Part III — Schramm–Loewner Evolutions
Theorems

Based on lectures by J. Miller
Notes taken by Dexter Chua

Lent 2018

Schramm–Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Contents

0 Introduction

1 Conformal transformations

1.1 Conformal transformations

Theorem (Riemann mapping theorem)**.** Let U be a simply connected domain with $U \neq \mathbb{C}$ and $z \in U$ be any point. Then there exists a unique conformal transformation $f : \mathbb{D} \to U$ such that $f(0) = z$, and $f'(0)$ is real and positive.

1.2 Brownian motion and harmonic functions

Theorem. Let u be a harmonic function on a bounded domain D which is continuous on $\bar{D}$. For $z \in D$, let $\mathbb{P}_z$ be the law of a complex Brownian motion starting from z, and let τ be the first hitting time of D. Then

$$u(z) = \mathbb{E}_z[u(B_\tau)]. \qquad \square$$

Corollary (Mean value property)**.** If u is a harmonic function, then, whenever it makes sense, we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \;\mathrm{d}\theta.$$

Corollary (Maximum principle)**.** Let u be harmonic in a domain D. If u attains its maximum at an interior point in D, then u is constant.

Corollary (Maximum modulus principle)**.** Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum in the interior of D, then f is constant.

Lemma (Schwarz lemma)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|f(z)| = |z|$ for some non-zero $z \in \mathbb{D}$, then $f(w) = \lambda w$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

1.3 Distortion estimates for conformal maps

Theorem (Koebe-1/4 theorem)**.** If $f \in \mathcal{U}$ and $0 < r \leq 1$, then $B(0, r/4) \subseteq f(r\mathbb{D})$.

Theorem. If $f \in \mathcal{U}$, then $|a_2| \leq 2$.

Corollary. Let $D, \tilde{D}$ be domains and $z \in D$, $\tilde{z} \in \tilde{D}$. If $f : D \to \tilde{D}$ is a conformal transformation with $f(z) = \tilde{z}$, then

$$\frac{\tilde{d}}{4d} \leq |f'(z)| \leq \frac{4\tilde{d}}{d},$$

where $d = \mathrm{dist}(z, \partial D)$ and $\tilde{d} = \mathrm{dist}(\tilde{z}, \partial\tilde{D})$.

Proposition. Let $f \in \mathcal{U}$. Then

$$\mathrm{area}(f(\mathbb{D})) = \pi \sum_{n=1}^\infty n |a_n|^2.$$

Proposition. If $K \in \mathcal{H}$, then

$$\mathrm{area}(K) = \pi \left(1 - \sum_{n=1}^\infty n |b_n|^2\right).$$

Lemma. Let $f \in \mathcal{U}$. Then there exists an odd function $h \in \mathcal{U}$ with $h(z)^2 = f(z^2)$.

1.4 Half-plane capacity

Proposition. For each $A \in \mathcal{Q}$, there exists a unique conformal transformation $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ with $|g_A(z) - z| \to 0$ as $z \to \infty$.

Theorem (Schwarz reflection principle)**.** Let $D \subseteq \mathbb{H}$ be a simply connected domain, and let $\phi : D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets and sends $\mathbb{R} \cap D$ to $\mathbb{R}$. Then ϕ extends by reflection to a conformal transformation on

$$D^* = D \cup \{\bar{z} : z \in D\} = D \cup \bar{D}$$

by setting $\phi(\bar{z}) = \overline{\phi(z)}$. □

Proposition.

(i) Scaling: If $r > 0$ and $A \in \mathcal{Q}$, then $\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A)$.

(ii) Translation invariance: If $x \in \mathbb{R}$ and $a \in \mathcal{Q}$, then $\operatorname{hcap}(A + x) = \operatorname{hcap}(A)$.

(iii) Monotonicity: If $A, \tilde{A} \in \mathcal{Q}$ are such that $A \subseteq \tilde{A}$. Then $\operatorname{hcap}(A) \leq \operatorname{hcap}(\tilde{A})$.

Proposition. Let $A \in \mathcal{Q}$ and B_t be complex Brownian motion. Define the stopping time

$$\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}.$$

Then

(i) For all $z \in \mathbb{H} \setminus A$, we have

$$\operatorname{im}(z - g_A(z)) = \mathbb{E}_z[\operatorname{im}(B_\tau)]$$

(ii)

$$\operatorname{hcap}(A) = \lim_{y \to \infty} y\, \mathbb{E}_y[\operatorname{im}(B_\tau)].$$

In particular, $\operatorname{hcap}(A) \geq 0$.

(iii) If $A \subseteq \bar{\mathbb{D}} \cap \mathbb{H}$, then

$$\operatorname{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}[\operatorname{im}(B_\tau)] \sin\theta \,\mathrm{d}\theta.$$

Theorem. Let $D, \tilde{D} \subseteq \mathbb{C}$ be domains, and $f : D \to \tilde{D}$ a conformal transformation. Let $B, \tilde{B}$ be Brownian motions starting from $z \in D, \tilde{z} \in \tilde{D}$ respectively, with $f(z) = \tilde{z}$. Let

$$\begin{aligned} \tau &= \inf\{t \geq 0 : B_t \notin D\} \\ \tilde{\tau} &= \inf\{t \geq 0 : \tilde{B}_t \notin \tilde{D}\} \end{aligned}$$

Set

$$\begin{aligned} \tau' &= \int_0^\tau |f'(B_s)|^2 \,\mathrm{d}s \\ \sigma(t) &= \inf\left\{s \geq 0 : \int_0^s |f'(B_r)|^2 \,\mathrm{d}r = t\right\} \\ B'_t &= f(B_{\sigma(t)}). \end{aligned}$$

Then $(B'_t : t < \tilde{\tau}')$ has the same distribution as $(\tilde{B}_t : t < \tilde{\tau})$.

2 Loewner's theorem

2.1 Key estimates

Proposition. Let $A \in \mathcal{Q}$ and B be a complex Brownian motion. Set

$$\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}.$$

Then

- If $x > \mathrm{Rad}(A)$, then

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}[B_\tau \in [x, \infty)]\right).$$

- If $x < -\mathrm{Rad}(A)$, then

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\mathbb{P}_{iy}[B_\tau \in (-\infty, x]] - \frac{1}{2}\right).$$

Corollary. If $A \in \mathcal{Q}$, $\mathrm{Rad}(A) \leq 1$, then

$$\begin{aligned} x \leq g_A(x) &\leq x + \frac{1}{x} && \text{if } x > 1 \\ x + \frac{1}{x} \leq g_A(x) &\leq x && \text{if } x < -1. \end{aligned}$$

Moreover, for all $A \in \mathcal{Q}$, we have

$$|g_A(z) - z| \leq 3\,\mathrm{Rad}(A).$$

Proposition. There is a constant $c > 0$ so that for every $A \in \mathcal{Q}$ and $|z| > 2\,\mathrm{Rad}(A)$, we have

$$\left| g_a(z) - \left(z + \frac{\mathrm{hcap}(A)}{z}\right)\right| \leq c \frac{\mathrm{Rad}(A) \cdot \mathrm{hcap}(A)}{|z|^2}.$$

Theorem (Beurling estimate)**.** There exists a constant $c > 0$ so that the following holds. Let B be a complex Brownian motion, and $A \subseteq \bar{\mathbb{D}}$ be connected, $0 \in A$, and $A \cap \partial\bar{\mathbb{D}} \neq \emptyset$. Then for $z \in \mathbb{D}$, we have

$$\mathbb{P}_z[B[0, \tau] \cap A = \emptyset] \leq c|z|^{1/2},$$

where $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{D}\}$. □

Proposition. There exists a constant $c > 0$ so that the following is true: Suppose $A, \tilde{A} \in \mathcal{Q}$ with $A \subseteq \tilde{A}$ and $\tilde{A} \setminus A$ is coonnected. Then

$$\mathrm{diam}(g_A(\tilde{A} \setminus A)) \leq c \begin{cases} (dr)^{1/2} & d \leq r \\ \mathrm{Rad}(\tilde{A}) & d > r \end{cases},$$

where

$$d = \mathrm{diam}(\tilde{A} \setminus A), \quad r = \sup\{\mathrm{im}(z) : z \in \tilde{A}\}.$$

Theorem. Suppose that $(A_t) \in \mathcal{A}$. Let $g_t = g_{A_t}$. Then there exists a continuous function $U : [0, \infty) \to \mathbb{R}$ so that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

2.2 Schramm–Loewner evolution

Theorem (Schramm)**.** If (A_t) satisfy the conformal Markov property, then there exists $\kappa \geq 0$ so that $U_t = \sqrt{\kappa} B_t$, where B is a standard Brownian motion.

Theorem (Rhode–Schramm, 2005)**.** If (A_t) is an SLE_κ with flow g_t and driving function U_t, then $g_t^{-1} : \mathbb{H} \to A_t$ extends to a map on $\bar{\mathbb{H}}$ for all $t \geq 0$ almost surely. Moreover, if we set $\gamma(t) = g_t^{-1}(U_t)$, then $\mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$. $\square$

3 Review of stochastic calculus

Theorem (Lévy characterization)**.** Let M_t is a continuous local martingale with $[M]_t = t$ for $t \geq 0$, then M_t is a standard Brownian motion.

4 Phases of SLE

Theorem. SLE_κ is a simple curve if $\kappa \leq 4$, and is self-intersecting if $\kappa > 4$.

Lemma.

$$\mathrm{d}Z_t = 2Z_t^{1/2}\,\mathrm{d}\tilde{B}_t + d \cdot \mathrm{d}t.$$

where $\tilde{B}$ is a standard Brownian motion.

Lemma.

$$\mathrm{d}U_t = \left(\frac{d-1}{2}\right) U_t^{-1}\,\mathrm{d}t + \mathrm{d}\tilde{B}_t.$$

Proposition. Let $d \in \mathbb{R}$, and U_t a BES^d.

(i) If $d < 2$, then U_t hits 0 almost surely.

(ii) If $d \geq 2$, then U_t doesn't hit 0 almost surely.

Theorem. SLE_κ is a simple curve if $\kappa \leq 4$, and is self-intersecting if $\kappa > 4$.

Theorem. If $\kappa \geq 8$, then SLE_κ is space-filling, but not if $\kappa \in (4, 8)$. □

Proposition. SLE_κ fills $\partial\mathbb{H}$ iff $\kappa \geq 8$.

Proposition. For $r > 1$, the event $\{\tau_r = \tau_1\}$ is equivalent to the event

$$\sup_{t<\tau_1} \frac{V_t^r - V_t^1}{V_t^1} < \infty. \tag{$*$}$$

5 Scaling limit of critical percolation

Proposition. The maps (ψ_t) satisfy

$$\partial_t \psi_t(z) = 2\left(\frac{(\psi_t'(U_t))^2}{\psi_t(z) - \psi_t(U_t)} - \frac{\psi_t'(U_t)}{z - U_t}\right).$$

In particular, at $z = U_t$, we have

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Theorem. If γ is an SLE_κ, then $\psi(\gamma)$ is an SLE_κ up until hitting $\psi(\partial D \setminus \partial \mathbb{H})$ if and only if $\kappa = 6$.

6 Scaling limit of self-avoiding walks

Proposition. Suppose A be a compact $\mathbb{H}$-hull and g_A is as usual. If $x \in \mathbb{R} \setminus A$, then
$$\mathbb{P}_x[\hat{B}[0, \infty) \cap A = \emptyset] = g'_A(x).$$

Lemma. Suppose there exists $\alpha > 0$ so that
$$\mathbb{P}[V_A] = (\psi'_A(0))^\alpha$$
for all $A \in \mathcal{Q}_\pm$, then SLE_κ satisfies restriction.

Lemma. $M_{t\wedge\tau}$ is a continuous martingale if
$$\kappa = \frac{8}{3}, \quad \alpha = \frac{5}{8}.$$

Lemma. $M_{t\wedge\tau} \to 1$ on V_A as $t \to \infty$.

Lemma. $M_{t\wedge\tau} \to 0$ as $t \to \infty$ on V_A^c.

Theorem. $\mathrm{SLE}_{8/3}$ satisfies the restriction property. Moreover, if $\gamma \sim \mathrm{SLE}_{8/3}$, then
$$\mathbb{P}[\gamma[0, \infty) \cap A = \emptyset] = (\psi'_A(0))^{5/8}.$$

7 The Gaussian free field

Proposition.

(i) Conformal invariance: Suppose $\varphi : D \to \tilde{D}$ is a conformal transformation, and $f, g \in C_0^\infty(D)$. Then

$$(f, g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla$$

In other words, the Dirichlet inner product is conformally invariant.

In other words, $\varphi^* : H_0^1(D) \to H_0^1(\tilde{D})$ given by $f \mapsto f \circ \varphi^{-1}$ is an isomorphism of Hilbert spaces.

(ii) Inclusion: Suppose $U \subseteq D$ is open. If $f \in C_0^\infty(U)$, then $f \in C_0^\infty(D)$. Therefore the inclusion map $i : H_0^1(U) \to H_0^1(D)$ is well-defined and associates $H_0^1(U)$ with a subspace of $H_0^1(D)$. We write the image as $H_{\mathrm{supp}}(U)$.

(iii) Orthogonal decomposition: If $U \subseteq D$, let

$$H_{\mathrm{harm}}(U) = \{f \in H_0^1(D) : f \text{ is harmonic on } U\}.$$

Then

$$H_0^1(D) = H_{\mathrm{supp}}(U) \oplus H_{\mathrm{harm}}(U)$$

is an orthogonal decomposition of $H_0^1(D)$. This is going to translate to a Markov property of the Gaussian free field.

Proposition.

(i) If $\varphi : D \to \tilde{D}$ is a conformal transformation and h is a Gaussian free field on D, then $h \circ \varphi^{-1}$ is a Gaussian free field on $\tilde{D}$.

(ii) Markov property: If $U \subseteq D$ is open, then we can write $h = h_1 + h_2$ with h_1 and h_2 independent where h_1 is a Gaussian free field on U_1 and h_2 is harmonic on U.

Proposition. Let $D, \tilde{D}$ be domains in $\mathbb{C}$ and φ is a conformal transformation $D \to \tilde{D}$. Then $G_D(x, y) = G_{\tilde{D}}(\varphi(x), \varphi(y))$. □

Theorem (Schramm–Sheffield)**.** Let $\lambda = \frac{\pi}{2}$. Let $\gamma \sim \mathrm{SLE}_4$ in $\mathbb{H}$ from 0 to ∞. Let g_t its Loewner evolution with driving function $U_t = \sqrt{\kappa} B_t = 2B_t$, and set $f_t = g_t - U_t$. Fix $W \subseteq \mathbb{H}$ open and let

$$\tau = \inf\{t \geq 0 : \gamma(t) \in W\}.$$

Let h be a Gaussian free field on $\mathbb{H}$, $\lambda > 0$, and $\hbar$ be the unique harmonic function on $\mathbb{H}$ with boundary values λ on $\mathbb{R}_{>0}$ and $-\lambda$ on $\mathbb{R}_{<0}$. Explicitly, it is given by

$$\hbar = \lambda - \frac{2\lambda}{\pi} \arg(\,\cdot\,).$$

Then

$$h + \hbar \overset{d}{=} (h + \hbar) \circ f_{t \wedge \tau},$$

where both sides are restricted to W.