Part III — Schramm–Loewner Evolutions
Theorems with proof

Based on lectures by J. Miller

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Schramm–Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Contents

0 Introduction

1 Conformal transformations

1.1 Conformal transformations

Theorem (Riemann mapping theorem)**.** Let U be a simply connected domain with $U \neq \mathbb{C}$ and $z \in U$ be any point. Then there exists a unique conformal transformation $f : \mathbb{D} \to U$ such that $f(0) = z$, and $f'(0)$ is real and positive.

1.2 Brownian motion and harmonic functions

Theorem. Let u be a harmonic function on a bounded domain D which is continuous on $\bar{D}$. For $z \in D$, let $\mathbb{P}_z$ be the law of a complex Brownian motion starting from z, and let τ be the first hitting time of D. Then

$$u(z) = \mathbb{E}_z[u(B_\tau)]. \qquad \square$$

Corollary (Mean value property)**.** If u is a harmonic function, then, whenever it makes sense, we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, \mathrm{d}\theta.$$

Corollary (Maximum principle)**.** Let u be harmonic in a domain D. If u attains its maximum at an interior point in D, then u is constant.

Corollary (Maximum modulus principle)**.** Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum in the interior of D, then f is constant.

Proof. Observe that if f is holomorphic, then $\log |f|$ is harmonic and attains its maximum in the interior of D. It then follows that $|f|$ is constant (if $|f|$ vanishes somewhere, then consider $\log |f + M|$ for some large M, and do some patching if necessary). It is then a standard result that a holomorphic function of constant modulus is constant. □

Lemma (Schwarz lemma)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|f(z)| = |z|$ for some non-zero $z \in \mathbb{D}$, then $f(w) = \lambda w$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Proof. Consider the map

$$g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}.$$

Then one sees that g is holomorphic and $|g(z)| \leq 1$ for all $z \in \partial\mathbb{D}$, hence for all $z \in \mathbb{D}$ by the maximum modulus principle. If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$, then g must be constant, so f is linear. □

1.3 Distortion estimates for conformal maps

Theorem (Koebe-1/4 theorem)**.** If $f \in \mathcal{U}$ and $0 < r \leq 1$, then $B(0, r/4) \subseteq f(r\mathbb{D})$.

Theorem. If $f \in \mathcal{U}$, then $|a_2| \leq 2$.

Proof of Koebe-1/4 theorem. Suppose $f : \mathbb{D} \to D$ is in $\mathcal{U}$, and $z_0 \notin D$. We shall show that $|z_0| \geq \frac{1}{4}$. Consider the function

$$\tilde{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)}.$$

Since $\tilde{f}$ is composition of conformal transformations, it is itself conformal, and a direct computation shows $\tilde{f} \in \mathcal{U}$. Moreover, if

$$f(z) = z + a_2 z^2 + \cdots,$$

then

$$\tilde{f}(z) = z + \left(a_2 + \frac{1}{z_0}\right) z^2 + \cdots.$$

So we obtain the bounds

$$|a_2|, \left|a_2 + \frac{1}{z_0}\right| \leq 2.$$

By the triangle inequality, we must have $|z_0^{-1}| \leq 4$, hence $|z_0| \geq \frac{1}{4}$. □

Corollary. Let $D, \tilde{D}$ be domains and $z \in D$, $\tilde{z} \in \tilde{D}$. If $f : D \to \tilde{D}$ is a conformal transformation with $f(z) = \tilde{z}$, then

$$\frac{\tilde{d}}{4d} \leq |f'(z)| \leq \frac{4\tilde{d}}{d},$$

where $d = \mathrm{dist}(z, \partial D)$ and $\tilde{d} = \mathrm{dist}(\tilde{z}, \partial\tilde{D})$.

Proof. By translation, scaling and rotation, we may assume that $z = \tilde{z} = 0$, $d = 1$ and $f'(0) = 1$. Then we have

$$\tilde{D} = f(D) \supseteq f(\mathbb{D}) \supseteq B(0, 1/4).$$

So $\tilde{d} \geq \frac{1}{4}$, as desired. The other bound follows by considering f^{-1}. □

Proposition. Let $f \in \mathcal{U}$. Then

$$\mathrm{area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

Proof. In the ideal world, we will have something that helps us directly compute $\mathrm{area}(f(\mathbb{D}))$. However, for the derivation to work, we need to talk about what f does to the boundary of $\mathbb{D}$, but that is not necessarily well-defined. So we compute $\mathrm{area}(f(r\mathbb{D}))$ for $r < 1$ and then take the limit $r \to 1$.

So fix $r \in (0, 1)$, and define the curve $\gamma(\theta) = f(re^{i\theta})$ for $\theta \in [0, 2\pi]$. Then we can compute

$$\begin{aligned}
\frac{1}{2i}\int_\gamma \bar{z}\;\mathrm{d}z &= \frac{1}{2i}\int_\gamma (x - iy)(\mathrm{d}x + i\;\mathrm{d}y)\\
&= \frac{1}{2i}\int_\gamma (x - iy)\;\mathrm{d}x + (ix + y)\;\mathrm{d}y\\
&= \frac{1}{2i}\iint_{f(r\mathbb{D})} 2i\;\mathrm{d}x\;\mathrm{d}y\\
&= \mathrm{area}(f(r\mathbb{D})),
\end{aligned}$$

using Green's theorem. We can also compute the left-hand integral directly as

$$\begin{aligned}\frac{1}{2i}\int_\gamma \bar{z}\,\mathrm{d}z &= \frac{1}{2i}\int_0^{2\pi} \overline{f(re^{i\theta})}f'(re^{i\theta})ire^{i\theta}\,\mathrm{d}\theta\\ &= \frac{1}{2i}\int_0^{2\pi}\left(\sum_{n=1}^\infty \bar{a}_n r^n e^{-i\theta n}\right)\left(\sum_{n=1}^\infty a_n n r^{n-1}e^{i\theta(n-1)}\right) ire^{i\theta}\,\mathrm{d}\theta\\ &= \pi\sum_{n=1}^\infty r^{2n}|a_n|^2 n.\end{aligned}$$

□

Proposition. If $K \in \mathcal{H}$, then

$$\mathrm{area}(K) = \pi\left(1 - \sum_{n=1}^\infty n|b_n|^2\right).$$

Proof. The proof is essentially the same as last time. Let $r > 1$, and let $K_r = F(r\bar{\mathbb{D}})$ (or, if you wish, $\mathbb{C} \setminus F(\mathbb{C} \setminus r\bar{\mathbb{D}})$), and $\gamma(\theta) = F(re^{i\theta})$. As in the previous proposition, we have

$$\begin{aligned}\mathrm{area}(K_r) &= \frac{1}{2i}\int_\gamma \bar{z}\,\mathrm{d}z\\ &= \frac{1}{2i}\int_0^{2\pi}\overline{F(re^{i\theta})}F'(re^{i\theta})ire^{i\theta}\,\mathrm{d}\theta\\ &= \pi\left(r^2 - \sum_{n=1}^\infty n|b_n|^2 r^{-2n}\right).\end{aligned}$$

Then take the limit as $r \to 1$. □

Lemma. Let $f \in \mathcal{U}$. Then there exists an odd function $h \in \mathcal{U}$ with $h(z)^2 = f(z^2)$.

Proof. Note that $f(0) = 0$ by assumption, so taking the square root can potentially be problematic, since 0 is a branch point. To get rid of the problem, define the function

$$\tilde{f}(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0\\ f'(0) & z = 0\end{cases}.$$

Then $\tilde{f}$ is non-zero and conformal in $\mathbb{D}$, and so there is a function g with $g(z)^2 = \tilde{f}(z)$. We then set

$$h(z) = zg(z^2).$$

Then h is odd and $h^2 = z^2 g(z^2)^2 = f(z^2)$. It is also immediate that $h(0) = 0$ and $h'(0) = 1$. We need to show h is injective on $\mathbb{D}$. If $z_1, z_2 \in \mathbb{D}$ with $h(z_1) = h(z_2)$, then

$$z_1 g(z_1^2) = z_2 g(z_2^2). \tag{$*$}$$

By squaring, we know

$$z_1^2\tilde{f}(z_1^2) = z_2^2\tilde{f}(z_2)^2.$$

Thus, $f(z_1^2) = f(z_2^2)$ and so $z_1^2 = z_2^2$. But then $(*)$ implies $z_1 = z_2$. So h is injective, and hence $h \in \mathcal{U}$. □

Proof of theorem. We can Taylor expand

$$h(z) = z + c_3 z^3 + c_5 z^5 + \cdots .$$

Then comparing $h(z)^2 = f(z^2)$ implies

$$c_3 = \frac{a_2}{2}.$$

Setting $g(z) = \frac{1}{h(1/z)}$, we find that the z^{-1} coefficient of g is $-\frac{a_2}{2}$, and as we previously noted, this must be ≤ 1. □

1.4 Half-plane capacity

Proposition. For each $A \in \mathcal{Q}$, there exists a unique conformal transformation $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ with $|g_A(z) - z| \to 0$ as $z \to \infty$.

Theorem (Schwarz reflection principle)**.** Let $D \subseteq \mathbb{H}$ be a simply connected domain, and let $\phi : D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets and sends $\mathbb{R} \cap D$ to $\mathbb{R}$. Then ϕ extends by reflection to a conformal transformation on

$$D^* = D \cup \{\bar{z} : z \in D\} = D \cup \bar{D}$$

by setting $\phi(\bar{z}) = \overline{\phi(z)}$. □

Proof of proposition. The Riemann mapping theorem implies that there exists a conformal transformation $g : \mathbb{H} \setminus A \to \mathbb{H}$. Then $g(z) \to \infty$ as $z \to \infty$. By the Schwarz reflection principle, extend g to a conformal transformation defined on $\mathbb{C} \setminus (A \cup \bar{A})$.

By Laurent expanding g at ∞, we can write

$$g(z) = \sum_{n=-\infty}^{N} b_{-N} z^N.$$

Since g maps the real line to the real line, all b_i must be real. Moreover, by injectivity, considering large z shows that $N = 1$. In other words, we can write

$$g(z) = b_{-1} z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n},$$

with $b_{-1} \neq 0$. We can then define

$$g_A(z) = \frac{g(z) - b_0}{b_{-1}}.$$

Since b_0 and b_{-1} are both real, this is still a conformal transformation, and $|g_A(z) - z| \to 0$ as $z \to \infty$.

To show uniqueness, suppose g_A, g'_A are two such functions. Then $g'_A \circ g_A^{-1} : \mathbb{H} \to \mathbb{H}$ is such a function for $A = \emptyset$. Thus, it suffices to show that if $g : \mathbb{H} \to \mathbb{H}$ is a conformal mapping such that $g(z) - z \to 0$ as $z \to \infty$, then in fact $g = z$. But we can expand $g(z) - z$ as

$$g(z) - z = \sum_{n=1}^{\infty} \frac{c_n}{z^n},$$

and this has to be holomorphic at 0. So $c_n = 0$ for all n, and we are done. □

Proposition.

(i) Scaling: If $r > 0$ and $A \in \mathcal{Q}$, then $\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A)$.

(ii) Translation invariance: If $x \in \mathbb{R}$ and $a \in \mathcal{Q}$, then $\operatorname{hcap}(A + x) = \operatorname{hcap}(A)$.

(iii) Monotonicity: If $A, \tilde{A} \in \mathcal{Q}$ are such that $A \subseteq \tilde{A}$. Then $\operatorname{hcap}(A) \leq \operatorname{hcap}(\tilde{A})$.

Proof.

(i) We have $g_{rA}(z) = r g_A(z/r)$.

(ii) Observe $g_{A+x}(z) = g_A(z - x) + x$.

(iii) We can write

$$g_{\tilde{A}} = g_{g_A(\tilde{A} \setminus A)} \circ g_A.$$

Thus, expanding out tells us

$$\operatorname{hcap}(\tilde{A}) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(\tilde{A} \setminus A)).$$

So the desired result follows if we know that the half-plane capacity is non-negative, which we will prove next. □

Proposition. Let $A \in \mathcal{Q}$ and B_t be complex Brownian motion. Define the stopping time

$$\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}.$$

Then

(i) For all $z \in \mathbb{H} \setminus A$, we have

$$\operatorname{im}(z - g_A(z)) = \mathbb{E}_z[\operatorname{im}(B_\tau)]$$

(ii)

$$\operatorname{hcap}(A) = \lim_{y \to \infty} y \, \mathbb{E}_y[\operatorname{im}(B_\tau)].$$

In particular, $\operatorname{hcap}(A) \geq 0$.

(iii) If $A \subseteq \bar{\mathbb{D}} \cap \mathbb{H}$, then

$$\operatorname{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}[\operatorname{im}(B_\tau)] \sin\theta \, \mathrm{d}\theta.$$

Proof.

(i) Let $\phi(z) = \operatorname{im}(z - g_A(z))$. Since $z - g_A(z)$ is holomorphic, we know ϕ is harmonic. Moreover, φ is continuous and bounded, as it $\to 0$ at infinity. These are exactly the conditions needed to solve the Dirichlet problem using Brownian motion.

Since $\operatorname{im}(g_A(z)) = 0$ when $z \in \partial(\mathbb{H} \setminus A)$, we know $\operatorname{im}(B_\tau) = \operatorname{im}(B_t - g_A(B_\tau))$. So the result follows.

(ii) We have

$$\begin{aligned}\mathrm{hcap}(A) &= \lim_{z\to\infty} z(g_A(z) - z)\\ &= \lim_{y\to\infty} (iy)(g_A(iy) - iy)\\ &= \lim_{y\to\infty} y\,\mathrm{im}(iy - g_A(iy))\\ &= \lim_{y\to\infty} y\,\mathbb{E}_{iy}[\mathrm{im}(B_\tau)]\end{aligned}$$

where we use the fact that $\mathrm{hcap}(A)$ is real, so we can take the limit of the real part instead.

(iii) See example sheet. □

Theorem. Let $D, \tilde{D} \subseteq \mathbb{C}$ be domains, and $f : D \to \tilde{D}$ a conformal transformation. Let $B, \tilde{B}$ be Brownian motions starting from $z \in D, \tilde{z} \in \tilde{D}$ respectively, with $f(z) = \tilde{z}$. Let

$$\begin{aligned}\tau &= \inf\{t \geq 0 : B_t \notin D\}\\ \tilde{\tau} &= \inf\{t \geq 0 : \tilde{B}_t \notin \tilde{D}\}\end{aligned}$$

Set

$$\begin{aligned}\tau' &= \int_0^\tau |f'(B_s)|^2\,\mathrm{d}s\\ \sigma(t) &= \inf\left\{s \geq 0 : \int_0^s |f'(B_r)|^2\,\mathrm{d}r = t\right\}\\ B'_t &= f(B_{\sigma(t)}).\end{aligned}$$

Then $(B'_t : t < \tilde{\tau}')$ has the same distribution as $(\tilde{B}_t : t < \tilde{\tau})$.

Proof. See Stochastic Calculus. □

2 Loewner's theorem

2.1 Key estimates

Proposition. Let $A \in \mathcal{Q}$ and B be a complex Brownian motion. Set

$$\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}.$$

Then

- If $x > \mathrm{Rad}(A)$, then

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}[B_\tau \in [x, \infty)] \right).$$

- If $x < -\mathrm{Rad}(A)$, then

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\mathbb{P}_{iy}[B_\tau \in (-\infty, x]] - \frac{1}{2} \right).$$

Proof. First consider the case $A = \emptyset$ and, by symmetry, $x > 0$. Then

$$\begin{aligned} &\lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}[B_\tau \in [x, \infty)] \right) \\ &= \lim_{y \to \infty} \pi y \, \mathbb{P}_{iy}[B_\tau \in [0, x)] \\ &= \lim_{y \to \infty} \pi y \int_0^x \frac{y}{\pi(s^2 + y^2)} \, \mathrm{d}s \\ &= x, \end{aligned}$$

where the first equality follows from the fact that Brownian motion exits through the positive reals with probability $\frac{1}{2}$; the second equality follows from the previously computed exit distribution; and the last follows from dominated convergence.

Now suppose $A \neq \emptyset$. We will use conformal invariance to reduce this to the case above. We write $g_A = u_A + iv_A$. We let

$$\sigma = \inf\{t > 0 : B_t \notin \mathbb{H}\}.$$

Then we know

$$\begin{aligned} \mathbb{P}_{iy}[B_\tau \in [x, \infty)] &= \mathbb{P}_{g_A(iy)}[B_\sigma \in [g_A(x), \infty)] \\ &= \mathbb{P}_{iv_A(iy)}[B_\sigma \in [g_A(x) - u_A(iy), \infty)]. \end{aligned}$$

Since $g_A(z) - z \to 0$ as $z \to \infty$, it follows that $\frac{v_A(iy)}{y} \to 1$ and $u_A(iy) \to 0$ as $y \to \infty$. So we have

$$\left| \mathbb{P}_{iv_A(iy)}\big[B_\sigma \in [g_A(x) - u_A(iy), \infty)\big] - \mathbb{P}_{iy}\big[B_\sigma \in [g_A(x), \infty)\big] \right| = o(y^{-1})$$

as $y \to \infty$. Combining with the case $A = \emptyset$, the the proposition follows. □

Corollary. If $A \in \mathcal{Q}$, $\mathrm{Rad}(A) \leq 1$, then

$$\begin{aligned} x \leq g_A(x) &\leq x + \frac{1}{x} && \text{if } x > 1 \\ x + \frac{1}{x} \leq g_A(x) &\leq x && \text{if } x < -1. \end{aligned}$$

Moreover, for all $A \in \mathcal{Q}$, we have

$$|g_A(z) - z| \leq 3\,\mathrm{Rad}(A).$$

Proof. Exercise on the first example sheet. Note that $z \mapsto z + \frac{1}{z}$ sends $\mathbb{H} \setminus \bar{\mathbb{D}}$ to $\mathbb{H}$. □

Proposition. There is a constant $c > 0$ so that for every $A \in \mathcal{Q}$ and $|z| > 2\,\mathrm{Rad}(A)$, we have

$$\left| g_a(z) - \left(z + \frac{\mathrm{hcap}(A)}{z}\right)\right| \leq c \frac{\mathrm{Rad}(A) \cdot \mathrm{hcap}(A)}{|z|^2}.$$

Proof. Performing a scaling if necessary, we can assume that $\mathrm{Rad}(A) = 1$. Let

$$h(z) = z + \frac{\mathrm{hcap}(A)}{z} - g_A(z).$$

We aim to control the imaginary part of this, and then use the Cauchy–Riemann equations to control the real part as well. We let

$$v(z) = \mathrm{im}(h(z)) = \mathrm{im}(z - g_A(z)) = \frac{\mathrm{im}(z)}{|z|^2} \mathrm{hcap}(A).$$

Let B be a complex Brownian motion, and let

$$\begin{aligned} \sigma &= \inf\{t \geq 0 : B_t \not\in \mathbb{H} \setminus \bar{\mathbb{D}}\} \\ \tau &= \inf\{t \geq 0 : B_t \not\in \mathbb{H}\}. \end{aligned}$$

Let $p(z, e^{i\theta})$ be the density with respect to the Lebesgue measure at $e^{i\theta}$ for B_σ. Then by the strong Markov property at the time σ, we have

$$\mathrm{im}(z - g_A(z)) = \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{im}(B_\tau)] p(z, e^{i\theta})\;\mathrm{d}\theta.$$

In the first example sheet Q3, we show that

$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\mathrm{im}(z)}{|z|^2} \sin\theta \left(1 + O\left(\frac{1}{|z|}\right)\right). \qquad (*)$$

We also have

$$\mathrm{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{im}(B_\tau)] \sin\theta\;\mathrm{d}\theta.$$

So

$$\begin{aligned} |v(z)| &= \left|\mathrm{im}(z - g_A(z)) - \frac{\mathrm{im}(z)}{|z|^2}\,\mathrm{hcap}(A)\right| \\ &= \left|\int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{im}(B_\tau)] p(z, e^{i\theta})\;\mathrm{d}\theta - \frac{\mathrm{im}(z)}{|z|^2} \cdot \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{im}(B_\tau)] \sin\theta\;\mathrm{d}\theta\right|. \end{aligned}$$

By applying $(*)$, we get

$$|v(z)| \leq c\frac{c\,\mathrm{hcap}(A)\,\mathrm{im}(z)}{|z|^3}.$$

where c is a constant.

Recall that v is harmonic as it is the imaginary part of a holomorphic function. By example sheet 1 Q9, we have

$$|\partial_x v(z)| \leq \frac{c\,\mathrm{hcap}(A)}{|z|^3}, \quad |\partial_y v(z)| \leq \frac{c\,\mathrm{hcap}(A)}{|z|^3}.$$

By the Cauchy–Riemann equations, $\mathrm{re}(h(z))$ satsifies the same bounds. So we know that

$$|h'(z)| \leq \frac{c\,\mathrm{hcap}(A)}{|z|^3}.$$

Then

$$h(iy) = \int_y^\infty h'(is)\;\mathrm{d}s,$$

since $h(iy) \to 0$ as $y \to \infty$. Taking absolute values, we have

$$\begin{aligned}|h(iy) &= \int_y^\infty |h'(is)|\;\mathrm{d}s\\ &\leq c\,\mathrm{hcap}(A)\int_y^\infty s^{-3}\;\mathrm{d}s\\ &\leq c'\,\mathrm{hcap}(A)y^{-z}.\end{aligned}$$

To get the desired bound for a general z, integrate h' along the boundary of the circle of radius $|z|$ to get

$$|h(z)| = |h(re^{i\theta})| \leq \frac{c\,\mathrm{hcap}(A)}{|z|^2} + h(iz).$$ □

Theorem (Beurling estimate)**.** There exists a constant $c > 0$ so that the following holds. Let B be a complex Brownian motion, and $A \subseteq \bar{\mathbb{D}}$ be connected, $0 \in A$, and $A \cap \partial\bar{\mathbb{D}} \neq \emptyset$. Then for $z \in \mathbb{D}$, we have

$$\mathbb{P}_z[B[0,\tau] \cap A = \emptyset] \leq c|z|^{1/2},$$

where $\tau = \inf\{t \geq 0 : B_t \not\in \mathbb{D}\}$. □

Proposition. There exists a constant $c > 0$ so that the following is true: Suppose $A, \tilde{A} \in \mathcal{Q}$ with $A \subseteq \tilde{A}$ and $\tilde{A} \setminus A$ is coonnected. Then

$$\mathrm{diam}(g_A(\tilde{A} \setminus A)) \leq c\begin{cases}(dr)^{1/2} & d \leq r\\ \mathrm{Rad}(\tilde{A}) & d > r\end{cases},$$

where

$$d = \mathrm{diam}(\tilde{A} \setminus A), \quad r = \sup\{\mathrm{im}(z) : z \in \tilde{A}\}.$$

Proof. By scaling, we can assume that $r = 1$.

– If $d \geq 1$, then the result follows since

$$|g_A(z) - z| \leq 3\mathrm{Rad}(A),$$

and so

$$\mathrm{diam}(g_A(\tilde{A} \setminus A)) \leq \mathrm{diam}(A) + 6\mathrm{Rad}(A) \leq 8\,\mathrm{diam}(\tilde{A}).$$

– If $d < 1$, fix $z \in \mathbb{H}$ so that $U = B(z, d) \supseteq \tilde{A} \setminus A$. It then suffices to bound the size of $g_A(U)$ (or, to be precise, $g_A(U \setminus A)$).

Let B be a complex Brownian motion starting from iy with $y \geq 2$. Let

$$\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}.$$

For $B[0, \tau]$ to reach U, it must

(i) Reach $B(z, 1)$ without leaving $\mathbb{H} \setminus A$, which occurs with probability at most c/y for some constant c, by example sheet.

(ii) It must then hit U before leaving $\mathbb{H} \setminus A$. By the Beurling estimate, this occurs with probability $\leq cd^{1/2}$.

Combining the two, we see that

$$\limsup_{y \to \infty} \mathbb{P}_{iy}[B[0, \tau] \cap U \neq \emptyset] \leq cd^{1/2}.$$

By the conformal invariance of Brownian motion, if $\sigma = \inf\{t \geq 0 : B_t \notin \mathbb{H}\}$, this implies

$$\limsup_{y \to \infty} y\,\mathbb{P}_y[B[0, \sigma] \cap g_A(\tilde{A} \setminus A) \neq \emptyset] \leq cd^{1/2}.$$

Since $g_A(\tilde{A} \setminus A)$ is connected, by Q10 of example sheet 1, we have

$$\mathrm{diam}(g_A(\tilde{A} \setminus A)) \leq cd^{1/2}. \qquad \square$$

Theorem. Suppose that $(A_t) \in \mathcal{A}$. Let $g_t = g_{A_t}$. Then there exists a continuous function $U : [0, \infty) \to \mathbb{R}$ so that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Proof. First note that since the hulls are locally growing, the intersection $\bigcap_{s \geq t} g_t(A_s)$ consists of exactly one point. Call this point U_t. Again by the locally growing property, U_t is in fact a continuous function in t.

Recall that if $A \in \mathcal{Q}$, then

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{\mathrm{hcap}(A)\mathrm{Rad}(A)}{|z|^2}\right).$$

If $x \in \mathbb{R}$, then as $g_{A+x}(z) - x = g_A(z - x)$, we have

$$g_A(z) = g_A(z + x) - x = z + \frac{\mathrm{hcap}(z)}{z + x} + O\left(\frac{\mathrm{hcap}(A)\mathrm{Rad}(A + x)}{|z + x|^2}\right). \qquad (*)$$

Fix $\varepsilon > 0$. For $0 \leq s \leq t$, let

$$g_{s,t} = g_t \circ g_s^{-1}.$$

Note that

$$\operatorname{hcap}(g_T(A_{t+\varepsilon} \setminus A_t)) = 2\varepsilon.$$

Apply $(*)$ with $A = g_t(A_{t+\varepsilon} \setminus A_t)$, $x = -U_t$, and use that $\operatorname{Rad}(A + x) = \operatorname{Rad}(A - U_t) \leq \operatorname{diam}(A)$ to see that

$$g_A(z) = g_{t,t+\varepsilon}(z) = z + \frac{2\varepsilon}{z - U_t} + 2\varepsilon \operatorname{diam}(g_t(A_{t+\varepsilon} \setminus A_t)) O\left(\frac{1}{|z - U_t|^2}\right).$$

So

$$\begin{aligned} g_{t+\varepsilon}(z) - g_t(z) &= (g_{t,t+\varepsilon} - g_{t,t}) \circ g_t(z) \\ &= \frac{2\varepsilon}{g_t(z) - U_t} + 2\varepsilon \operatorname{diam}(g_t(A_{t+\varepsilon} \setminus A_t)) O\left(\frac{1}{|g_t(z) - U_t|^2}\right). \end{aligned}$$

Dividing both sides by ε and taking the limit as $\varepsilon \to 0$, the desired result follows since $\operatorname{diam}(g_t(A_{t+\varepsilon} \setminus A_t)) \to 0$. □

2.2 Schramm–Loewner evolution

Theorem (Schramm)**.** If (A_t) satisfy the conformal Markov property, then there exists $\kappa \geq 0$ so that $U_t = \sqrt{\kappa} B_t$, where B is a standard Brownian motion.

Proof. The first property is exactly the same thing as saying that given $\mathcal{F}_t$, we have

$$(U_{t+s} - U_t)_{s \geq 0} \overset{d}{=} (U_s).$$

So U_t is a continuous process with stationary, independent increments. This implies there exists $\kappa \geq 0$ and $a \in \mathbb{R}$ such that

$$U_t = \sqrt{\kappa} B_t + at,$$

where B is a standard Brownian motion. Then the second part says

$$(rU_{t/r^2})_{t \geq 0} \overset{d}{=} (U_t)_{t \geq 0}.$$

So U satisfies Brownian scaling. Plugging this in, we know

$$r\sqrt{\kappa} B_{t/r^2} + at/r \overset{d}{=} r\sqrt{\kappa} B_t + at.$$

Since $r\sqrt{\kappa} B_{t/r^2} \overset{d}{=} r\sqrt{\kappa} B_t$, we must have $a = 0$. □

Theorem (Rhode–Schramm, 2005)**.** If (A_t) is an SLE_κ with flow g_t and driving function U_t, then $g_t^{-1} : \mathbb{H} \to A_t$ extends to a map on $\bar{\mathbb{H}}$ for all $t \geq 0$ almost surely. Moreover, if we set $\gamma(t) = g_t^{-1}(U_t)$, then $\mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$. □

3 Review of stochastic calculus

Theorem (Lévy characterization)**.** Let M_t is a continuous local martingale with $[M]_t = t$ for $t \geq 0$, then M_t is a standard Brownian motion.

Proof sketch. Use Itô's formula with the exponential moment generating function $e^{i\theta M_t + \theta^2/2[M]_t}$. □

4 Phases of SLE

Theorem. SLE_κ is a simple curve if $\kappa \leq 4$, and is self-intersecting if $\kappa > 4$.

Lemma.
$$\mathrm{d}Z_t = 2Z_t^{1/2}\,\mathrm{d}\tilde{B}_t + d\cdot \mathrm{d}t.$$
where $\tilde{B}$ is a standard Brownian motion.

Lemma.
$$\mathrm{d}U_t = \left(\frac{d-1}{2}\right)U_t^{-1}\,\mathrm{d}t + \mathrm{d}\tilde{B}_t.$$

Proposition. Let $d \in \mathbb{R}$, and U_t a BES^d.

(i) If $d < 2$, then U_t hits 0 almost surely.

(ii) If $d \geq 2$, then U_t doesn't hit 0 almost surely.

Proof. The proof is similar to the proof of recurrence of Brownian motion. For $a \in \mathbb{R}_{>0}$, we define
$$\tau_a = \inf\{t \geq 0 : U_t = a\}.$$
We then consider $\mathbb{P}[\tau_b < \tau_a]$, and take the limit $a \to 0$ and $b \to \infty$. To do so, we claim

Claim. U_t^{2-d} is a continuous local martingale.

To see this, we simply compute using Itô's formula to get
$$\begin{aligned}\mathrm{d}U_t^{2-d} &= (2-d)U_t^{1-d}\,\mathrm{d}U_t + \frac{1}{2}(2-d)(1-d)U_t^{-d}\,\mathrm{d}[U]_t\\ &= (2-d)U_t^{1-d}\,\mathrm{d}\tilde{B}_t + \frac{(2-d)(d-1)}{2U_t}U_t^{1-d}\,\mathrm{d}t + \frac{1}{2}(2-d)(1-d)U_t^{-d}\,\mathrm{d}t\\ &= (2-d)U_t^{1-d}\,\mathrm{d}\tilde{B}_t.\end{aligned}$$

Therefore U_t is a continuous local martingale. Since $U_{t\wedge\tau_a\wedge\tau_b}$ is bounded, it is a true martingale, and so optional stopping tells us
$$U_0^{2-d} = \mathbb{E}[U_{\tau_a\wedge\tau_b}^{2-d}] = a^{2-d}\mathbb{P}[\tau_a < \tau_b] + b^{2-d}\mathbb{P}[\tau_b < \tau_a].$$

– If $d < 2$, we set $a = 0$, and then
$$U_0^{2-d} = b^{2-d}\mathbb{P}[\tau_b < \tau_0].$$
Dividing both sides as b, we find that
$$\left(\frac{U_0}{b}\right)^{2-d} = \mathbb{P}[\tau_b < \tau_0].$$
Taking the limit $b \to \infty$, we see that U_t hits 0 almost surely.

– If $d > 2$, then we have
$$\mathbb{P}[\tau_a < \tau_b] = \left(\frac{U_0}{a}\right)^{2-d} - \left(\frac{b}{a}\right)^{2-d}\mathbb{P}[\tau_b < \tau_a] \to 0$$
as $a \to 0$ for any b and $U_0 > 0$. So we are done in this case.

– If $d = 2$, then our martingale is just constant, and this analysis is useless. In this case, we consider $\log U_t$ and perform the same analysis to obtain the desired conclusion. □

Theorem. SLE_κ is a simple curve if $\kappa \leq 4$, and is self-intersecting if $\kappa > 4$.

Proof. If $\kappa \leq 4$, consider the probability that $\gamma(t+n)$ hits $\gamma([0,n])$. This is equivalently the probability that $\gamma(t+n)$ hits ∂A_n. This is bounded above by the probably that $g_n(\gamma(t+n)) - U_n$ hits $\partial \mathbb{H}$. But by the conformal Markov property, $g_n(\gamma(t+n)) - U_n$ is an SLE_κ. So this probability is 0.

If $\kappa > 4$, we want to reverse the above argument, but we need to be a bit careful in taking limits. We have

$$\lim_{n\to\infty} \mathbb{P}[\gamma(t) \in [-n,n] \text{ for some } t] = 1.$$

On the other hand, for any fixed n, we have

$$\lim_{m\to\infty} \mathbb{P}[g_m(A_m) - U_m \supseteq [-n,n]] = 1.$$

The probability that SLE_κ self-intersects is

$$\geq \mathbb{P}[g_m(A_m) - U_m \supseteq [-n,n] \text{ and } g_m(\gamma(t+m)) - U_m \in [-n,n] \text{ for some } t].$$

By the conformal Markov property, $g_m(\gamma(t+m)) - U_m$ is another SLE_κ given $\mathcal{F}_m$. So this factors as

$$\mathbb{P}[g_m(A_m) - U_m \supseteq [-n,n]]\mathbb{P}[\gamma(t) \in [-n,n] \text{ for some } t].$$

Since this is true for all m and n, we can take the limit $m \to \infty$, then the limit $n \to \infty$ to obtain the desired result. □

Theorem. If $\kappa \geq 8$, then SLE_κ is space-filling, but not if $\kappa \in (4,8)$. □

Proposition. SLE_κ fills $\partial\mathbb{H}$ iff $\kappa \geq 8$.

Proposition. For $r > 1$, the event $\{\tau_r = \tau_1\}$ is equivalent to the event

$$\sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty. \tag{$*$}$$

Proof. If $(*)$ happens, then we cannot have that $\tau_1 < \tau_r$, or else the supremum is infinite. So $(*) \subseteq \{\tau_1 = \tau_r\}$. The prove the proposition, we have to show that

$$\mathbb{P}\left[\tau_1 = \tau_r, \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} = \infty\right] = 0.$$

For $M > 0$, we define

$$\sigma_M = \inf\left\{t \geq 0 : \frac{V_t^r - V_t^1}{V_t^1} \geq M\right\}.$$

It then suffices to show that

$$P_M \equiv \mathbb{P}\left[\tau_1 = \tau_r \,\middle|\, \sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} \geq M\right] = \mathbb{P}[\tau_1 = \tau_r \mid \sigma_M < \tau_1] \to 0 \text{ as } M \to \infty.$$

But at time σ_M, we have $V_t^r = (M+1)V_t^1$, and $\tau_1 = \tau_r$ if these are cut off at the same time. Thus, the conformal Markov property implies

$$P_M = g(1, M+1).$$

So we are done. □

5 Scaling limit of critical percolation

Proposition. The maps (ψ_t) satisfy

$$\partial_t \psi_t(z) = 2\left(\frac{(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \frac{\psi_t'(U_t)}{z - U_t}\right).$$

In particular, at $z = U_t$, we have

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Proof. These are essentially basic calculus computations. □

Theorem. If γ is an SLE_κ, then $\psi(\gamma)$ is an SLE_κ up until hitting $\psi(\partial D \setminus \partial \mathbb{H})$ if and only if $\kappa = 6$.

6 Scaling limit of self-avoiding walks

Proposition. Suppose A be a compact $\mathbb{H}$-hull and g_A is as usual. If $x \in \mathbb{R} \setminus A$, then

$$\mathbb{P}_x[\hat{B}[0,\infty) \cap A = \emptyset] = g_A'(x).$$

Proof. This is a straightforward computation. Take $z = x + i\varepsilon$ with $\varepsilon > 0$, and let B_t be a Brownian motion. Define

$$\sigma_R = \inf\{t \geq 0 : \mathrm{im}(B_t) = R\}.$$

Then the desired probability is

$$\lim_{\varepsilon \to 0} \lim_{R \to \infty} \mathbb{P}_z[B[0, \sigma_R] \cap A = \emptyset \mid B[0, \sigma_R] \cap \mathbb{R} = \emptyset].$$

By Bayes' theorem, this is equal to

$$\lim_{\varepsilon \to 0} \lim_{R \to \infty} \frac{\mathbb{P}_z[B[0, \sigma_R] \cap (A \cup \mathbb{R}) = \emptyset]}{\mathbb{P}[B[0, \sigma_R] \cap \mathbb{R} = \emptyset]}.$$

We understand the numerator and denominator separately. The gambler's ruin estimate says the denominator is just ε/R, and to bound the numerator, recall that for $z \in \mathbb{H} \setminus A$, we have

$$|g_A(z) - z| \leq 3\mathrm{Rad}(A).$$

Thus, using conformal invariance, we can bound

$$\begin{aligned}\mathbb{P}_{g_A(z)}[B[0, \sigma_{R + 3\mathrm{Rad}(A)}] \cap \mathbb{R} = \emptyset] &\leq \mathbb{P}_z[B[0, \sigma_R] \cap (A \cup \mathbb{R}) = \emptyset] \\ &\leq \mathbb{P}_{g_A(z)}[B[0, \sigma_{R - 3\mathrm{Rad}(A)}] \cap \mathbb{R} = \emptyset].\end{aligned}$$

So we get

$$\frac{\mathrm{im}(g_A(z))}{R + 3\mathrm{Rad}(A)} \leq \text{numerator} \leq \frac{\mathrm{im}(g_A(z))}{R - 3\mathrm{Rad}(Z)}.$$

Combining, we find that the desired probability is

$$\lim_{\varepsilon \to 0} \frac{\mathrm{im}(g_A(x + i\varepsilon))}{\varepsilon} = g_A'(x). \qquad \square$$

Lemma. Suppose there exists $\alpha > 0$ so that

$$\mathbb{P}[V_A] = (\psi_A'(0))^\alpha$$

for all $A \in \mathcal{Q}_\pm$, then SLE_κ satisfies restriction.

Proof. Suppose the assertion in the lemma is true. Suppose that $A, B \in \mathcal{Q}_\pm$. Then we have that

$$\begin{aligned}\mathbb{P}[\psi_A(\gamma[0,\infty)) \cap B = \emptyset \mid V_A] &= \frac{\mathbb{P}[\gamma[0,\infty) \cap (\psi_A^{-1}(B) \cup A) = \emptyset]}{\mathbb{P}[\gamma[0,\infty) \cap A = \emptyset]} \\ &= \frac{(\psi'_{(\psi_A^{-1}(B) \cup A)}(0))^\alpha}{(\psi_A'(0))^\alpha} \\ &= \frac{(\psi_B'(0))^\alpha (\psi_A'(0))^\alpha}{(\psi_A'(0))^\alpha} \\ &= (\psi_B'(0))^\alpha \\ &= \mathbb{P}[V_B],\end{aligned}$$

where we used that

$$\psi_{\psi_A^{-1}(B)\cup A} = \psi_B \circ \psi_A.$$

So the law of $\psi_A(\gamma)$ given V_A is the law of γ. □

Lemma. $M_{t\wedge\tau}$ is a continuous martingale if

$$\kappa = \frac{8}{3}, \quad \alpha = \frac{5}{8}.$$

Proof. Recall that we showed that g_A' is a probability involving Brownian excursion, and in particular is bounded in $[0,1]$. So the same is true for ψ_A', and hence $M_{t\wedge\tau}$. So it suffices to show that $M_{t\wedge\tau}$ is a continuous local martingale. Observe that

$$M_{t\wedge\tau} = (\psi'_{g_{t\wedge\tau}(A)-g_{t\wedge\tau}(0)}(0))^\alpha$$

So if we define

$$N_t = (\psi'_{g_t(A)-g_t(0)}(0))^\alpha,$$

then it suffices to show that N_t is a continuous local martingale by optional stopping. We write

$$\psi_t = \tilde{g}_t \circ \psi_A \circ g_t^{-1},$$

where $\tilde{g}_t = g_{\psi_A(\gamma(0,t])}$. We then have

$$N_t = (\psi_t'(U_t))^\alpha.$$

In the example sheet, we show that

$$\partial_t \psi_t'(U_t) = \frac{\psi_t''(U_t)^2}{2\psi_t'(U_t)} - \frac{4}{3}\psi_t'''(U_t).$$

By Itô's formula, we get

$$\mathrm{d}N_t = \alpha N_t \left[\frac{(\alpha-1)\kappa + 1}{2}\frac{\psi_t''(U_t)^2}{\psi_t'(U_t)^2} + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\frac{\psi_t'''(U_t)}{\psi_t'(U_t)}\right]\,\mathrm{d}t$$
$$+ \alpha N_t \frac{\psi_t''(U_t)}{\psi_t'(U_t)} \cdot \sqrt{\kappa}\,\mathrm{d}B_t.$$

Picking $\kappa = \frac{8}{3}$ ensures the second $\mathrm{d}t$ term vanishes, and then setting $\alpha = \frac{5}{8}$ kills the first $\mathrm{d}t$ term as well, and we are done. □

Lemma. $M_{t\wedge\tau} \to 1$ on V_A as $t \to \infty$.

Proof. Let $\hat{B}$ be a Brownian excursion in $\mathbb{H}\setminus\gamma[0,\sigma_r]$ from $\gamma(\sigma_r)$ to ∞. Let B be a complex Brownian motion, and

$$\tau_R = \inf\{t \geq 0 : \mathrm{im}(B_t) = R\}, \quad z = \gamma(\sigma_r) + i\varepsilon.$$

Then the probability that $\hat{B}$ hits A is

$$1 - \psi'_{\sigma_r}(U_{\sigma_r}) = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\frac{\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H}\setminus\gamma[0,\sigma_r], B[0,\tau_R]\cap A \neq \emptyset]}{\mathbb{P}_z[B[0,\tau_R]\subseteq \mathbb{H}\setminus\gamma[0,\sigma_r]]}, \quad (*)$$

We will show that this expression is $\leq Cr^{-1/2}$ for some constant $C > 0$. Then we know that $M_{\sigma_r \wedge \tau} \to 1$ as $r \to \infty$ on V_A. This is convergence along a subsequence, but since we already know that $M_{t\wedge\tau}$ converges this is enough.

We first tackle the denominator, which we want to bound from below. The idea is to bound the probability that the Brownian motion reaches the lime $\operatorname{im}(z) = r + 1$ without hitting $\mathbb{R} \cup \gamma[0, \sigma_r]$. Afterwards, the gambler's ruin estimate tells us the probability of reaching $\operatorname{im}(z) = R$ without going below the $\operatorname{im}(z) = r$ line is $\frac{1}{R-r}$.

In fact, we shall consider the box $S = [-1, 1]^2 + \gamma(\sigma_r)$ of side length 2 centered at $\gamma(\sigma_r)$. Let η be the first time B leaves S, and we want this to leave via the top edge ℓ. By symmetry, if we started right at $\gamma(\sigma_r)$, then the probability of leaving at ℓ is exactly $\frac{1}{4}$. Thus, if we are at $z = \gamma(\sigma_r) + i\varepsilon$, then the probability of leaving via ℓ is $> \frac{1}{4}$.

What we would want to show is that

$$\mathbb{P}_z[B(\eta) \in \ell \mid B[0, \eta] \cap \gamma[0, \sigma_r] = \emptyset] > \frac{1}{4}. \tag{$\dagger$}$$

We then have the estimate

$$\text{denominator} \geq \frac{1}{4} \cdot \mathbb{P}_z[B[0, \eta] \cap \gamma[0, \sigma_r] = \emptyset] \cdot \frac{1}{R - r}.$$

Intuitively, (†) must be true, because $\gamma[0, \sigma_r]$ lies below $\operatorname{im}(z) = r$, and so if $B[0, \eta]$ doesn't hit $\gamma[0, \sigma_r]$, then it is more likely to go upwards. To make this rigorous, we write

$$\begin{aligned}
\frac{1}{4} &< \mathbb{P}_z[B(\eta) \in \ell] \\
&= \mathbb{P}_z[B(\eta) \in \ell \mid B[0, \eta] \cap \gamma[0, \sigma_r] = \emptyset]\ \mathbb{P}[B[0, \eta] \cap \gamma[0, \sigma_r] = \emptyset] \\
&\quad + \mathbb{P}_z[B(\eta) \in \ell \mid B[0, \eta] \cap \gamma[0, \sigma_r] \neq \emptyset]\ \mathbb{P}[B[0, \eta] \cap \gamma[0, \sigma_r] \neq \emptyset]
\end{aligned}$$

To prove (†), it suffices to observe that the first factor of the second term is is $\leq \frac{1}{4}$, which follows from the strong Markov property, since $\mathbb{P}_w[B(\eta) \in \ell] \leq \frac{1}{4}$ whenever $\operatorname{im}(w) \leq r$, which in particular is the case when $w \in \gamma[0, \sigma_r]$.

To bound the numerator, we use the strong Markov property and the Beurling estimate to get

$$\mathbb{P}_z[B \text{ hits } A \text{ without hitting } \mathbb{R} \cup \gamma[0, \sigma_r]] \leq \mathbb{P}_z[B[0, \eta] \cap \gamma[0, \sigma_r]] \cdot Cr^{-1/2}.$$

Combining, we know the numerator in $(*)$ is

$$\leq \frac{1}{R} C \cdot r^{-1/2} \cdot \mathbb{P}[B[0, \eta] \cap \gamma[0, \sigma_r] = \emptyset].$$

These together give the result. □

Lemma. $M_{t\wedge\tau} \to 0$ as $t \to \infty$ on V_A^c.

Proof. By the example sheet, we may assume that A is bounded by a smooth, simple curve $\beta : (0, 1) \to \mathbb{H}$.

Note that $\gamma(\tau) = \beta(s)$ for some $s \in (0, 1)$. We need to show that

$$\lim_{t \to \tau} \psi_t'(U_t) = 0.$$

For $m \in \mathbb{N}$, let

$$t_m = \inf\left\{t \geq 0 : |\gamma(t) - \beta(s)| = \frac{1}{m}\right\}$$

Since β is smooth, there exists $\delta > 0$ so that

$$\ell = [\beta(s), \beta(s) + \delta \mathbf{n}] \subseteq A,$$

where $\mathbf{n}$ is the unit inward pointing normal at $\beta(s)$. Let

$$L_t = g_t(\ell) - U_t.$$

Note that a Brownian motion starting from a point on ℓ has a uniformly positive chance of exiting $\mathbb{H} \setminus \gamma[0, t_m]$ on the left side of $\gamma[0, t_m]$ and on the right side as well.

On the second example sheet, we see that this implies that

$$L_{t_m} \subseteq \{w : \operatorname{im}(w) \geq a |\operatorname{Re}(w)|\}$$

for some $a > 0$, using the conformal invariance of Brownian motion. Intuitively, this is because after applying $g_t - U_t$, we have uniformly positive probability of exiting via the positive or real axis, and so we cannot be too far away in one directionn.

Again by the second example sheet, the Brownian excursion in $\mathbb{H}$ from 0 to ∞ hits L_{t_m} with probability $\to 1$ as $m \to \infty$. $\square$

Theorem. $\mathrm{SLE}_{8/3}$ satisfies the restriction property. Moreover, if $\gamma \sim \mathrm{SLE}_{8/3}$, then

$$\mathbb{P}[\gamma[0, \infty) \cap A = \emptyset] = (\psi'_A(0))^{5/8}.$$

7 The Gaussian free field

Proposition.

(i) Conformal invariance: Suppose $\varphi : D \to \tilde{D}$ is a conformal transformation, and $f, g \in C_0^\infty(D)$. Then

$$(f, g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla$$

In other words, the Dirichlet inner product is conformally invariant.

In other words, $\varphi^* : H_0^1(D) \to H_0^1(\tilde{D})$ given by $f \mapsto f \circ \varphi^{-1}$ is an isomorphism of Hilbert spaces.

(ii) Inclusion: Suppose $U \subseteq D$ is open. If $f \in C_0^\infty(U)$, then $f \in C_0^\infty(D)$. Therefore the inclusion map $i : H_0^1(U) \to H_0^1(D)$ is well-defined and associates $H_0^1(U)$ with a subspace of $H_0^1(D)$. We write the image as $H_{\mathrm{supp}}(U)$.

(iii) Orthogonal decomposition: If $U \subseteq D$, let

$$H_{\mathrm{harm}}(U) = \{f \in H_0^1(D) : f \text{ is harmonic on } U\}.$$

Then

$$H_0^1(D) = H_{\mathrm{supp}}(U) \oplus H_{\mathrm{harm}}(U)$$

is an orthogonal decomposition of $H_0^1(D)$. This is going to translate to a Markov property of the Gaussian free field.

Proof. Conformal invariance is a routine calculation, and inclusion does not require proof. To prove orthogonality, suppose $f \in H_{\mathrm{supp}}(U)$ and $g \in H_{\mathrm{harm}}(U)$. Then

$$(f, g) = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x)\, \mathrm{d}x = -\frac{1}{2\pi} \int f(x) \Delta g(x)\, \mathrm{d}x = 0.$$

since f is supported on U and Δg is supported outside of U.

To prove that they span, suppose $f \in H_0^1(D)$, and f_0 the orthogonal projection of f onto $H_{\mathrm{supp}}(U)$. Let $g_0 = f - f_0$. We want to show that g_0 is harmonic. It would be a straightforward manipulation if we can take Δ, but there is no guarantee that f_0 is smooth.

We shall show that g_0 is weakly harmonic, and then it is a standard analysis result (which is also on the example sheet) that g_0 is in fact genuinely harmonic.

Suppose $\varphi \in C_0^\infty(U)$. Then since $g_0 \perp H_{\mathrm{supp}}(U)$, we have

$$0 = (g_0, \varphi) = \frac{1}{2\pi} \int \nabla g_0(x) \cdot \nabla \varphi(x)\, \mathrm{d}x = -\frac{1}{2\pi} \int g_0(x) \Delta \varphi(x)\, \mathrm{d}x.$$

This implies g_0 is C^∞ on U and harmonic. □

Proposition.

(i) If $\varphi : D \to \tilde{D}$ is a conformal transformation and h is a Gaussian free field on D, then $h \circ \varphi^{-1}$ is a Gaussian free field on $\tilde{D}$.

(ii) Markov property: If $U \subseteq D$ is open, then we can write $h = h_1 + h_2$ with h_1 and h_2 independent where h_1 is a Gaussian free field on U_1 and h_2 is harmonic on U.

Proof.

(i) Clear.

(ii) Take h_1 to be the projection onto $H_{\mathrm{supp}}(U)$. This works since we can take the orthonormal basis (f_n) to be the union of an orthonormal basis of $H_{\mathrm{supp}}(U)$ plus an orthonormal basis of $H_{\mathrm{harm}}(U)$. □

Proposition. Let $D, \tilde{D}$ be domains in $\mathbb{C}$ and φ is a conformal transformation $D \to \tilde{D}$. Then $G_D(x, y) = G_{\tilde{D}}(\varphi(x), \varphi(y))$. □

Theorem (Schramm–Sheffield)**.** Let $\lambda = \frac{\pi}{2}$. Let $\gamma \sim \mathrm{SLE}_4$ in $\mathbb{H}$ from 0 to ∞. Let g_t its Loewner evolution with driving function $U_t = \sqrt{\kappa} B_t = 2B_t$, and set $f_t = g_t - U_t$. Fix $W \subseteq \mathbb{H}$ open and let

$$\tau = \inf\{t \geq 0 : \gamma(t) \in W\}.$$

Let h be a Gaussian free field on $\mathbb{H}$, $\lambda > 0$, and $\hbar$ be the unique harmonic function on $\mathbb{H}$ with boundary values λ on $\mathbb{R}_{>0}$ and $-\lambda$ on $\mathbb{R}_{<0}$. Explicitly, it is given by

$$\hbar = \lambda - \frac{2\lambda}{\pi} \arg(\,\cdot\,).$$

Then

$$h + \hbar \overset{d}{=} (h + \hbar) \circ f_{t \wedge \tau},$$

where both sides are restricted to W.

Proof. We want to show that if $\varphi \in C_0^\infty(W)$,

$$((h + \hbar) \circ f_{t\wedge\tau}, \varphi) \overset{d}{=} (h + \hbar, \varphi).$$

In other words, writing

$$m_t(\varphi) = (\hbar \circ f_t, \varphi), \quad \sigma_0^2(\varphi) = \iint \varphi(x) G_{\mathbb{H}}(f_t(x), f_t(y)) \varphi(y)\,\mathrm{d}x\,\mathrm{d}y,$$

we want to show that

$$((h + \hbar) \circ f_{t\wedge\tau}, \varphi) \sim N(m_0(\varphi), \sigma_0^2(\varphi)).$$

This is the same as proving that

$$\mathbb{E}\left[e^{i\theta((h+\hbar)\circ f_{t\wedge\tau}, \varphi)}\right] = \exp\left[i\theta m_0(\varphi) - \frac{\theta^2}{2}\sigma_0^2(\varphi)\right].$$

Let $\mathcal{F}_t = \sigma(U_s : s \leq t)$ be the filtration of U_t. Then

$$\begin{aligned}\mathbb{E}\left[e^{i\theta((h+\hbar)\circ f_{t\wedge\tau}, \varphi)} \,\middle|\, \mathcal{F}_{t\wedge\tau}\right] &= \mathbb{E}\left[e^{i\theta(h\circ f_{t\wedge\tau}, \varphi)} \mid \mathcal{F}_{t\wedge\tau}\right] e^{i\theta m_{t\wedge\tau}(\varphi)}\\ &= \exp\left[i\theta m_{t\wedge\tau}(\varphi) - \frac{\theta^2}{2}\sigma_{t\wedge\tau}^2(\varphi)\right],\end{aligned}$$

If we knew that
$$\exp\left[i\theta m_t(\varphi) - \frac{\theta^2}{2}\sigma_t^2(\varphi)\right]$$
is a martingale, then taking the expectation of the above equation yields the desired results.

Note that this looks exactly like the form of an exponential martingale, which in particular is a martingale. So it suffices to show that $m_t(\varphi)$ is a martingale with
$$[m_{\bullet}(\varphi)]_t = \sigma_0^2(\varphi) - \sigma_t^2(\varphi).$$

To check that $m_t(\varphi)$ is a martingale, we expand it as
$$\hbar \circ f_t(z) = \lambda - \frac{2\lambda}{\pi}\arg(f_t(z)) = \lambda - \frac{2\lambda}{\pi}\operatorname{im}(\log(g_t(z) - U_t)).$$

So it suffices to check that $\log(g_t(z) - U_t)$ is a martingale. We apply Itô's formula to get
$$\mathrm{d}\log(g_t(z) - U_t) = \frac{1}{g_t(z) - U_t}\cdot\frac{2}{g_t(z) - U_t}\,\mathrm{d}t - \frac{1}{g_t(z) - U_t}\,\mathrm{d}U_t - \frac{\kappa/2}{(g_t(z) - U_t)^2}\,\mathrm{d}t,$$
and so this is a continuous local martingale when $\kappa = 4$. Since $m_t(\varphi)$ is bounded, it is a genuine martingale.

We then compute the derivative of the quadratic variation
$$\mathrm{d}[m_{\bullet}(\varphi)]_t = \int \varphi(x)\operatorname{im}\left(\frac{2}{g_t(x) - U_t}\right)\operatorname{im}\left(\frac{2}{g_t(y) - U_t}\right)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

To finish the proof, we need to show that $\mathrm{d}\sigma_t^2(\varphi)$ takes the same form. Recall that the Green's function can be written as
$$G_{\mathbb{H}}(x, y) = -\log|x - y| + \log|x - \bar{y}| = -\operatorname{Re}(\log(x - y) - \log(x - \bar{y})).$$

Since we have
$$\log(f_t(x) - f_t(y)) = \log(g_t(x) - g_t(y)),$$
we can compute
$$\begin{aligned}\mathrm{d}\log(g_t(x) - g_t(y)) &= \frac{1}{g_t(x) - g_t(y)}\left[\frac{2}{g_t(x) - U_t} - \frac{2}{g_t(y) - U_t}\right]\,\mathrm{d}t\\ &= \frac{-2}{(g_t(x) - U_t)(g_t(y) - U_t)}\,\mathrm{d}t.\end{aligned}$$

Similarly, we have
$$\mathrm{d}\log(g_t(x) - \overline{g_t(y)}) = \frac{-2}{(g_t(x) - U_t)(\overline{g_t(y)} - U_t)}\,\mathrm{d}t.$$

So we have
$$\mathrm{d}G_t(x, y) = -\operatorname{im}\left(\frac{2}{g_t(x) - U_t}\right)\operatorname{im}\left(\frac{2}{g_t(y) - U_t}\right)\,\mathrm{d}t.$$

This is exactly what we wanted it to be. □