$

We use the martingale property and orthogonal increments the rearrange the off-diagonal term as

$$\mathbb{E}\left((M_{i2^{-n}} - M_{(i-1)2^{-n}})(M_{N2^{-n}} - M_{i2^{-n}})^2\right).$$

Taking some sups, we get

$$\begin{aligned}
\mathbb{E}\left((\langle M\rangle_T^{(n)})^2\right) &\leq 12C^2\mathbb{E}\left(\sum_{i=1}^N (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\right) \\
&= 12C^2\mathbb{E}\left((M_{N2^{-n}} - M_0)^2\right) \\
&\leq 12C^2 \cdot 4C^2.
\end{aligned}$$

So done.

So we now have $X^n \to X$ in M_c^2 for some $X \in M_c^2$. In particular, we have

$$\left\|\sup_t |X_t^n - X_t|\right\|_{L^2} \to 0$$

So we know that

$$\sup_t |X_t^n - X_t| \to 0$$

almost surely along a subsequence Λ.

Let $N \subseteq \Omega$ be the events on which this convergence fails. We define

$$A_t^{(T)} = \begin{cases} M_t^2 - 2X_t & \omega \in \Omega \setminus N \\ 0 & \omega \in N \end{cases}.$$

Then $A^{(T)}$ is continuous, adapted since M and X are, and $(M_{t\wedge T}^2 - A_{t\wedge T}^{(T)})_t$ is a martingale since X is. Finally, $A^{(T)}$ is increasing since $M_t^2 - X_t^n$ is increasing on $2^{-n}\mathbb{Z} \cap [0, T]$ and the limit is uniform. So this $A^{(T)}$ basically satisfies all the properties we want $\langle M \rangle_t$ to satisfy, except we have the stopping time T.

We next observe that for any $T \geq 1$, $A_{t\wedge T}^{(T)} = A_{t\wedge T}^{(T+1)}$ for all t almost surely. This essentially follows from the same uniqueness argument as we had at the beginning of the proof. Thus, there is a process $(\langle M \rangle_t)_{t\geq 0}$ such that

$$\langle M \rangle_t = A_t^{(T)}$$

for all $t \in [0, T]$ and $T \in \mathbb{N}$, almost surely. Then this is the desired process. So we have constructed $\langle M \rangle$ in the case where M is bounded.

Claim. $\langle M \rangle^{(n)} \to \langle M \rangle$ u.c.p.

Recall that

$$\langle M \rangle_t^{(n)} = M_{2^{-n}\lfloor 2^n t\rfloor}^2 - 2X_{2^{-n}\lfloor 2^n t\rfloor}^n.$$

We also know that

$$\sup_{t\leq T} |X_t^n - X_t| \to 0$$

in L^2, hence also in probability. So we have

$$\begin{aligned} |\langle M \rangle_t - \langle M \rangle_t^{(n)}| &\leq \sup_{t\leq T} |M_{2^{-n}\lfloor 2^n t\rfloor}^2 - M_t^2| \\ &\quad + \sup_{t\leq T} |X_{2^{-n}\lfloor 2^n t\rfloor}^n - X_{2^{-n}\lfloor 2^n t\rfloor}| + \sup_{t\leq T} |X_{2^{-n}\lfloor 2^n t\rfloor} - X_t|. \end{aligned}$$

The first and last terms $\to 0$ in probability since M and X are uniformly continuous on $[0, T]$. The second term converges to zero by our previous assertion. So we are done.

Claim. The theorem holds for M any continuous local martingale.

We let $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then (T_n) reduces M and M^{T_n} is a bounded martingale. So in particular M^{T_n} is a bounded continuous martingale. We set

$$A^n = \langle M^{T_n} \rangle.$$

Then (A_t^n) and $(A_{t\wedge T_n}^{n+1})$ are indistinguishable for $t < T_n$ by the uniqueness argument. Thus there is a process $\langle M \rangle$ such that $\langle M \rangle_{t\wedge T_n} = A_t^n$ are indistinguishable for all n. Clearly, $\langle M \rangle$ is increasing since the A^n are, and $M_{t\wedge T_n}^2 - \langle M \rangle_{t\wedge T_n}$ is a martingale for every n, so $M_t^2 - \langle M \rangle_t$ is a continuous local martingale.

Claim. $\langle M \rangle^{(n)} \to \langle M \rangle$ u.c.p.

We have seen

$$\langle M^{T_k} \rangle^{(n)} \to \langle M^{T_k} \rangle \text{ u.c.p.}$$

for every k. So

$$\mathbb{P}\left(\sup_{t\leq T} |\langle M \rangle_t^{(n)} - \langle M_t \rangle| > \varepsilon\right) \leq \mathbb{P}(T_k < T) + \mathbb{P}\left(\sup_{t\leq T} |\langle M^{T_k} \rangle_t^{(n)} - \langle M^{T_k} \rangle_t > \varepsilon\right).$$

So we can fisrt pick k large enough such that the first term is small, then pick n large enough so that the second is small. □

Proof. Since $M_t^2 - \langle M\rangle_t$ is a continuous local martingle, so is $M_{t\wedge T}^2 - \langle M\rangle_{t\wedge T} = (M^T)_t^2 - \langle M\rangle_{t\wedge T}$. So we are done by uniqueness. $\square$

Proof. If $M = 0$, then $\langle M\rangle = 0$. Conversely, if $\langle M\rangle = 0$, then M^2 is a continuous local martingale and positive. Thus $\mathbb{E}M_t^2 \leq \mathbb{E}M_0^2 = 0$. $\square$

Proposition. Let $M \in \mathcal{M}_c^2$. Then $M^2 - \langle M\rangle$ is a uniformly integrable martingale, and

$$\|M - M_0\|_{\mathcal{M}^2} = (\mathbb{E}\langle M\rangle_\infty)^{1/2}.$$

Proof. We will show that $\langle M\rangle_\infty \in L^1$. This then implies

$$|M_t^2 - \langle M\rangle_t| \leq \sup_{t\geq 0} M_t^2 + \langle M\rangle_\infty.$$

Then the right hand side is in L^1. Since $M^2 - \langle M\rangle$ is a local martingale, this implies that it is in fact a uniformly integrable martingale.

To show $\langle M\rangle_\infty \in L^1$, we let

$$S_n = \inf\{t \geq 0 : \langle M\rangle_t \geq n\}.$$

Then $S_n \to \infty$, S_n is a stopping time and moreover $\langle M\rangle_{t\wedge S_n} \leq n$. So we have

$$M_{t\wedge S_n}^2 - \langle M\rangle_{t\wedge S_n} \leq n + \sup_{t\geq 0} M_t^2,$$

and the second term is in L^1. So $M_{t\wedge S_n}^2 - \langle M\rangle_{t\wedge S_n}$ is a true martingale.

So

$$\mathbb{E}M_{t\wedge S_n}^2 - \mathbb{E}M_0^2 = \mathbb{E}\langle M\rangle_{t\wedge S_n}.$$

Taking the limit $t \to \infty$, we know $\mathbb{E}M_{t\wedge S_n}^2 \to \mathbb{E}M_{S_n}^2$ by dominated convergence. Since $\langle M\rangle_{t\wedge S_n}$ is increasing, we also have $\mathbb{E}\langle M\rangle_{t\wedge S_n} \to \mathbb{E}\langle M\rangle_{S_n}$ by *monotone* convergence. We can take $n \to \infty$, and by the same justification, we have

$$\mathbb{E}\langle M\rangle \leq \mathbb{E}M_\infty^2 - \mathbb{E}M_0^2 = \mathbb{E}(M_\infty - M_0)^2 < \infty. \qquad \square$$

2.5 Covariation

Proposition.

(i) $\langle M, N\rangle$ is the unique (up to indistinguishability) finite variation process such that $M_tN_t - \langle M, N\rangle_t$ is a continuous local martingale.

(ii) The mapping $(M, N) \mapsto \langle M, N\rangle$ is bilinear and symmetric.

(iii)

$$\langle M, N\rangle_t = \lim_{n\to\infty} \langle M, N\rangle_t^{(n)} \text{ u.c.p.}$$

$$\langle M, N\rangle_t^{(n)} = \sum_{i=1}^{\lceil 2^n t\rceil} (M_{i2^{-n}-M_{(i-1)2^{-n}}})(N_{i2^{-n}} - N_{(i-1)}2^{-n}).$$

(iv) For every stopping time T,

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_{t\wedge T}.$$

(v) If $M, N \in \mathcal{M}_c^2$, then $M_t N_t - \langle M, N\rangle_t$ is a uniformly integrable martingale, and
$$\langle M - M_0, N - N_0\rangle_{\mathcal{M}^2} = \mathbb{E}\langle M, N\rangle_\infty. \qquad \square$$

Proof. Assume $B_0 = B_0' = 0$. Then $X_\pm = \frac{1}{\sqrt{2}}(B \pm B')$ are Brownian motions, and so $\langle X_\pm\rangle = t$. So their difference vanishes. $\square$

Proposition (Kunita–Watanabe)**.** Let M, N be continuous local martingales and let H, K be two (previsible) processes. Then almost surely
$$\int_0^\infty |H_s||K_s||\mathrm{d}\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2 \;\mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_0^\infty H_s^2 \langle N\rangle_s\right)^{1/2}.$$

Proof. For convenience, we write
$$\langle M, N\rangle_s^t = \langle M, N\rangle_t - \langle M, N\rangle_s.$$

Claim. For all $0 \leq s \leq t$, we have
$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}.$$

By continuity, we can assume that s, t are dyadic rationals. Then
$$\begin{aligned}
|\langle M, N\rangle_s^t| &= \lim_{n\to\infty} \left|\sum_{i=2^n s+1}^{2^n t} (M_{i2^{-n}} - M_{(i-1)2^{-n}})(N_{i2^{-n}} - N_{(i-1)2^{-n}})\right| \\
&\leq \lim_{n\to\infty} \left|\sum_{i=2^n s+1}^{2^n t} (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\right|^{1/2} \times \\
&\qquad \left|\sum_{i=2^n s+1}^{2^n t} (N_{i2^{-n}} - N_{(i-1)2^{-n}})^2\right|^{1/2} \qquad \text{(Cauchy–Schwarz)} \\
&= \left(\langle M, M\rangle_s^t\right)^{1/2} \left(\langle N, N\rangle_s^t\right)^{1/2},
\end{aligned}$$

where all equalities are u.c.p.

Claim. For all $0 \leq s < t$, we have
$$\int_s^t |\mathrm{d}\langle M, N\rangle_u| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}.$$

Indeed, for any subdivision $s = t_0 < t_1 < \cdots t_n = t$, we have
$$\begin{aligned}
\sum_{i=1}^n |\langle M, N\rangle_{t_{i-1}}^{t_i}| &\leq \sum_{i=1}^n \sqrt{\langle M, M\rangle_{t_{i-1}}^{t_i}}\sqrt{\langle N, N\rangle_{t_{i-1}}^{t_i}} \\
&\leq \left(\sum_{i=1}^n \langle M, M\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \left(\sum_{i=1}^n \langle N, N\rangle_{t_{i-1}}^{t_i}\right)^{1/2}. \\
&\qquad \text{(Cauchy–Schwarz)}
\end{aligned}$$

Taking the supremum over all subdivisions, the claim follows.

Claim. For all bounded Borel sets $B \subseteq [0, \infty)$, we have

$$\int_B |\mathrm{d}\langle M, N\rangle_u| \leq \sqrt{\int_B \mathrm{d}\langle M\rangle_u}\sqrt{\int_B \mathrm{d}\langle N\rangle_u}.$$

We already know this is true if B is an interval. If B is a finite union of integrals, then we apply Cauchy–Schwarz. By a monotone class argument, we can extend to all Borel sets.

Claim. The theorem holds for

$$H = \sum_{\ell=1}^k h_\ell \mathbf{1}_{B_\ell}, \quad K = \sum_{\ell=1}^n k_\ell \mathbf{1}_{B_\ell}$$

for $B_\ell \subseteq [0, \infty)$ bounded Borel sets with disjoint support.

We have

$$\begin{aligned}\int |H_s K_s|\ |\mathrm{d}\langle M, N\rangle_s| &\leq \sum_{\ell=1}^n |h_\ell k_\ell| \int_{B_\ell} |\mathrm{d}\bar{M}, N\rangle_s| \\ &\leq \sum_{\ell=1}^n |h_\ell k_\ell| \left(\int_{B_\ell} \mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_{B_\ell} \mathrm{d}\langle N\rangle_s\right)^{1/2} \\ &\leq \left(\sum_{\ell=1}^n h_\ell^2 \int_{B_\ell} \mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\sum_{\ell=1}^n k_\ell^2 \int_{B_\ell} \mathrm{d}\langle N\rangle_s\right)^{1/2}\end{aligned}$$

To finish the proof, approximate general H and K by step functions and take the limit. □

2.6 Semi-martingale

3 The stochastic integral

3.1 Simple processes

Proposition. If $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$, then $H \cdot M \in \mathcal{M}_c^2$ and

$$\|H \cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \, \mathrm{d}\langle M \rangle_s\right). \tag{$*$}$$

Proof. We first show that $H \cdot M \in \mathcal{M}_c^2$. By linearity, we only have to check it for

$$X_t^i = H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$$

We have to check that $\mathbb{E}(X_t^i \mid \mathcal{F}_s) = 0$ for all $t > s$, and the only non-trivial case is when $t > t_{i-1}$.

$$\mathbb{E}(X_t^i \mid \mathcal{F}_s) = H_{i-1}\mathbb{E}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t} \mid \mathcal{F}_s) = 0.$$

We also check that

$$\|X^i\|_{\mathcal{M}^2} \leq 2\|H\|_\infty \|M\|_{\mathcal{M}^2}.$$

So it is bounded. So $H \cdot M \in \mathcal{M}_c^2$.

To prove ($*$), we note that the X^i are orthogonal and that

$$\langle X^i \rangle_t = H_{i-1}^2(\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t}).$$

So we have

$$\langle H \cdot M, H \cdot M \rangle = \sum \langle X^i, X^i \rangle = \sum H_{i-1}^2(\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t}) = \int_0^t H_s^2 \, \mathrm{d}\langle M \rangle_s.$$

In particular,

$$\|H \cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\langle H \cdot M \rangle_\infty = \mathbb{E}\left(\int_0^\infty H_s^2 \, \mathrm{d}\langle M \rangle_s\right). \qquad \square$$

Proposition. Let $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$. Then

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$$

for all $N \in \mathcal{M}^2$.

Proof. Write $H \cdot M = \sum X^i = \sum H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$ as before. Then

$$\langle X^i, N \rangle_t = H_{i-1}\langle M_{t_i \wedge t} - M_{t_{i-1} \wedge t}, N \rangle = H_{i-1}(\langle M, N \rangle_{t_i \wedge t} - \langle M, N \rangle_{t_{i-1} \wedge t}). \qquad \square$$

3.2 Itô isometry

Proposition. Let $M \in \mathcal{M}_c^2$. Then $\mathcal{E}$ is dense in $L^2(M)$.

Proof. Since $L^2(M)$ is a Hilbert space, it suffices to show that if $(K, H) = 0$ for all $H \in \mathcal{E}$, then $K = 0$.

So assume that $(K, H) = 0$ for all $H \in \mathcal{E}$ and

$$X_t = \int_0^t K_s \, \mathrm{d}\langle M \rangle_s,$$

Then X is a well-defined finite variation process, and $X_t \in L^1$ for all t. It suffices to show that $X_t = 0$ for all t, and we shall show that X_t is a continuous martingale.

Let $0 \leq s < t$ and $F \in \mathcal{F}_s$ bounded. We let $H = F1_{(s,t]} \in \mathcal{E}$. By assumption, we know

$$0 = (K, H) = \mathbb{E}\left(F \int_s^t K_u \, \mathrm{d}\langle M \rangle_u\right) = \mathbb{E}(F(X_t - X_S)).$$

Since this holds for all $\mathcal{F}_s$ measurable F, we have shown that

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s.$$

So X is a (continuous) martingale, and we are done. □

Theorem. Let $M \in \mathcal{M}_c^2$. Then

(i) The map $H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}_c^2$ extends uniquely to an isometry $L^2(M) \to \mathcal{M}_c^2$, called the *Itô isometry*.

(ii) For $H \in L^2(M)$, $H \cdot M$ is the unique martingale in $\mathcal{M}_c^2$ such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$$

for all $N \in \mathcal{M}_c^2$, where the integral on the LHS is the stochastic integral (as above) and the RHS is the finite variation integral.

(iii) If T is a stopping time, then $(1_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T$.

Proof.

(i) We have already shown that this map is an isometry when restricted to $\mathcal{E}$. So extend by completeness of $\mathcal{M}_c^2$ and denseness of $\mathcal{E}$.

(ii) Again the equation to show is known for simple H, and we want to show it is preserved under taking limits. Suppose $H^n \to H$ in $L^2(M)$ with $H^n \in L^2(M)$. Then $H^n \cdot M \to H \cdot M$ in $\mathcal{M}_c^2$. We want to show that

$$\langle H \cdot M, N \rangle_\infty = \lim_{n \to \infty} \langle H^n \cdot M, N \rangle_\infty \text{ in } L^1.$$
$$H \cdot \langle M, N \rangle = \lim_{n \to \infty} H^n \cdot \langle M, N \rangle \text{ in } L^1.$$

for all $N \in \mathcal{M}_c^2$.

To show the first holds, we use the Kunita–Watanabe inequality to get

$$\mathbb{E}|\langle H \cdot M - H^n \cdot M, N \rangle_\infty| \leq \mathbb{E}\left(\langle H \cdot M - H^n \cdot M \rangle_\infty\right)^{1/2} \left(\mathbb{E}\langle N \rangle_\infty\right)^{1/2},$$

and the first factor is $\|H \cdot M - H^n \cdot M\|_{\mathcal{M}^2} \to 0$, while the second is finite since $N \in \mathcal{M}_c^2$. The second follows from

$$\mathbb{E}\left|((H - H^n) \cdot \langle M, N \rangle)_\infty\right| \leq \|H - H^n\|_{L^2(M)} \|N\|_{\mathcal{M}^2} \to 0.$$

So we know that $\langle H \cdot M, N \rangle_\infty = (H \cdot \langle M, N \rangle)_\infty$. We can then replace N by the stopped process N^t to get $\langle H \cdot M, N \rangle_t = (H \cdot \langle M, N \rangle)_t$.

To see uniqueness, suppose $X \in \mathcal{M}_c^2$ is another such martingale. Then we have $\langle X - H \cdot M, N \rangle = 0$ for all N. Take $N = X - H \cdot M$, and then we are done.

(iii) For $N \in \mathcal{M}_c^2$, we have

$$\langle (H \cdot M)^T, N\rangle_t = \langle H \cdot M, N\rangle_{t \wedge T} = H \cdot \langle M, N\rangle_{t\wedge T} = (H 1_{[0,T]} \cdot \langle M, N\rangle)_t$$

for every N. So we have shown that

$$(H \cdot M)^T = (1_{[0,T]} H \cdot M)$$

by (ii). To prove the second equality, we have

$$\langle H \cdot M^T, N\rangle_t = H \cdot \langle M^T, N\rangle_t = H \cdot \langle M, N\rangle_{t\wedge T} = ((H 1_{[0,T]} \cdot \langle M, N\rangle)_t. \quad \square$$

Corollary.

$$\langle H \cdot M, K \cdot N\rangle = H \cdot (K \cdot \langle M, N\rangle) = (HK) \cdot \langle M, N\rangle.$$

In other words,

$$\left\langle \int_0^{(-)} H_s \,\mathrm{d}M_s, \int_0^{(-)} K_s \,\mathrm{d}N_s \right\rangle_t = \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s. \quad \square$$

Corollary. Since $H \cdot M$ and $(H \cdot M)(K \cdot N) - \langle H \cdot M, K \cdot N\rangle$ are martingales starting at 0, we have

$$\mathbb{E}\left(\int_0^t H \,\mathrm{d}M_s\right) = 0$$

$$\mathbb{E}\left(\left(\int_0^t H_s \,\mathrm{d}M_s\right)\left(\int_0^t K_s \,\mathrm{d}N_s\right)\right) = \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s. \quad \square$$

Corollary. Let $H \in L^2(M)$, then $HK \in L^2(M)$ iff $K \in L^2(H \cdot M)$, in which case

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Proof. We have

$$\mathbb{E}\left(\int_0^\infty K_s^2 H_s^2 \,\mathrm{d}\langle M_s\rangle\right) = \mathbb{E}\left(\int_0^\infty K_s^2 \langle H \cdot M\rangle_s\right),$$

so $\|K\|_{L^2(H\cdot M)} = \|HK\|_{L^2(M)}$. For $N \in \mathcal{M}_c^2$, we have

$$\langle (KH) \cdot M, N\rangle_t = (KH \cdot \langle M, N\rangle)_t = (K \cdot (H \cdot \langle M, N\rangle))_t = (K \cdot \langle H \cdot M, N\rangle)_t. \quad \square$$

3.3 Extension to local martingales

Theorem. Let M be a continuous local martingale.

(i) For every $H \in L^2_{bc}(M)$, there is a unique continuous local martingale $H \cdot M$ with $(H \cdot M)_0 = 0$ and

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$$

for all N, M.

(ii) If T is a stopping time, then

$$(\mathbf{1}_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

(iii) If $H \in L^2_{loc}(M)$, K is previsible, then $K \in L^2_{loc}(H \cdot M)$ iff $HK \in L^2_{loc}(M)$, and then

$$K \cdot (H \cdot M) = (KH) \cdot M.$$

(iv) Finally, if $M \in \mathcal{M}_c^2$ and $H \in L^2(M)$, then the definition is the same as the previous one.

Proof. Assume $M_0 = 0$, and that $\int_0^t H_s^2 \, \mathrm{d}\langle M \rangle_s < \infty$ for all $\omega \in \Omega$ (by setting $H = 0$ when this fails). Set

$$S_n = \inf \left\{ t \geq 0 : \int_0^t (1 + H_s^2) \, \mathrm{d}\langle M \rangle_s \geq n \right\}.$$

These S_n are stopping times that tend to infinity. Then

$$\langle M^{S_n}, M^{S_n} \rangle_t = \langle M, M \rangle_{t \wedge S_n} \leq n.$$

So $M^{S_n} \in \mathcal{M}_c^2$. Also,

$$\int_0^\infty H_s \, \mathrm{d}\langle M^{S_n} \rangle_s = \int_0^{S_n} H_s^2 \, \mathrm{d}\langle M \rangle_s \leq n.$$

So $H \in L^2(M^{S_n})$, and we have already defined what $H \cdot M^{S_n}$ is. Now notice that

$$H \cdot M^{S_n} = (H \cdot M^{S_m})^{S_n} \text{ for } m \geq n.$$

So it makes sense to define

$$H \cdot M = \lim_{n \to \infty} H \cdot M^{S_n}.$$

This is the unique process such that $(H \cdot M)^{S_n} = H \cdot M^{S_n}$. We see that $H \cdot M$ is a continuous adapted local martingale with reducing sequence S_n.

Claim. $\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$.

Indeed, assume that $N_0 = 0$. Set $S_n' = \inf\{t \geq 0 : |N_t| \geq n\}$. Set $T_n = S_n \wedge S_n'$. Observe that $N^{S_n'} \in \mathcal{M}_c^2$. Then

$$\langle H \cdot M, N \rangle^{T_n} = \langle H \cdot M^{S_n}, N^{S_n'} \rangle = H \cdot \langle M^{S_n}, N^{S_n'} \rangle = H \cdot \langle M, N \rangle^{T_n}.$$

Taking the limit $n \to \infty$ gives the desired result.

The proofs of the other claims are the same as before, since they only use the characterizing property $\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$. □

3.4 Extension to semi-martingales

Proposition.

(i) $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H and K are locally bounded.

(iii) $(H \cdot X)^T = H1_{[0,T]} \cdot X = H \cdot X^T$ for every stopping time T.

(iv) If X is a continuous local martingale (resp. a finite variation process), then so is $H \cdot X$.

(v) If $H = \sum_{i=1}^n H_{i-1} \mathbf{1}_{(t_{i-1}, t_i]}$ and $H_{i-1} \in \mathcal{F}_{t_{i-1}}$ (not necessarily bounded), then

$$(H \cdot X)_t = \sum_{i=1}^n H_{i-1}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}).$$

Proof. (i) to (iv) follow from analogous properties for $H \cdot M$ and $H \cdot A$. The last part is also true by definition if the H_i are uniformly bounded. If H_i is not bounded, then the finite variation part is still fine, since for each fixed $\omega \in \Omega$, $H_i(\omega)$ is a fixed number. For the martingale part, set

$$T_n = \inf\{t \geq 0 : |H_t| \geq n\}.$$

Then T_n are stopping times, $T_n \to \infty$, and $H1_{[0,T_n]} \in \mathcal{E}$. Thus

$$(H \cdot M)_{t \wedge T_n} = \sum_{i=1}^n H_{i-1} T_{[0,T_n]}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}).$$

Then take the limit $n \to \infty$. □

Proposition (Stochastic dominated convergence theorem)**.** Let X be a continuous semi-martingale. Let H, H_s be previsible and locally bounded, and let K be previsible and non-negative. Let $t > 0$. Suppose

(i) $H_s^n \to H_s$ as $n \to \infty$ for all $s \in [0, t]$.

(ii) $|H_s^n| \leq K_s$ for all $s \in [0, t]$ and $n \in \mathbb{N}$.

(iii) $\int_0^t K_s^2 \, \mathrm{d}\langle M \rangle_s < \infty$ and $\int_0^t K_s \, |\mathrm{d}A_s| < \infty$ (note that both conditions are okay if K is locally bounded).

Then

$$\int_0^t H_s^n \, \mathrm{d}X_s \to \int_0^t H_s \, \mathrm{d}X_s \text{ in probability.}$$

Proof. For the finite variation part, the convergence follows from the usual dominated convergence theorem. For the martingale part, we set

$$T_m = \inf\left\{t \geq 0 : \int_0^t K_s^2 \, \mathrm{d}\langle M \rangle_s \geq m\right\}.$$

So we have

$$\mathbb{E}\left(\left(\int_0^{T_m\wedge t} H_s^n \,\mathrm{d}M_s - \int_0^{T_n\wedge t} H_s \,\mathrm{d}M_s\right)^2\right) \leq \mathbb{E}\left(\int_0^{T_n\wedge t} (H_s^n - H_s)^2 \,\mathrm{d}\langle M\rangle_s\right) \to 0.$$

using the usual dominated convergence theorem, since $\int_0^{T_n\wedge t} K_s^2 \,\mathrm{d}\langle M\rangle_s \leq m$.

Since $T_n \wedge t = t$ eventually as $n \to \infty$ almost surely, hence in probability, we are done. □

Proposition. Let X be a continuous semi-martingale, and let H be an adapted bounded left-continuous process. Then for every subdivision $0 < t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)}$ of $[0, t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$, then

$$\int_0^t H_s \,\mathrm{d}X_s = \lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^{(m)}}(X_{t_i^{(m)}} - X_{t_{i-1}^{(m)}})$$

in probability.

Proof. We have already proved this for the Lebesgue–Stieltjes integral, and all we used was dominated convergence. So the same proof works using stochastic dominated convergence theorem. □

3.5 Itô formula

Theorem (Integration by parts)**.** Let X, Y be a continuous semi-martingale. Then almost surely,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \,\mathrm{d}Y_s + \int_0^t Y_s \,\mathrm{d}X_s + \langle X, Y\rangle_t$$

The last term is called the *Itô correction*.

Proof. We have

$$X_t Y_t - X_s Y_s = X_s(Y_t - Y_s) + (X_t - X_s)Y_s + (X_t - X_s)(Y_t - Y_s).$$

When doing usual calculus, we can drop the last term, because it is second order. However, the quadratic variation of martingales is in general non-zero, and so we must keep track of this. We have

$$\begin{aligned} X_{k2^{-n}}Y_{k2^{-n}} - X_0Y_0 &= \sum_{i=1}^k (X_{i2^{-n}}Y_{i2^{-n}} - X_{(i-1)2^{-n}}Y_{(i-1)2^{-n}}) \\ &= \sum_{i=1}^n \Big(X_{(i-1)2^{-n}}(Y_{i2^{-n}} - Y_{(i-1)2^{-n}}) \\ &\quad + Y_{(i-1)2^{-n}}(X_{i2^{-n}} - X_{(i-1)2^{-n}}) \\ &\quad + (X_{i2^{-n}} - X_{(i-1)^{2^{-n}}})(Y_{i2^{-n}} - Y_{(i-1)2^{-n}})\Big) \end{aligned}$$

Taking the limit $n \to \infty$ with $k2^{-n}$ fixed, we see that the formula holds for t a dyadic rational. Then by continuity, it holds for all t. □

Theorem (Itô's formula)**.** Let $X^1, \ldots, X^p$ be continuous semi-martingales, and let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Then, writing $X = (X^1, \ldots, X^p)$, we have, almost surely,

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, \mathrm{d}X_s^i + \frac{1}{2}\sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, \mathrm{d}\langle X^i, X^j\rangle_s.$$

In particular, $f(X)$ is a semi-martingale.

Proof.

Claim. Itô's formula holds when f is a polynomial.

It clearly does when f is a constant! We then proceed by induction. Suppose Itô's formula holds for some f. Then we apply integration by parts to

$$g(x) = x^k f(x).$$

where x^k denotes the kth component of x. Then we have

$$g(X_t) = g(X_0) + \int_0^t X_s^k\, \mathrm{d}f(X_s) + \int_0^t f(X_s)\, \mathrm{d}X_s^k + \langle X^k, f(X)\rangle_t$$

We now apply Itô's formula for f to write

$$\begin{aligned}\int_0^t X_s^k\, \mathrm{d}f(X_s) = \sum_{i=1}^p \int_0^t X_s^k \frac{\partial f}{\partial x^i}(X_s)\, \mathrm{d}X_s^i \\ + \frac{1}{2}\sum_{i,j=1}^p \int_0^t X_s^k \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, \mathrm{d}\langle X^i, X^j\rangle_s.\end{aligned}$$

We also have

$$\langle X^k, f(X)\rangle_t = \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, \mathrm{d}\langle X^k, X^i\rangle_s.$$

So we have

$$g(X_t) = g(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial g}{\partial x^i}(X_s)\, \mathrm{d}X_s^i + \frac{1}{2}\sum_{i,j=1}^p \int_0^t \frac{\partial^2 g}{\partial x^i \partial x^j}(X_s)\, \mathrm{d}\langle X^i, X^j\rangle_s.$$

So by induction, Itô's formula holds for all polynomials.

Claim. Itô's formula holds for all $f \in C^2$ if $|X_t(\omega)| \leq n$ and $\int_0^t |\mathrm{d}A_s| \leq n$ for all (t, ω).

By the Weierstrass approximation theorem, there are polynomials p_k such that

$$\sup_{|x|\leq k}\left(|f(x) - p_k(x)| + \max_i \left|\frac{\partial f}{\partial x^i} - \frac{\partial p}{\partial x^i}\right| + \max_{i,j}\left|\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial p_k}{\partial x^i \partial x^j}\right|\right) \leq \frac{1}{k}.$$

By taking limits, in probability, we have

$$\begin{aligned}f(X_t) - f(X_0) &= \lim_{k\to\infty}(p_k(X_t) - p_k(X_0)) \\ \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, \mathrm{d}X_s^i &= \lim_{k\to\infty}\frac{\partial p_k}{\partial x^i}(X_s)\, \mathrm{d}X_s^i\end{aligned}$$

by stochastic dominated convergence theorem, and by the regular dominated convergence, we have

$$\int_0^t \frac{\partial f}{\partial x^i \partial x^j} \, \mathrm{d}\langle X^i, X^j \rangle_s = \lim_{k \to \infty} \int_0^t \frac{\partial^2 p_k}{\partial x^i \partial x^j} \, \mathrm{d}\langle X^i, X^j \rangle.$$

Claim. Itô's formula holds for all X.

Let

$$T_n = \inf\left\{t \geq 0 : |X_t| \geq n \text{ or } \int_0^t |\mathrm{d}A_s| \geq n\right\}$$

Then by the previous claim, we have

$$\begin{aligned} f(X_t^{T_n}) &= f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X_s^{T_n}) \, \mathrm{d}(X_i)_s^{T_n} \\ &\quad + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^{T_n}) \, \mathrm{d}\langle (X_i)^{T_n}, (X^j)^{T_n} \rangle_s \\ &= f(X_0) + \sum_{i=1}^p \int_0^{t \wedge T_n} \frac{\partial f}{\partial x^i}(X_s) \, \mathrm{d}(X_i)_s \\ &\quad + \frac{1}{2} \sum_{i,j} \int_0^{t \wedge T_n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) \, \mathrm{d}\langle (X_i), (X^j) \rangle_s. \end{aligned}$$

Then take $T_n \to \infty$. □

3.6 The Lévy characterization

Theorem (Lévy's characterization of Brownian motion)**.** Let $(X^1, \ldots, X^d)$ be continuous local martingales. Suppose that $X_0 = 0$ and that $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for all $i, j = 1, \ldots, d$ and $t \geq 0$. Then $(X^1, \ldots, X^d)$ is a standard d-dimensional Brownian motion.

Proof. Let $0 \leq s < t$. It suffices to check that $X_t - X_s$ is independent of $\mathcal{F}_s$ and $X_t - X_s \sim N(0, (t-s)I)$.

Claim. $\mathbb{E}(e^{i\theta \cdot (X_t - X_s)} \mid \mathcal{F}_s) = e^{-\frac{1}{2}|\theta|^2 (t-s)}$ for all $\theta \in \mathbb{R}^d$ and $s < t$.

This is sufficient, since the right-hand side is independent of $\mathcal{F}_s$, hence so is the left-hand side, and the Fourier transform characterizes the distribution.

To check this, for $\theta \in \mathbb{R}^d$, we define

$$Y_t = \theta \cdot X_t = \sum_{i=1}^d \theta^i X_t^i.$$

Then Y is a continuous local martingale, and we have

$$\langle Y \rangle_t = \langle Y, Y \rangle_t = \sum_{i,j=1}^d \theta^j \theta^k \, \langle X^j, X^k \rangle_t = |\theta|^2 t.$$

by assumption. Let

$$Z_t = e^{iY_t + \frac{1}{2}\langle Y\rangle_t} = e^{i\theta\cdot X_t + \frac{1}{2}|\theta|^2 t}.$$

By Itô's formula, with $X = iY + \frac{1}{2}\langle Y\rangle_t$ and $f(x) = e^x$, we get

$$\mathrm{d}Z_t = Z_t\left(i\mathrm{d}Y_t - \frac{1}{2}\mathrm{d}\langle Y\rangle_t + \frac{1}{2}\mathrm{d}\langle Y\rangle_t\right) = iZ_t\,\mathrm{d}Y_t.$$

So this implies Z is a continuous local martingale. Moreover, since Z is bounded on bounded intervals of t, we know Z is in fact a martingale, and $Z_0 = 1$. Then by definition of a martingale, we have

$$\mathbb{E}(Z_t \mid \mathcal{F}_s) = Z_s,$$

And unwrapping the definition of Z_t shows that the result follows. □

Theorem (Dubins–Schwarz)**.** Let M be a continuous local martingale with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$. Let

$$T_s = \inf\{t \geq 0 : \langle M\rangle_t > s\},$$

the right-continuous inverse of $\langle M\rangle_t$. Let $B_s = M_{T_s}$ and $\mathcal{G}_s = \mathcal{F}_{T_s}$. Then T_s is a $(\mathcal{F}_t)$ stopping time, $\langle M\rangle_{T_s} = s$ for all $s \geq 0$, B is a $(\mathcal{G}_s)$-Brownian motion, and

$$M_t = B_{\langle M\rangle_t}.$$

Proof. Since $\langle M\rangle$ is continuous and adapted, and $\langle M\rangle_\infty = \infty$, we know T_s is a stopping time and $T_s < \infty$ for all $s \geq 0$.

Claim. $(\mathcal{G}_s)$ is a filtration obeying the usual conditions, and $\mathcal{G}_\infty = \mathcal{F}_\infty$

Indeed, if $A \in \mathcal{G}_s$ and $s < t$, then

$$A \cap \{T_t \leq u\} = A \cap \{T_s \leq u\} \cap \{T_t \leq u\} \in \mathcal{F}_u,$$

using that $A \cap \{T_s \leq u\} \in \mathcal{F}_u$ since $A \in \mathcal{G}_s$. Then right-continuity follows from that of $(\mathcal{F}_t)$ and the right-continuity of $s \mapsto T_s$.

Claim. B is adapted to $(\mathcal{G}_s)$.

In general, if X is cádlág and T is a stopping time, then $X_T \mathbf{1}_{\{T<\infty\}} \in \mathcal{F}_T$. Apply this is with $X = M$, $T = T_s$ and $\mathcal{F}_T = \mathcal{G}_s$. Thus $B_s \in \mathcal{G}_s$.

Claim. B is continuous.

Here this is actually something to verify, because $s \mapsto T_s$ is only right continuous, not necessarily continuous. Thus, we only know B_s is right continuous, and we have to check it is left continuous.

Now B is left-continuous at s iff $B_s = B_{s^-}$, iff $M_{T_s} = M_{T_{s-}}$. Now we have

$$T_{s-} = \inf\{t \geq 0 : \langle M\rangle_t \geq s\}.$$

If $T_s = T_{s-}$, then there is nothing to show. Thus, we may assume $T_s > T_{s-}$. Then we have $\langle M\rangle_{T_s} = \langle M\rangle_{T_{s-}}$. Since $\langle M\rangle_t$ is increasing, it means $\langle M\rangle_{T_s}$ is constant in $[T_{s-}, T_s]$. We will later prove that

Lemma. M is constant on $[a,b]$ iff $\langle M\rangle$ being constant on $[a,b]$.

So we know that if $T_s > T_{s-}$, then $M_{T_s} = M_{T_s-}$. So B is left continuous. We then have to show that B is a martingale.

Claim. $(M^2 - \langle M\rangle)^{T_s}$ is a uniformly integrable martingale.

To see this, observe that $\langle M^{T_s}\rangle_\infty = \langle M\rangle_{T_s} = s$, and so M^{T_s} is bounded. So $(M^2 - \langle M\rangle)^{T_s}$ is a uniformly integrable martingale.

We now apply the optional stopping theorem, which tells us

$$\mathbb{E}(B_s \mid \mathcal{G}_r) = \mathbb{E}(M_\infty^{T_s} \mid \mathcal{G}_s) = M_{T_t} = B_t.$$

So B_t is a martingale. Moreover,

$$\mathbb{E}(B_s^2 - s \mid \mathcal{G}_r) = \mathbb{E}((M^2 - \langle M\rangle)^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r}^2 - \langle M\rangle_{T_r} = B_r^2 - r.$$

So $B_t^2 - t$ is a martingale, so by the characterizing property of the quadratic variation, $\langle B\rangle_t = t$. So by Lévy's criterion, this is a Brownian motion in one dimension. □

Lemma. M is constant on $[a,b]$ iff $\langle M\rangle$ being constant on $[a,b]$.

Proof. It is clear that if M is constant, then so is $\langle M\rangle$. To prove the converse, by continuity, it suffices to prove that for any fixed $a < b$,

$$\{M_t = M_a \text{ for all } t \in [a,b]\} \supseteq \{\langle M\rangle_b = \langle M\rangle_a\} \text{ almost surely.}$$

We set $N_t = M_t - M_t \wedge a$. Then $\langle N\rangle_t = \langle M\rangle_t - \langle M\rangle_{t\wedge a}$. Define

$$T_\varepsilon = \inf\{t \geq 0 : \langle N\rangle_t \geq \varepsilon\}.$$

Then since $N^2 - \langle N\rangle$ is a local martingale, we know that

$$\mathbb{E}(N_{t\wedge T_\varepsilon}^2) = \mathbb{E}(\langle N\rangle_{t\wedge T_\varepsilon}) \leq \varepsilon.$$

Now observe that on the event $\{\langle M\rangle_b = \langle M\rangle_a\}$, we have $\langle N\rangle_b = 0$. So for $t \in [a,b]$, we have

$$\mathbb{E}(1_{\{\langle M\rangle_b = \langle M\rangle_a\}} N_t^2) = \mathbb{E}(1_{\{\langle M\rangle_b = \langle M\rangle_a} N_{t\wedge T_\varepsilon}^2) = \mathbb{E}(\langle N\rangle_{t\wedge T_\varepsilon}) = 0.$$ □

3.7 Girsanov's theorem

Proposition. Let M be a continuous local martingale with $M_0 = 0$. Then $\mathcal{E}(M) = Z$ satisfies

$$\mathrm{d}Z_t = Z_t \,\mathrm{d}M,$$

i.e.

$$Z_t = 1 + \int_0^t Z_s \,\mathrm{d}M_s.$$

In particular, $\mathcal{E}(M)$ is a continuous local martingale. Moreover, if $\langle M\rangle$ is uniformly bounded, then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. By Itô's formula with $X = M - \frac{1}{2}\langle M\rangle$, we have

$$\mathrm{d}Z_t = Z_t\mathrm{d}\left(M_t - \frac{1}{2}\mathrm{d}\langle M\rangle_t\right) + \frac{1}{2}Z_t\mathrm{d}\langle M\rangle_t = Z_t\;\mathrm{d}M_t.$$

Since M is a continuous local martingale, so is $\int Z_s\;\mathrm{d}M_s$. So Z is a continuous local martingale.

Now suppose $\langle M\rangle_\infty \leq b < \infty$. Then

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq a\right) = \mathbb{P}\left(\sup_{t\geq 0} M_t \geq a,\ \langle M\rangle_\infty \leq b\right) \leq e^{-a^2/2b},$$

where the final equality is an exercise on the third example sheet, which is true for general continuous local martingales. So we get

$$\begin{aligned}\mathbb{E}\left(\exp\left(\sup_t M_t\right)\right) &= \int_0^\infty \mathbb{P}(\exp(\sup M_t) \geq \lambda)\;\mathrm{d}\lambda\\ &= \int_0^\infty \mathbb{P}(\sup M_t \geq \log\lambda)\;\mathrm{d}\lambda\\ &\leq 1 + \int_1^\infty e^{-(\log\lambda)^2/2b}\;\mathrm{d}\lambda < \infty.\end{aligned}$$

Since $\langle M\rangle \geq 0$, we know that

$$\sup_{t\geq 0}\mathcal{E}(M)_t \leq \exp\left(\sup M_t\right),$$

So $\mathcal{E}(M)$ is a uniformly integrable martingale. □

Theorem (Girsanov's theorem)**.** Let M be a continuous local martingale with $M_0 = 0$. Suppose that $\mathcal{E}(M)$ is a uniformly integrable martingale. Define a new probability measure

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(M)_\infty$$

Let X be a continuous local martingale with respect to $\mathbb{P}$. Then $X - \langle X, M\rangle$ is a continuous local martingale with respect to $\mathbb{Q}$.

Proof. Define the stopping time

$$T_n = \inf\{t \geq 0 : |X_t - \langle X, M\rangle_t| \geq n\},$$

and $\mathbb{P}(T_n \to \infty) = 1$ by continuity. Since $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, we know that $\mathbb{Q}(T_n \to \infty) = 1$. Thus it suffices to show that $X^{T_n} - \langle X^{T_n}, M\rangle$ is a continuous martingale for any n. Let

$$Y = X^{T_n} - \langle X^{T_n}, M\rangle,\quad Z = \mathcal{E}(M).$$

Claim. ZY is a continuous local martingale with respect to $\mathbb{P}$.

We use the product rule to compute

$$\begin{aligned}\mathrm{d}(ZY) &= Y_t\;\mathrm{d}Z_t + Z_t\;\mathrm{d}Y_t + \mathrm{d}\langle Y, Z\rangle_t\\ &= YZ_t\;\mathrm{d}M_t + Z_t(\mathrm{d}X^{T_n} - \mathrm{d}\langle X^{T_n}, M\rangle_t) + Z_t\;\mathrm{d}\langle M, X^{T_n}\rangle\\ &= YZ_t\;\mathrm{d}M_t + Z_t\;\mathrm{d}X^{T_n}\end{aligned}$$

So we see that ZY is a stochastic integral with respect to a continuous local martingale. Thus ZY is a continuous local martingale.

Claim. ZY is uniformly integrable.

Since Z is a uniformly integrable martingale, $\{Z_T : T \text{ is a stopping time}\}$ is uniformly integrable. Since Y is bounded, $\{Z_T Y_T : T \text{ is a stopping time}\}$ is also uniformly integrable. So ZY is a true martingale (with respect to $\mathbb{P}$).

Claim. Y is a martingale with respect to $\mathbb{Q}$.

We have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(Y_t - Y_s \mid \mathcal{F}_s) &= \mathbb{E}^{\mathbb{P}}(Z_\infty Y_t - Z_\infty Y_s \mid \mathcal{F}_s)\\ &= \mathbb{E}^{\mathbb{P}}(Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s) = 0.\end{aligned}$$

□

4 Stochastic differential equations

4.1 Existence and uniqueness of solutions

Theorem (Yamada–Watanabe)**.** Assume weak existence and pathwise uniqueness holds. Then

(i) Uniqueness in law holds.

(ii) For every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B and any $x \in \mathbb{R}^d$, there is a unique strong solution to $E_x(a, b)$. □

Theorem. Assume b, σ are Lipschitz in x. Then there is pathwise uniqueness for the $E(\sigma, b)$ and for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and every $(\mathcal{F}_t)$-Brownian motion B, for every $x \in \mathbb{R}^d$, there exists a unique strong solution to $E_x(\sigma, b)$.

Proof. To simplify notation, we assume $m = d = 1$.

We first prove pathwise uniqueness. Suppose X, X' are two solutions with $X_0 = X'_0$. We will show that $\mathbb{E}[(X_t - X'_t)^2] = 0$. We will actually put some bounds to control our variables. Define the stopping time

$$S = \inf\{t \geq 0 : |X_t| \geq n \text{ or } |X'_t| \geq n\}.$$

By continuity, $S \to \infty$ as $n \to \infty$. We also fix a deterministic time $T > 0$. Then whenever $t \in [0, T]$, we can bound, using the identity $(a + b)^2 \leq 2a^2 + 2b^2$,

$$\begin{aligned}\mathbb{E}((X_{t\wedge S} - X'_{t\wedge S})^2) \leq 2\mathbb{E}\left(\left(\int_0^{t\wedge S} (\sigma(s, X_s) - \sigma(s, X'_s))\;\mathrm{d}B_s\right)^2\right)\\ + 2\mathbb{E}\left(\left(\int_0^{t\wedge S} (b(s, X_s) - b(s, X'_s))\;\mathrm{d}s\right)^2\right).\end{aligned}$$

We can apply the Lipschitz bound to the second term immediately, while we can simplify the first term using the (corollary of the) Itô isometry

$$\mathbb{E}\left(\left(\int_0^{t\wedge S} (\sigma(s, X_s) - \sigma(s, X'_s))\;\mathrm{d}B_s\right)^2\right) = \mathbb{E}\left(\int_0^{t\wedge S} (\sigma(s, X_s) - \sigma(s, X'_s))^2\;\mathrm{d}s\right).$$

So using the Lipschitz bound, we have

$$\begin{aligned}\mathbb{E}((X_{t\wedge S} - X'_{t\wedge S})^2) &\leq 2K^2(1 + T)\mathbb{E}\left(\int_0^{t\wedge S} |X_s - X'_s|^2\;\mathrm{d}s\right)\\ &\leq 2K^2(1 + T)\int_0^t \mathbb{E}(|X_{s\wedge S} - X'_{s\wedge S}|^2)\;\mathrm{d}s.\end{aligned}$$

We now use Grönwall's lemma:

Lemma. Let $h(t)$ be a function such that

$$h(t) \leq c\int_0^t h(s)\;\mathrm{d}s$$

for some constant c. Then

$$h(t) \leq h(0)e^{ct}.$$

□

Applying this to

$$h(t) = \mathbb{E}((X_{t\wedge S} - X'_{t\wedge S})^2),$$

we deduce that $h(t) \le h(0)e^{ct} = 0$. So we know that

$$\mathbb{E}(|X_{t\wedge S} - X'_{t\wedge S}|^2) = 0$$

for every $t \in [0, T]$. Taking $n \to \infty$ and $T \to \infty$ gives pathwise uniqueness.

We next prove existence of solutions. We fix $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$ and B, and define

$$F(X)_t = X_0 + \int_0^t \sigma(s, X_s)\;\mathrm{d}B_s + \int_0^t b(s, X_s)\;\mathrm{d}s.$$

Then X is a solution to $E_x(a, b)$ iff $F(X) = X$ and $X_0 = x$. To find a fixed point, we use Picard iteration. We fix $T > 0$, and define the T-norm of a continuous adapted process X as

$$\|X\|_T = \mathbb{E}\left(\sup_{t\le T} |X_t|^2\right)^{1/2}.$$

In particular, if X is a martingale, then this is the same as the norm on the space of L^2-bounded martingales by Doob's inequality. Then

$$B = \{X : \Omega \times [0, T] \to \mathbb{R} : \|X\|_T < \infty\}$$

is a Banach space.

Claim. $\|F(0)\|_T < \infty$, and

$$\|F(X) - F(Y)\|_T^2 \le (2T + 8)K^2 \int_0^T \|X - Y\|_t^2\;\mathrm{d}t.$$

We first see how this claim implies the theorem. First observe that the claim implies F indeed maps B into itself. We can then define a sequence of processes X_t^i by

$$X_t^0 = x, \quad X^{i+1} = F(X^i).$$

Then we have

$$\|X^{i+1} - X^i\|_T^2 \le CT \int_0^T \|X^i - X^{i-1}\|_t^2\;\mathrm{d}t \le \cdots \le \|X^1 - X^0\|_T^2 \left(\frac{CT^i}{i!}\right).$$

So we find that

$$\sum_{i=1}^\infty \|X^i - X^{i-1}\|_T^2 < \infty$$

for all T. So X^i converges to X almost surely and uniformly on $[0, T]$, and $F(X) = X$. We then take $T \to \infty$ and we are done.

To prove the claim, we write

$$\|F(0)\|_T \le |X_0| + \left\|\int_0^t b(s, 0)\;\mathrm{d}s\right\| + \left\|\int_0^t \sigma(s, 0)\;\mathrm{d}B_s\right\|_T.$$

The first two terms are constant, and we can bound the last by Doob's inequality and the Itô isometry:

$$\left\|\int_0^t \sigma(s,0)\,\mathrm{d}B_s\right\|_T \leq 2\mathbb{E}\left(\left|\int_0^T \sigma(s,0)\,\mathrm{d}B_s\right|^2\right) = 2\int_0^T \sigma(s,0)^2\,\mathrm{d}s.$$

To prove the second part, we use

$$\begin{aligned}\|F(X)-F(Y)\|^2 \leq 2\mathbb{E}&\left(\sup_{t\leq T}\left|\int_0^t b(s,X-s)-b(s,Y_s)\,\mathrm{d}s\right|^2\right)\\ &+2\mathbb{E}\left(\sup_{t\leq T}\left|\int_0^t (\sigma(s,X_s)-\sigma(s,Y_s))\,\mathrm{d}B_s\right|^2\right).\end{aligned}$$

We can bound the first term with Cauchy–Schwartz by

$$T\mathbb{E}\left(\int_0^T |b(s,X_s)-b(s,Y_s)|^2\right) \leq TK^2\int_0^T \|X-Y\|_t^2\,\mathrm{d}t,$$

and the second term with Doob's inequality by

$$\mathbb{E}\left(\int_0^T |\sigma(s,X_s)-\sigma(s,Y_s)|^2\,\mathrm{d}s\right) \leq 4K^2\int_0^T \|X-Y\|_t^2\,\mathrm{d}t.$$ □

4.2 Examples of stochastic differential equations

Proof. We only have to compute the covariance. By the Itô isometry, we have

$$\begin{aligned}\mathbb{E}((X_t-\mathbb{E}X_t)(X_s-\mathbb{E}X_s)) &= \mathbb{E}\left(\int_0^t e^{-\lambda(t-u)}\,\mathrm{d}B_u\int_0^s e^{-\lambda(s-u)}\,\mathrm{d}B_u\right)\\ &= e^{-\lambda(t+s)}\int_0^{t\wedge s} e^{\lambda u}\,\mathrm{d}u.\end{aligned}$$ □

Theorem. The eigenvalues $\lambda_1(t)\leq\cdots\leq\lambda_N(t)$ satisfies

$$\mathrm{d}\lambda_t^i = \left(-\frac{\lambda^i}{2}+\frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)\mathrm{d}t+\sqrt{\frac{2}{N\beta}}\,\mathrm{d}B^i.$$

Here $\beta=1$, but if we replace symmetric matrices by Hermitian ones, we get $\beta=2$; if we replace symmetric matrices by symplectic ones, we get $\beta=4$.

4.3 Representations of solutions to PDEs

Proposition. Let $x\in\mathbb{R}^d$, and X a solution to $E_x(\sigma,b)$. Then for every $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ that is C^1 in $\mathbb{R}_+$ and C^2 in $\mathbb{R}^d$, the process

$$M_t^f = f(t,X_t)-f(0,X_0)-\int_0^t\left(\frac{\partial}{\partial s}+L\right)f(s,X_s)\,\mathrm{d}s$$

is a continuous local martingale. □

Theorem. Assume U has a smooth boundary (or satisfies the exterior cone condition), a, b are Hölder continuous and a is uniformly elliptic. Then for every Hölder continuous $f : \bar{U} \to \mathbb{R}$ and any continuous $g : \partial U \to \mathbb{R}$, the Dirichlet–Poisson process has a solution. □

Theorem. Let σ and b be bounded measurable and $\sigma\sigma^T$ uniformly elliptic, $U \subseteq \mathbb{R}^d$ as above. Let u be a solution to the Dirichlet–Poisson problem and X a solution to $E_x(\sigma, b)$ for some $x \in \mathbb{R}^d$. Define the stopping time

$$T_U = \inf\{t \geq 0 : X_t \not\in U\}.$$

Then $\mathbb{E}T_U < \infty$ and

$$u(x) = \mathbb{E}_x \left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\;\mathrm{d}s\right).$$

In particular, the solution to the PDE is unique.

Proof. Our previous proposition applies to functions defined on all of $\mathbb{R}^n$, while u is just defined on U. So we set

$$U_n = \left\{x \in U : \mathrm{dist}(x, \partial U) > \frac{1}{n}\right\}, \quad T_n = \inf\{t \geq 0 : X_t \not\in U_n\},$$

and pick $u_n \in C_b^2(\mathbb{R}^d)$ such that $u|_{U_n} = u_n|_{U_n}$. Recalling our previous notation, let

$$M_t^n = (M^{u_n})_t^{T_n} = u_n(X_{t \wedge T_n}) - u_n(X_0) - \int_0^{t \wedge T_n} Lu_n(X_s)\;\mathrm{d}s.$$

Then this is a continuous local martingale that is bounded by the proposition, and is bounded, hence a true martingale. Thus for $x \in U$ and n large enough, the martingale property implies

$$\begin{aligned} u(x) = u_n(x) = \mathbb{E}&\left(u(X_{t \wedge T_n}) - \int_0^{t \wedge T_n} Lu(X_s)\;\mathrm{d}s\right) \\ &= \mathbb{E}\left(u(X_{t \wedge T_n}) + \int_0^{t \wedge T_n} f(X_s)\;\mathrm{d}s\right). \end{aligned}$$

We would be done if we can take $n \to \infty$. To do so, we first show that $\mathbb{E}[T_U] < \infty$.

Note that this does not depend on f and g. So we can take $f = 1$ and $g = 0$, and let v be a solution. Then we have

$$\mathbb{E}(t \wedge T_n) = \mathbb{E}\left(-\int_0^{t \wedge T_n} Lv(X_s)\;\mathrm{d}s\right) = v(x) - \mathbb{E}(v(X_{t \wedge T_n})).$$

Since v is bounded, by dominated/monotone convergence, we can take the limit to get

$$\mathbb{E}(T_U) < \infty.$$

Thus, we know that $t \wedge T_n \to T_U$ as $t \to \infty$ and $n \to \infty$. Since

$$\mathbb{E}\left(\int_0^{T_U} |f(X_s)|\;\mathrm{d}s\right) \leq \|f\|_\infty \mathbb{E}[T_U] < \infty,$$

the dominated convergence theorem tells us

$$\mathbb{E}\left(\int_0^{t\wedge T_n} f(X_s)\,\mathrm{d}s\right) \to \mathbb{E}\left(\int_0^{T_U} f(X_s)\,\mathrm{d}s\right).$$

Since u is continuous on $\bar{U}$, we also have

$$\mathbb{E}(u(X_{t\wedge T_n})) \to \mathbb{E}(u(T_u)) = \mathbb{E}(g(T_u)). \qquad \square$$

Theorem. For every $f \in C_b^2(\mathbb{R}^d)$, there exists a solution to the Cauchy problem. $\square$

Theorem. Let u be a solution to the Cauchy problem. Let X be a solution to $E_x(\sigma, b)$ for $x \in \mathbb{R}^d$ and $0 \leq s \leq t$. Then

$$\mathbb{E}_x(f(X_t) \mid \mathcal{F}_s) = u(t-s, X_s).$$

In particular,

$$u(t,x) = \mathbb{E}_x(f(X_t)).$$

Proof. The martingale has $\frac{\partial}{\partial t} + L$, but the heat equation has $\frac{\partial}{\partial t} - L$. So we set $g(s,x) = u(t-s,x)$. Then

$$\left(\frac{\partial}{\partial s} + L\right) g(s,x) = -\frac{\partial}{\partial t} u(t-s,x) + Lu(t-s,x) = 0.$$

So $g(s,X_s) - g(0,X_0)$ is a martingale (boundedness is an exercise), and hence

$$u(t-s,X_s) = g(s,X_s) = \mathbb{E}(g(t,X_t) \mid \mathcal{F}_s) = \mathbb{E}(u(0,X_t) \mid \mathcal{F}_s) = \mathbb{E}(f(X_t) \mid X_s).$$

$\square$

Theorem (Feynman–Kac formula)**.** Let $f \in C_b^2(\mathbb{R}^d)$ and $V \in C_b(\mathbb{R}^d)$ and suppose that $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu + Vu &&\text{on } \mathbb{R}_+ \times \mathbb{R}^d\\ u(0,\,\cdot\,) &= f &&\text{on } \mathbb{R}^d, \end{aligned}$$

where $Vu = V(x)u(x)$ is given by multiplication.

Then for all $t > 0$ and $x \in \mathbb{R}^d$, and X a solution to $\mathbb{E}_x(\sigma, b)$. Then

$$u(t,x) = \mathbb{E}_x\left(f(X_t)\exp\left(\int_0^t V(X_s)\,\mathrm{d}s\right)\right). \qquad \square$$