Part III — Stochastic Calculus and Applications
Theorems

Based on lectures by R. Bauerschmidt

Notes taken by Dexter Chua

Lent 2018

- *Brownian motion.* Existence and sample path properties.
- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita–Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins–Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, Yamada–Watanabe theorem, strong Markov property, and relation to second order partial differential equations.

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with discrete-time martingales and Brownian motion.

Contents

0 Introduction

Proposition. Let H be any separable Hilbert space. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Gaussian subspace $S \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ and an isometry $I : H \to S$. In other words, for any $f \in H$, there is a corresponding random variable $I(f) \sim N(0, (f, f)_H)$. Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ and $(f, g)_H = \mathbb{E}[I(f)I(g)]$.

Proposition.

- For $A \subseteq \mathbb{R}_+$ with $|A| < \infty$, $WN(A) \sim N(0, |A|)$.
- For disjoint $A, B \subseteq \mathbb{R}_+$, the variables $WN(A)$ and $WN(B)$ are independent.
- If $A = \bigcup_{i=1}^\infty A_i$ for disjoint sets $A_i \subseteq \mathbb{R}_+$, with $|A| < \infty, |A_i| < \infty$, then

$$WN(A) = \sum_{i=1}^\infty WN(A_i) \text{ in } L^2 \text{ and a.s.}$$

1 The Lebesgue–Stieltjes integral

Theorem. For any two finite measures μ_1, μ_2, there is a signed measure μ with $\mu(A) = \mu_1(A) - \mu_2(A)$.

Theorem. There is a bijection

$$\left\{ \text{signed measures on } [0, T] \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{càdlàg functions of bounded} \\ \text{variation } a : [0, T] \to \mathbb{R} \end{array} \right\}$$

that sends a signed measure μ to $a(t) = \mu([0, t])$. To construct the inverse, given a, we define

$$a_\pm = \frac{1}{2}(V_a \pm a).$$

Then $a_\pm$ are both positive, and $a = a_+ - a_-$. We can then define $\mu_\pm$ by

$$\begin{aligned} \mu_\pm[0, t] &= a_\pm(t) - a_\pm(0) \\ \mu &= \mu_+ - \mu_- \end{aligned}$$

Moreover, $V_{\mu[0,t]} = |\mu|[0, t]$.

Proposition. Let a be càdlàg and BV on $[0, t]$, and h bounded and left-continuous. Then

$$\begin{aligned} \int_0^t h(s)\, \mathrm{d}a(s) &= \lim_{m \to \infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \Big(a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \Big) \\ \int_0^t h(s)\, |\mathrm{d}a(s)| &= \lim_{m \to \infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \Big| a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \Big| \end{aligned}$$

for any sequence of subdivisions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ of $[0, t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$.

2 Semi-martingales

2.1 Finite variation processes

Proposition. The total variation process V of a càdlàg adapted process A is also càdlàg, finite variation and adapted, and it is also increasing.

Proposition. Let A be a finite variation process, and H previsible such that

$$\int_0^t |H(\omega, s)|\, |\mathrm{d}A(\omega, s)| < \infty \text{ for all } (\omega, t) \in \Omega \times [0, \infty).$$

Then $H \cdot A$ is a finite variation process.

2.2 Local martingale

Theorem (Optional stopping theorem)**.** Let X be a càdlàg adapted integrable process. Then the following are equivalent:

(i) X is a martingale, i.e. $X_t \in L^1$ for every t, and

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \text{ for all } t > s.$$

(ii) The *stopped process* $X^T = (X^T_t) = (X_{T\wedge t})$ is a martingale for all stopping times T.

(iii) For all stopping times T, S with T bounded, $X_T \in L^1$ and $\mathbb{E}(X_T \mid \mathcal{F}_S) = X_{T\wedge S}$ almost surely.

(iv) For all bounded stopping times T, $X_T \in L^1$ and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

For X uniformly integrable, (iii) and (iv) hold for all stopping times.

Proposition. Let X be a local martingale and $X_t \geq 0$ for all t. Then X is a supermartingale.

Proposition. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then the set

$$\chi = \{\mathbb{E}(X \mid \mathcal{G}) : G \subseteq \mathcal{F} \text{ a sub-}\sigma\text{-algebra}\}$$

is uniformly integrable, i.e.

$$\sup_{Y \in \chi} \mathbb{E}(|Y| \mathbf{1}_{|Y|>\lambda}) \to 0 \text{ as } \lambda \to \infty.$$

Theorem (Vitali theorem)**.** $X_n \to X$ in L^1 iff (X_n) is uniformly integrable and $X_n \to X$ in probability.

Proposition. The following are equivalent:

(i) X is a martingale.

(ii) X is a local martingale, and for all $t \geq 0$, the set

$$\chi_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$

is uniformly integrable.

Corollary. If $Z \in L^1$ is such that $|X_t| \leq Z$ for all t, then X is a martingale. In particular, every bounded local martingale is a martingale.

Proposition. Let X be a *continuous* local martingale with $X_0 = 0$. Define

$$S_n = \inf\{t \geq 0 : |X_t| = n\}.$$

Then S_n is a stopping time, $S_n \to \infty$ and X^{S_n} is a bounded martingale. In particular, (S_n) reduces X.

Theorem. Let X be a continuous local martingale with $X_0 = 0$. If X is also a finite variation process, then $X_t = 0$ for all t.

2.3 Square integrable martingales

Theorem (Doob's inequality)**.** Let $X \in \mathcal{M}^2$. Then

$$\mathbb{E}\left(\sup_{t \geq 0} X_t^2\right) \leq 4\mathbb{E}(X_\infty^2).$$

Theorem. $\mathcal{M}^2$ is a Hilbert space and $\mathcal{M}_c^2$ is a closed subspace.

2.4 Quadratic variation

Theorem. Let M be a continuous local martingale with $M_0 = 0$. Then there exists a unique (up to indistinguishability) continuous adapted increasing process $(\langle M \rangle_t)_{t \geq 0}$ such that $\langle M \rangle_0 = 0$ and $M_t^2 - \langle M \rangle_t$ is a continuous local martingale. Moreover,

$$\langle M \rangle_t = \lim_{n \to \infty} \langle M \rangle_t^{(n)}, \quad \langle M \rangle_t^{(n)} = \sum_{i=1}^{\lceil 2^n t \rceil} (M_{t2^{-n}} - M_{(i-1)2^{-n}})^2,$$

where the limit u.c.p.

Proposition. Let $M \in \mathcal{M}_c^2$. Then $M^2 - \langle M \rangle$ is a uniformly integrable martingale, and

$$\|M - M_0\|_{\mathcal{M}^2} = (\mathbb{E}\langle M \rangle_\infty)^{1/2}.$$

2.5 Covariation

Proposition.

(i) $\langle M, N \rangle$ is the unique (up to indistinguishability) finite variation process such that $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.

(ii) The mapping $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.

(iii)

$$\langle M, N \rangle_t = \lim_{n \to \infty} \langle M, N \rangle_t^{(n)} \text{ u.c.p.}$$

$$\langle M, N \rangle_t^{(n)} = \sum_{i=1}^{\lceil 2^n t \rceil} (M_{i2^{-n}} - M_{(i-1)2^{-n}})(N_{i2^{-n}} - N_{(i-1)}2^{-n}).$$

(iv) For every stopping time T,

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_{t\wedge T}.$$

(v) If $M, N \in \mathcal{M}_c^2$, then $M_t N_t - \langle M, N\rangle_t$ is a uniformly integrable martingale, and

$$\langle M - M_0, N - N_0\rangle_{\mathcal{M}^2} = \mathbb{E}\langle M, N\rangle_\infty.$$

□

Proposition (Kunita–Watanabe)**.** Let M, N be continuous local martingales and let H, K be two (previsible) processes. Then almost surely

$$\int_0^\infty |H_s||K_s||\mathrm{d}\langle M, N\rangle_s| \leq \left(\int_0^\infty H_s^2 \,\mathrm{d}\langle M\rangle_s\right)^{1/2} \left(\int_0^\infty H_s^2 \langle N\rangle_s\right)^{1/2}.$$

2.6 Semi-martingale

3 The stochastic integral

3.1 Simple processes

Proposition. If $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$, then $H \cdot M \in \mathcal{M}_c^2$ and

$$\|H \cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \,\mathrm{d}\langle M\rangle_s\right). \tag{$*$}$$

Proposition. Let $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$. Then

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$$

for all $N \in \mathcal{M}^2$.

3.2 Itô isometry

Proposition. Let $M \in \mathcal{M}_c^2$. Then $\mathcal{E}$ is dense in $L^2(M)$.

Theorem. Let $M \in \mathcal{M}_c^2$. Then

(i) The map $H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}_c^2$ extends uniquely to an isometry $L^2(M) \to \mathcal{M}_c^2$, called the *Itô isometry*.

(ii) For $H \in L^2(M)$, $H \cdot M$ is the unique martingale in $\mathcal{M}_c^2$ such that

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$$

for all $N \in \mathcal{M}_c^2$, where the integral on the LHS is the stochastic integral (as above) and the RHS is the finite variation integral.

(iii) If T is a stopping time, then $(1_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T$.

Corollary.

$$\langle H \cdot M, K \cdot N\rangle = H \cdot (K \cdot \langle M, N\rangle) = (HK) \cdot \langle M, N\rangle.$$

In other words,

$$\left\langle \int_0^{(-)} H_s \,\mathrm{d}M_s, \int_0^{(-)} K_s \,\mathrm{d}N_s \right\rangle_t = \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s. \qquad \square$$

Corollary. Since $H \cdot M$ and $(H \cdot M)(K \cdot N) - \langle H \cdot M, K \cdot N\rangle$ are martingales starting at 0, we have

$$\mathbb{E}\left(\int_0^t H \,\mathrm{d}M_s\right) = 0$$

$$\mathbb{E}\left(\left(\int_0^t H_s \,\mathrm{d}M_s\right)\left(\int_0^t K_s \,\mathrm{d}N_s\right)\right) = \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s. \qquad \square$$

Corollary. Let $H \in L^2(M)$, then $HK \in L^2(M)$ iff $K \in L^2(H \cdot M)$, in which case

$$(KH) \cdot M = K \cdot (H \cdot M).$$

3.3 Extension to local martingales

Theorem. Let M be a continuous local martingale.

(i) For every $H \in L^2_{bc}(M)$, there is a unique continuous local martingale $H \cdot M$ with $(H \cdot M)_0 = 0$ and

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$$

for all N, M.

(ii) If T is a stopping time, then

$$(\mathbf{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

(iii) If $H \in L^2_{loc}(M)$, K is previsible, then $K \in L^2_{loc}(H \cdot M)$ iff $HK \in L^2_{loc}(M)$, and then

$$K \cdot (H \cdot M) = (KH) \cdot M.$$

(iv) Finally, if $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$, then the definition is the same as the previous one.

3.4 Extension to semi-martingales

Proposition.

(i) $(H, X) \mapsto H \cdot X$ is bilinear.

(ii) $H \cdot (K \cdot X) = (HK) \cdot X$ if H and K are locally bounded.

(iii) $(H \cdot X)^T = H 1_{[0,T]} \cdot X = H \cdot X^T$ for every stopping time T.

(iv) If X is a continuous local martingale (resp. a finite variation process), then so is $H \cdot X$.

(v) If $H = \sum_{i=1}^n H_{i-1} \mathbf{1}_{(t_{i-1}, t_i]}$ and $H_{i-1} \in \mathcal{F}_{t_{i-1}}$ (not necessarily bounded), then

$$(H \cdot X)_t = \sum_{i=1}^n H_{i-1}(X_{t_i \wedge t} - X_{t_{i-1} \wedge t}).$$

Proposition (Stochastic dominated convergence theorem)**.** Let X be a continuous semi-martingale. Let H, H_s be previsible and locally bounded, and let K be previsible and non-negative. Let $t > 0$. Suppose

(i) $H^n_s \to H_s$ as $n \to \infty$ for all $s \in [0, t]$.

(ii) $|H^n_s| \leq K_s$ for all $s \in [0, t]$ and $n \in \mathbb{N}$.

(iii) $\int_0^t K_s^2 \, \mathrm{d}\langle M\rangle_s < \infty$ and $\int_0^t K_s \, |\mathrm{d}A_s| < \infty$ (note that both conditions are okay if K is locally bounded).

Then

$$\int_0^t H^n_s \, \mathrm{d}X_s \to \int_0^t H_s \, \mathrm{d}X_s \text{ in probability.}$$

Proposition. Let X be a continuous semi-martingale, and let H be an adapted bounded left-continuous process. Then for every subdivision $0 < t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)}$ of $[0,t]$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$, then

$$\int_0^t H_s \, \mathrm{d}X_s = \lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^{(m)}} (X_{t_i^{(m)}} - X_{t_{i-1}^{(m)}})$$

in probability.

3.5 Itô formula

Theorem (Integration by parts)**.** Let X, Y be a continuous semi-martingale. Then almost surely,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \mathrm{d}Y_s + \int_0^t Y_s \, \mathrm{d}X_s + \langle X, Y\rangle_t$$

The last term is called the *Itô correction*.

Theorem (Itô's formula)**.** Let $X^1, \ldots, X^p$ be continuous semi-martingales, and let $f : \mathbb{R}^p \to \mathbb{R}$ be C^2. Then, writing $X = (X^1, \ldots, X^p)$, we have, almost surely,

$$f(X_t) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x_i}(X_s) \, \mathrm{d}X_s^i + \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) \, \mathrm{d}\langle X^i, X^j\rangle_s.$$

In particular, $f(X)$ is a semi-martingale.

3.6 The Lévy characterization

Theorem (Lévy's characterization of Brownian motion)**.** Let $(X^1, \ldots, X^d)$ be continuous local martingales. Suppose that $X_0 = 0$ and that $\langle X^i, X^j\rangle_t = \delta_{ij} t$ for all $i, j = 1, \ldots, d$ and $t \geq 0$. Then $(X^1, \ldots, X^d)$ is a standard d-dimensional Brownian motion.

Theorem (Dubins–Schwarz)**.** Let M be a continuous local martingale with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$. Let

$$T_s = \inf\{t \geq 0 : \langle M\rangle_t > s\},$$

the right-continuous inverse of $\langle M\rangle_t$. Let $B_s = M_{T_s}$ and $\mathcal{G}_s = \mathcal{F}_{T_s}$. Then T_s is a $(\mathcal{F}_t)$ stopping time, $\langle M\rangle_{T_s} = s$ for all $s \geq 0$, B is a $(\mathcal{G}_s)$-Brownian motion, and

$$M_t = B_{\langle M\rangle_t}.$$

Lemma. M is constant on $[a,b]$ iff $\langle M\rangle$ being constant on $[a,b]$.

Lemma. M is constant on $[a,b]$ iff $\langle M\rangle$ being constant on $[a,b]$.

3.7 Girsanov's theorem

Proposition. Let M be a continuous local martingale with $M_0 = 0$. Then $\mathcal{E}(M) = Z$ satisfies

$$\mathrm{d}Z_t = Z_t\,\mathrm{d}M,$$

i.e.

$$Z_t = 1 + \int_0^t Z_s\,\mathrm{d}M_s.$$

In particular, $\mathcal{E}(M)$ is a continuous local martingale. Moreover, if $\langle M\rangle$ is uniformly bounded, then $\mathcal{E}(M)$ is a uniformly integrable martingale.

Theorem (Girsanov's theorem)**.** Let M be a continuous local martingale with $M_0 = 0$. Suppose that $\mathcal{E}(M)$ is a uniformly integrable martingale. Define a new probability measure

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(M)_\infty$$

Let X be a continuous local martingale with respect to $\mathbb{P}$. Then $X - \langle X, M\rangle$ is a continuous local martingale with respect to $\mathbb{Q}$.

4 Stochastic differential equations

4.1 Existence and uniqueness of solutions

Theorem (Yamada–Watanabe)**.** Assume weak existence and pathwise uniqueness holds. Then

(i) Uniqueness in law holds.

(ii) For every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B and any $x \in \mathbb{R}^d$, there is a unique strong solution to $E_x(a, b)$. □

Theorem. Assume b, σ are Lipschitz in x. Then there is pathwise uniqueness for the $E(\sigma, b)$ and for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and every $(\mathcal{F}_t)$-Brownian motion B, for every $x \in \mathbb{R}^d$, there exists a unique strong solution to $E_x(\sigma, b)$.

Lemma. Let $h(t)$ be a function such that

$$h(t) \leq c \int_0^t h(s)\, \mathrm{d}s$$

for some constant c. Then

$$h(t) \leq h(0) e^{ct}.$$

□

4.2 Examples of stochastic differential equations

Theorem. The eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ satisfies

$$\mathrm{d}\lambda_t^i = \left(-\frac{\lambda^i}{2} + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}\right) \mathrm{d}t + \sqrt{\frac{2}{N\beta}}\, \mathrm{d}B^i.$$

Here $\beta = 1$, but if we replace symmetric matrices by Hermitian ones, we get $\beta = 2$; if we replace symmetric matrices by symplectic ones, we get $\beta = 4$.

4.3 Representations of solutions to PDEs

Proposition. Let $x \in \mathbb{R}^d$, and X a solution to $E_x(\sigma, b)$. Then for every $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ that is C^1 in $\mathbb{R}_+$ and C^2 in $\mathbb{R}^d$, the process

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s, X_s)\, \mathrm{d}s$$

is a continuous local martingale. □

Theorem. Assume U has a smooth boundary (or satisfies the exterior cone condition), a, b are Hölder continuous and a is uniformly elliptic. Then for every Hölder continuous $f : \bar{U} \to \mathbb{R}$ and any continuous $g : \partial U \to \mathbb{R}$, the Dirichlet–Poisson process has a solution. □

Theorem. Let σ and b be bounded measurable and $\sigma\sigma^T$ uniformly elliptic, $U \subseteq \mathbb{R}^d$ as above. Let u be a solution to the Dirichlet–Poisson problem and X a solution to $E_x(\sigma, b)$ for some $x \in \mathbb{R}^d$. Define the stopping time

$$T_U = \inf\{t \geq 0 : X_t \not\in U\}.$$

Then $\mathbb{E}T_U < \infty$ and

$$u(x) = \mathbb{E}_x \left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\;\mathrm{d}s \right).$$

In particular, the solution to the PDE is unique.

Theorem. For every $f \in C_b^2(\mathbb{R}^d)$, there exists a solution to the Cauchy problem. □

Theorem. Let u be a solution to the Cauchy problem. Let X be a solution to $E_x(\sigma, b)$ for $x \in \mathbb{R}^d$ and $0 \leq s \leq t$. Then

$$\mathbb{E}_x(f(X_t) \mid \mathcal{F}_s) = u(t - s, X_s).$$

In particular,

$$u(t, x) = \mathbb{E}_x(f(X_t)).$$

Theorem (Feynman–Kac formula)**.** Let $f \in C_b^2(\mathbb{R}^d)$ and $V \in C_b(\mathbb{R}^d)$ and suppose that $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= Lu + Vu && \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0, \cdot\,) &= f && \text{on } \mathbb{R}^d, \end{aligned}$$

where $Vu = V(x)u(x)$ is given by multiplication.

Then for all $t > 0$ and $x \in \mathbb{R}^d$, and X a solution to $\mathbb{E}_x(\sigma, b)$. Then

$$u(t, x) = \mathbb{E}_x \left(f(X_t) \exp\left(\int_0^t V(X_s)\;\mathrm{d}s \right) \right).$$ □