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Problem 1. Let $f\colon [0,\infty) \to \mathbb{R}$ is absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for all} \quad t \geq 0$$

for an integrable function f'. Let $v(t)$ be the total variation of f on $[0,t]$. Show that

$$v(t) = |f(0)| + \int_0^t |f'(s)|ds.$$

Problem 2. Let $f, g\colon [0,\infty) \to \mathbb{R}$ be bounded and measurable and let $a\colon [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$f \cdot (g \cdot a) = (fg) \cdot a$$

where $\cdot$ denotes the Lebesgue-Stieltjes integral.

Problem 3. Suppose that $(f^n)_{n\geq 1}$ is a sequence of càdlàg functions such that $f^n \to f$ uniformly on $[0,t]$. Show that f is càdlàg on $[0,t]$. Let f be a right-continuous function of bounded variation. Show that v_f is right-continuous.

Problem 4. Let H be a previsible process. Let $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s : s < t)$. Show that H_t is $\mathcal{F}_{t^-}$-measurable, for any $t > 0$.

Problem 5. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, let T be a stopping time, and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}.$$

(i) Show that $\mathcal{F}_T$ is a σ-algebra.

(ii) Show that T is $\mathcal{F}_T$-measurable.

(iii) Suppose that X is a càdlàg, adapted process. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 6. Let $(T_n)_{n\geq 1}$ denote a sequence of stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$.

(i) Show that $\sup_{n\geq 1} T_n$ is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$.

(ii) Show that T is stopping time for the filtration $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ iff $\{T < t\} \in \mathcal{F}_{t^+}$ for all t.

(iii) Show that $\inf_{n\geq 1} T_n$ is a stopping time for $(\mathcal{F}_{t^+})_{t\geq 0}$.

Problem 7. Let B be a standard Brownian motion.

(i) Let $T = \inf\{t \geq 0 : B_t = 1\}$. Show that H defined by $H_t = \mathbf{1}\{T \geq t\}$ is previsible.

(ii) Let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Show that $(\operatorname{sgn}(B_t))_{t\geq 0}$ is a previsible process which is neither left nor right continuous.

Problem 8. Let N be a Poisson process of rate 1, and let $X_t = N_t - t$ for $t \geq 0$. Show that X is of finite variation. Show that both X and $X_t^2 - t$ are martingales.

Problem 9. Let T and ξ denote two independent random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(T \le t) = t \quad \text{for } t \in [0,1], \qquad \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2.$$

Define $X_t = \xi \mathbf{1}_{t \ge T}$ and $\mathcal{F}_t = \sigma(X_s : s \le t)$. Show that X is a martingale with respect to $(\mathcal{F}_t)_{t \in [0,1]}$, and that it is of finite variation. Define pathwise

$$Y_t(\omega) := \int_{(0,t]} X_s(\omega) dX_s(\omega) \qquad \text{for all } \omega \in \Omega,$$

where the right-hand side is a Lebesgue-Stieltjes integral. Show that $Y = (Y_t)_{t \in [0,1]}$ is not a martingale. Is X previsible?

Problem 10. Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family

$$\mathcal{X} = \{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \quad \text{is a } \sigma\text{-algebra}\} \qquad \text{is UI.}$$

Problem 11. Let X be a continuous local martingale. Show that if

$$\mathbb{E}\Big(\sup_{0 \le s \le t} |X_s| \Big) < \infty \qquad \forall t \ge 0$$

then X is a martingale.

Problem 12. Let B be a standard Brownian motion and fix $t \ge 0$. For $n \ge 1$, let $\Delta_n = \{0 : t_0(n) < t_1(n) < \cdots < t_{m_n}(n) = t\}$ be a partition of $[0,t]$ such that

$$\max_{1 \le i \le m_n} \big(t_i(n) - t_{i-1}(n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Show that

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 = t$$

in L^2. Show that if the subdivision is dyadic, then the convergence is also almost sure.

Bonus: Show that the convergence is almost sure for any nested subdivision. Hint: see Question 6, `https://www.maths.cam.ac.uk/sites/www.maths.cam.ac.uk/files/pre2014/postgrad/mathiii/pastpapers/2011/PaperIII_28.pdf`.

Problem 13. (A silly martingale) Construct a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$, a L^∞-bounded martingale $(M_t)_{t=0}^{t=1}$ and a stopping time T taking values in $[0,1]$, such that

$$\mathbb{E}(M_T) \ne \mathbb{E}(M_0).$$

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Problem 1. Suppose that M is a continuous local martingale with $M_0 = 0$. Show that M is an L^2-bounded martingale if and only if $\mathbb{E}(\langle M\rangle_\infty) < \infty$.

Problem 2.

(i) Suppose that M, N are independent continuous local martingales. Show that $\langle M, N\rangle = 0$. In particular, if $B^{(1)}$ and $B^{(2)}$ are the coordinates of a standard Brownian motion in $\mathbb{R}^2$, this shows that $\langle B^{(1)}, B^{(2)}\rangle_t = 0$ for all $t \geq 0$.

(ii) Let B be a standard Brownian motion in $\mathbb{R}$ and let T be a stopping time which is a.s. not constant. By considering B^T and $B - B^T$, show that the converse to the previous part is false. Hint: show that T is measurable with respect to the σ-algebras generated by both B^T and $B - B^T$.

Problem 3. (Burkholder inequality) Fix $p \geq 2$ and let M be a continuous local martingale with $M_0 = 0$. Use Itô's formula, Doob's inequality, and Hölder's inequality to show that there exists a constant $C_p > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |M_s|^p\right) \leq C_p \mathbb{E}\big(\langle M\rangle_t^{p/2}\big).$$

Problem 4. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function. Show that if f has finite variation then it has zero quadratic variation. Conversely, show that if f has finite and positive quadratic variation then it must be of infinite variation.

Problem 5. Let B be a standard Brownian motion. Use Itô's formula to show that the following are martingales with respect to the filtration generated by B.

(i) $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$

(ii) $X_t = (B_t + t)\exp(-B_t - t/2)$

(iii) $X_t = \exp(B_t - t/2)$

Problem 6. Let $h\colon [0,\infty) \to \mathbb{R}$ be a measurable function which is square-integrable when restricted to $[0,t]$ for each $t > 0$ and let B be a standard Brownian motion. Show that the process $H_t = \int_0^t h(s)dB_s$ is Gaussian and compute its covariance. (A real-valued process (X_t) is Gaussian if for any finite family $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian).

Problem 7. Show that convergence in $(\mathcal{M}_c^2, \|\cdot\|)$ implies ucp convergence.

Problem 8. Show that the covariation $\langle\cdot,\cdot\rangle$ is symmetric and bilinear. That is, if M_1, M_2, M_3 are continuous local martingales and $a \in \mathbb{R}$, then

$$\langle aM_1 + M_2, M_3\rangle = a\langle M_1, M_3\rangle + \langle M_2, M_3\rangle.$$

Problem 9. Let B be a standard Brownian motion and let

$$\widehat{B}_t = B_t - \int_0^t \frac{B_s}{s}ds.$$

(i) Show that $\widehat{B}$ is not a martingale in the filtration generated by B.

(ii) Show that $\widehat{B}$ is a martingale in its own filtration by showing that it is a Brownian motion. [Hint: show that $\widehat{B}$ is a continuous Gaussian process and identify its mean and covariance.]

Problem 10. Fix $d \geq 3$ and let B be a Brownian motion in $\mathbb{R}^d$ starting at $B_0 = \overline{x} = (x, 0, \ldots, 0) \in \mathbb{R}^d$ for some $x > 0$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. For each $a > 0$, let $\tau_a = \inf\{t > 0 : |B_t| = a\}$.

(i) Let $D = \mathbb{R}^d \setminus \{0\}$ and let $h\colon D \to \mathbb{R}$ be defined by $h(x) = |x|^{2-d}$. Show that h is harmonic on D and that $M_t = |B_t^{\tau_a}|^{2-d}$ is a local martingale for all $a \geq 0$. For which values of x is M a true martingale?

(ii) Use the previous part to show that for any $a < b$ such that $0 < a < x < b$,

$$\mathbb{P}_{\overline{x}}[\tau_a < \tau_b] = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

where $\phi(u) = u^{2-d}$. Conclude that if $x > a > 0$, then

$$\mathbb{P}_x[\tau_a < \infty] = (a/x)^{d-2}.$$

Problem 11.

(i) Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale in $\mathbb{R}^2$. Show further that M is a time-change of Brownian motion in $\mathbb{R}^2$.

(ii) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix $z \in \mathbb{D}$. What is the hitting distribution for Z on ∂D in the case that $Z_0 = 0$? By applying a Möbius transformation $\mathbb{D} \to \mathbb{D}$ and using the previous part, determine the hitting distribution for Z on $\partial\mathbb{D}$.

Problem 12. **($\star$)** Let $U \subset \mathbb{R}^d$ be an open set. We say that a function $u \in L^\infty_{\text{loc}}(U)$ satisfies the *mean value property* if, whenever $S(x, r) \subset U$, we have

$$u(x) = \int_{S(x,r)} u(y)\mu_{x,r}(dy) \tag{1}$$

where we write $\mu_{x,r}$ for the uniform distribution on the sphere $S(x, r) = \partial B(x, r)$.

(i) Suppose $u \in C^2(U)$ is harmonic. Show that u satisfies (1).

(ii) Suppose, conversely, that u satisfies (1). For any compact $K \subset U$, express $u|_K$ as a convolution, and deduce that $u \in C^\infty(U)$.

(iii) Suppose u satisfies (1). Fix $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$. Let B be a d-dimensional Brownian Motion started at x, and let $\tau_r = \inf\{t > 0 : |x - B_t| = r\}$. Show that

$$\forall t \geq 0, \qquad \mathbb{E}\left(\int_0^{t\wedge\tau_r} \triangle u(B_s)ds\right) = 0.$$

Deduce that u is harmonic. Hence (1) is an equivalent characterisation of harmonic functions.

Problem 13. **($\star$)** (Liouville's Theorem.) Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is bounded and harmonic. Let B be a Brownian motion starting at 0.

(i) Show that $M_t = u(B_t)$ is a bounded martingale. Conclude that M_t converges, almost surely and in L^1, to a random variable M_∞.

(ii) Recall Blumenthal's $0 - 1$ law. Deduce that the *tail σ-algebra*

$$\tau = \cap_{t\geq 0}\, \sigma(B_s : s \geq t)$$

contains only events of probability 0 and 1. Deduce that M_∞ is almost surely constant.

(iii) Using the relationship between M_∞ and M_1, deduce that M_1 is almost surely constant. Conclude that u is constant.

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Problem 1. Suppose that $(Z_t)_{t\ge 0}$ is a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale M such that $Z = \mathcal{E}(M)$, where

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t).$$

Problem 2. Let M be a continuous local martingale with $M_0 = 0$. For any $a, b > 0$, show that

$$\mathbb{P}\left(\sup_{t\ge 0} M_t \ge a, \langle M\rangle_\infty \le b\right) \le \exp\left(-\frac{a^2}{2b}\right).$$

Problem 3. Let B be a standard Brownian motion and, for $a, b > 0$, let $\tau_{a,b} = \inf\{t \ge 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2}\exp(-(a-bt)^2/2t).$$

Problem 4. Suppose that M is a continuous local martingale with $\langle M\rangle_t \to \infty$ almost surely as $t \to \infty$. Show that $M_t/\langle M\rangle_t \to 0$ as $t \to \infty$ and conclude that $\mathcal{E}(M)_t \to 0$ almost surely.

Problem 5. [Gronwall's lemma] Let $T > 0$ and let f be a non-negative, bounded, measurable function on $[0, T]$. Suppose that there exist $a, b \ge 0$ such that

$$f(t) \le a + b\int_0^t f(s)ds \quad \text{for all} \quad t \in [0, T].$$

Show that $f(t) \le ae^{bt}$ for all $t \in [0, T]$.

Problem 6. Suppose that X is a continuous local martingale with quadratic variation

$$\langle X\rangle_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\ge 0}$. Show that there exists a Brownian motion B (possibly defined on a larger probability space) such that

$$X_t = \int_0^t A_s^{1/2} dB_s.$$

Problem 7. Suppose that σ and b are Lipschitz. Explain why uniqueness in law holds for the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$.

Problem 8. Suppose that $\mathbb{Q} \ll \mathbb{P}$. Show that if $X_n \to X$ in probability with respect to $\mathbb{P}$, then $X_n \to X$ in probability with respect to $\mathbb{Q}$.

Problem 9. Suppose that σ, b and σ_n, b_n for $n \in \mathbb{N}$ are Lipschitz with constant K uniformly in n. Suppose that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Suppose that X and X^n are defined by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x \tag{1}$$

$$dX_t^n = \sigma_n(X_t^n)dB_t + b_n(X_t^n)dt, \quad X_0^n = x. \tag{2}$$

Show for each $t > 0$ that

$$\mathbb{E}\left(\sup_{0\le s\le t} |X_s^n - X_s|^2\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Bonus: Suppose that b_n, σ_n are continuous, and b, σ are Lipschitz. Suppose that X^n still satisfy (1). What happens now?

Problem 10. Let b be bounded and σ be bounded and continuous.

(i) Suppose that X is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for all $f \in C^2$.

(ii) Let X be a continuous, adapted process such that

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for each $f \in C^2$. Suppose that $\sigma(x) > 0$ for all x. Show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. (Hint: use Problem 6.)

Problem 11. Let W be a standard Brownian motion.

(i) Let $B_t = W_t - tW_1$. Show that $(B_t)_{t\in[0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $\mathbb{E}[B_s B_t]$?

(ii) Is B adapted to the filtration generated by W?

(iii) Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s} \quad \text{for} \quad 0 \le t < 1.$$

Show that $X_t \to 0$ as $t \uparrow 1$.

(iv) Show that X is a continuous, mean-zero Gaussian process with the same covariance as B, i.e., X is a Brownian bridge.

Problem 12 ($\star$). Using the results of this course, give a *short* proof of the reflection principle: if T is a stopping time and B is a standard Brownian motion, then

$$W_t = \begin{cases} B_t & t \le T; \\ 2B_T - B_t & t > T. \end{cases}$$

is also a standard Brownian Motion.

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Problem 1. Let X be a continuous semimartingale under $\mathbb{P}$, and let $\tilde{\mathbb{P}}$ be another probability measure on the same space such that $\tilde{\mathbb{P}} \ll \mathbb{P}$. Suppose that X is also a semimartingale under $\tilde{\mathbb{P}}$. Show that X has the same quadratic variation process under $\mathbb{P}$ and under $\tilde{\mathbb{P}}$.

Problem 2. Let b be bounded and measurable. Use Girsanov's theorem to construct a weak solution to the SDE

$$dX_t = b(X_t)dt + dB_t$$

over the finite (non-random) time interval $[0, T]$.

Problem 3. Show that the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0$$

has strong existence but not pathwise uniqueness.

Problem 4. Find the unique strong solution to the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dB_t, \quad X_0 = x.$$

(Hint: consider the change of variables $Y_t = \sinh^{-1}(X_t)$.)

Problem 5. Construct a filtered probability space on which a Brownian motion B and an adapted process X are defined and such that

$$dX_t = \frac{X_t}{t}dt + dB_t, \quad X_0 = 0.$$

Is X adapted to the filtration generated by B? Is B a Brownian motion in the filtration generated by X?

Problem 6. Let X be a solution of the SDE

$$dX_t = X_t g(X_t)dB_t$$

where g is bounded and $X_0 = x > 0$ is non-random.

(i) Show that $\mathbb{P}(X_t > 0 \text{ for all } t \geq 0) = 1$. Hint: apply Ito's formula to

$$X_t \exp\left(-\int_0^t g(X_s)dB_s + \frac{1}{2}\int_0^t g^2(X_s)ds\right).$$

(ii) Show that $\mathbb{E}(X_t) = X_0$ for all $t \geq 0$.

(iii) Fix a non-random time horizon $T > 0$. Show that there exists a measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ which is mutually absolutely continuous with respect to $\mathbb{P}$ and a $\widehat{\mathbb{P}}$-Brownian motion $\widehat{B}$ such that

$$dY_t = Y_t g(1/Y_t)d\widehat{B}_t$$

where $Y_t = 1/X_t$.

Problem 7. Consider the Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}\Delta V - \frac{1}{2}|\nabla V|^2 + k \quad \text{on} \quad (0, \infty) \times \mathbb{R}^d,$$

with $V(0,x) = 0$ for $x \in \mathbb{R}^d$ where $k\colon \mathbb{R}^d \to [0,\infty)$ is a continuous function. Show that the only solution $V\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ which is continuous on its domain, of class $C^{1,2}$ on $(0,\infty) \times \mathbb{R}^d$, and satisfies the quadratic growth condition for every $T > 0$:

$$-V(t,x) \le C + a|x|^2, \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

where $T > 0$ is arbitrary and $0 < a < 1/(2Td)$ is given by

$$V(t,x) = -\log \mathbb{E}_x \left[\exp\left(-\int_0^t k(W_s)ds \right) \right]$$

for $t \ge 0$ and $x \in \mathbb{R}^d$.

Problem 8. Let $b\colon \mathbb{R}^d \to \mathbb{R}$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be bounded and continuous. For each n, j, set $t_j^n = n2^{-j}$ and $\psi_n(t) = t_j^n$ if $t \in [t_j^n, t_{j+1}^n)$. Assume that (X_0^n) is a tight sequence, and that X^n solves

$$X_t^n = X_0^n + \int_0^t b(X^n_{\psi_n(u)})du + \int_0^t \sigma(X^n_{\psi_n(u)})dB_u.$$

Show that for each $m, T > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}[|X_t^n - X_s^n|^{2m}] \le C(t-s)^m \quad \text{for all} \quad 0 \le s < t \le T. \tag{*}$$

Explain what it means for the sequence (X^n) to be tight in the space $C([0,T],\mathbb{R}^d)$ and explain why (*) implies that (X^n) is tight. (Hint: look at the proof of Kolmogorov's continuity criterion.)

Problem 9. Consider the SDE

$$dX_t = X_t^2 dB_t.$$

(i) By considering the process $\widetilde{X}_t = 1/|B_t - \xi|$ where B is a three-dimensional Brownian motion and ξ is a standard Gaussian in $\mathbb{R}^3$ independent of B, show that the SDE has a weak solution.

(ii) Let $\Phi(s) = \int_{-\infty}^s e^{-t^2/2}\, dt/\sqrt{2\pi}$ be the Gaussian distribution function. Verify that both

$$u^1(t,x) = x\left(2\Phi(1/(x\sqrt{t})) - 1\right) \quad \text{and} \quad u^2(x,t) = x$$

solve the PDE

$$\frac{\partial u}{\partial t} = \frac{x^4}{2} \cdot \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = x.$$

(iii) Which of these solutions corresponds to $u(t,x) = \mathbb{E}_x(X_t)$?