SCHRAMM-LOEWNER EVOLUTIONS, LENT 2018, EXAMPLE SHEET 1

Please send corrections to jpmiller@statslab.cam.ac.uk

Problem 1.

- Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is a conformal transformation (i.e., f is a conformal automorphism of $\mathbb{D}$). Use the Schwarz lemma to show that there exists $z \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ so that
$$f(w) = \lambda \frac{z-w}{\overline{z}w-1}.$$
- Suppose that $f\colon \mathbb{H} \to \mathbb{H}$ is a conformal transformation (i.e., f is a conformal automorphism of $\mathbb{H}$). Show that there exists $a,b,c,d \in \mathbb{R}$ with $ad-bc=1$ so that
$$f(z) = \frac{az+b}{cz+d}.$$
Deduce that if f fixes 0 and ∞ then there exists $a>0$ so that $f(z)=az$.

Problem 2.

- Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbb{D}$ on the unit circle is given by
$$p(z,e^{i\theta}) = \frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2} \quad \text{for} \quad \theta \in [0,2\pi).$$
You may assume that the hitting density is given by the uniform distribution on $\partial\mathbb{D}$ when $z=0$.
- Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbb{H}$ on the real line $\partial\mathbb{H}$ is given by
$$p(z,u) = \frac{1}{\pi}\frac{y}{(x-u)^2+y^2} \quad \text{where} \quad z = x+iy, \quad u \in \partial\mathbb{H}.$$
(Note that $p(i,\cdot)$ is the Cauchy distribution on $\mathbb{R}$.)

Problem 3.

- Show that $f(z) = z + 1/z$ is a conformal transformation from $\mathbb{H} \setminus \overline{\mathbb{D}}$ to $\mathbb{H}$.
- Using the conformal invariance of Brownian motion, show that the density $p(z,e^{i\theta})$, $\theta \in [0,\pi]$, for the first exit distribution (with respect to Lebesgue measure) of a complex Brownian motion on $\mathbb{H} \cap \partial\mathbb{D}$ starting from $z \in \mathbb{H} \setminus \overline{\mathbb{D}}$ satisfies:
$$p(z,e^{i\theta}) = \frac{2}{\pi}\frac{\mathrm{Im}(z)}{|z|^2}\sin(\theta)\left(1+O(|z|^{-1})\right) \quad \text{as} \quad z \to \infty.$$

Problem 4. Using the previous problem, show that if $A \in \mathcal{Q}$ then
$$\mathrm{hcap}(A) = \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]\sin(\theta)d\theta$$

where τ is the first time that a complex Brownian motion B exits $\mathbb{H} \setminus A$ and $\mathbb{E}_z$ denotes the expectation with respect to the law under which B starts from z.

Problem 5. (Schwarz reflection for harmonic functions) Suppose that $u\colon \overline{\mathbb{H} \cap \mathbb{D}} \to \mathbb{R}$ is harmonic in $\mathbb{H} \cap \mathbb{D}$, continuous in $\overline{\mathbb{H} \cap \mathbb{D}}$, and vanishes on $[-1, 1]$. Show that u extends to a harmonic function on $\mathbb{D}$ by odd reflection, i.e., by taking $u(\overline{z}) = -u(z)$.

Problem 6. Suppose that D is a domain in $\mathbb{C}$ and f is holomorphic and non-zero on D. Show that $\log|f|$ is harmonic.

Problem 7.

- Consider the rectangle $A_r = [-r, r] \times (0, 1]$ in $\mathbb{H}$. Show that there exists a constant $c > 0$ such that $\mathrm{hcap}(A_r) \leq cr$ for all $r \geq 1$.
- Find a sequence of compact $\mathbb{H}$-hulls (A_n) such that $\mathrm{diam}(A_n) \to \infty$ but $\mathrm{hcap}(A_n) \to 0$.

Problem 8. Suppose that u is a harmonic function on a domain $D \subseteq \mathbb{C}$. Show that for each $n \in \mathbb{N} = \{1, 2, \ldots\}$ there exists a constant $c_n > 0$ such that for all $j, k \in \mathbb{N}_0 = \{0, 1, \ldots\}$ with $j + k = n$ and $z = x + iy \in D$ we have that

$$\left|\partial_x^j \partial_y^k u(z)\right| \leq \frac{c_n}{\mathrm{dist}(z, \partial D)^n} \|u\|_\infty.$$

Hint: use the first part of Problem 2.

Problem 9. Suppose that $A \in \mathcal{Q}$ with $\mathrm{rad}(A) = \sup\{|z| : z \in A\} \leq 1$. Show that

$$x \leq g_A(x) \leq x + \frac{1}{x} \quad \text{for all} \quad x > 1$$
$$x + \frac{1}{x} \leq g_A(x) \leq x \quad \text{for all} \quad x < -1.$$

Show also that for all $A \in \mathcal{Q}$ and $A \in \mathbb{H} \setminus A$ we have that $|g_A(z) - z| \leq 3\mathrm{rad}(A)$. Hint: for $x > 1$, show that $g_A(x)$ is increasing in A and recall the first part of Problem 3.

Problem 10. Suppose that $A \in \mathcal{Q}$ is connected. Let B be a complex Brownian motion and let $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$. Show that there exists constants $c_1, c_2 > 0$ such that

$$c_1 \mathrm{diam}(A) \leq \lim_{y \to \infty} y \mathbb{P}_{iy}[B_\tau \in A] \leq c_2 \mathrm{diam}(A).$$

Problem 11. Suppose that $\gamma\colon [0, T] \to \overline{\mathbb{H}}$ is a simple curve (i.e., $s \neq t$ implies $\gamma(s) \neq \gamma(t)$) with $\gamma(0) = 0$ and $\gamma(t) \in \mathbb{H}$ for all $t \in (0, T]$. Show that $A_t = \gamma((0, t])$ for $t \in [0, T]$ is a family of locally growing compact $\mathbb{H}$-hulls. Show, moreover, that there exists a homeomorphism $\phi\colon [0, T] \to [0, \frac{1}{2}\mathrm{hcap}(A_T)]$ so that $\mathrm{hcap}(A_{\phi^{-1}(t)}) = 2t$ for all $t \in [0, \frac{1}{2}\mathrm{hcap}(A_T)]$. (This is the so-called capacity parameterization of γ.)

Problem 12. Suppose that $U\colon [0, T] \to \mathbb{R}$ is a continuous function. Let $g_t(z)$ solve the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Show for each $t \in [0, T]$ that g_t is a conformal transformation from its domain onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$ using the following steps.

- Show that $t \mapsto \mathrm{Im}(g_t(z))$ is decreasing in t, hence for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is defined up until $\tau_z = \sup\{t \geq 0 : \mathrm{Im}(g_t(z)) > 0\}$. Conclude that $H_t = \{z : \tau_z > t\}$ is the domain of g_t.
- Show for each $t \in [0, T]$ that $z \mapsto g_t(z)$ is complex differentiable on H_t.

- Show for each $t \in [0,T]$ that $z \mapsto g_t(z)$ has an inverse defined on $\mathbb{H}$ by showing that $g_t(f_t(w)) = w$ for all $w \in \mathbb{H}$ where f_s for $s \in [0,t]$ solves the so-called *reverse chordal Loewner equation*
$$\partial_s f_s(w) = -\frac{2}{f_s(w) - U_{t-s}}, \quad f_0(w) = w.$$

Optional problems: Riemann mapping theorem

The purpose of this sequence of problems is to prove the Riemann mapping theorem.

Optional Problem 1. Prove the Harnack inequality: suppose that u is a positive harmonic function defined on a domain D. Then for each $K \subseteq D$ compact there exists a constant $M > 0$ (independent of u) such that
$$\frac{\sup_{z \in K} u(z)}{\inf_{z \in K} u(z)} \leq M.$$

Optional Problem 2. Deduce from Problem 1 that if $f, \widetilde{f}$ are conformal transformations $D \to \mathbb{D}$ taking z to 0 and with positive derivative at z, then $f = \widetilde{f}$.

Optional Problem 3. Suppose that D is a simply connected domain with $D \neq \mathbb{C}$. Suppose that $z \in D$. Show that there exists a unique conformal transformation $f \colon D \to \mathbb{D}$ with $f(z) = 0$ and $f'(z) > 0$ using the following steps.

- Let $\mathcal{C}$ be the collection of conformal transformations f from D into a subset of $\mathbb{D}$ with $f(z) = 0$ and $f'(z) > 0$. Deduce from the Schwarz lemma that if $f \in \mathcal{C}$ then $f'(z) \leq (\mathrm{dist}(z, \partial D))^{-1}$.
- Show that $\mathcal{C}$ is non-empty.
- Suppose that (f_n) is a sequence in $\mathcal{C}$ such that, for each $K \subseteq D$ compact, we have that $f_n|_K \to f|_K$ uniformly where f is conformal on D. Show that f is either constant or injective.
- Let $M = \sup\{f'(z) : z \in \mathcal{C}\}$. Let (f_n) be a sequence of functions in $\mathcal{C}$ with $f_n'(z)$ increasing to M. Explain why there exists a subsequence (f_{n_k}) of (f_n) which converges uniformly to a map $f \colon D \to \mathbb{D}$. (Hint: use Problem 7, the Harnack inequality, and the Arzela-Ascoli theorem.) Explain why $f'(z) = M$ and deduce from the previous part that f is injective.
- Show that f is surjective onto $\mathbb{D}$. (Hint: argue by contradiction that if f is not surjective then $f'(z) < M$.)

SCHRAMM-LOEWNER EVOLUTIONS, LENT 2018, EXAMPLE SHEET 2

Please send corrections to jpmiller@statslab.cam.ac.uk

Problem 1. Suppose that $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion and let (g_t) solve

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

- (Markov property) Suppose that τ is a stopping time for U which is almost surely finite and let $\widetilde{g}_t = g_{\tau+t}(g_\tau^{-1}(z + U_\tau)) - U_\tau$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps (g_t).
- (Scale invariance) Fix $r > 0$ and let $\widetilde{g}_t(z) = r g_{t/r^2}(z/r)$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps (g_t).

Suppose that D is a simply connected domain, $x, y \in \partial D$ are distinct, and $\varphi \colon \mathbb{H} \to D$ is a conformal transformation with $\varphi(0) = x$ and $\varphi(\infty) = y$. Explain why the definition of SLE_κ given by $\varphi(\gamma)$ where γ is an SLE_κ in $\mathbb{H}$ from 0 to ∞ is well-defined.

Problem 2.

- Suppose that B is a standard Brownian motion and $a < 0$. Show that $\sup_{t \geq 0}(B_t + at) < \infty$ almost surely.
- Suppose that (g_t) is the family of conformal maps which solve the Loewner equation with driving function $U_t = \sqrt{\kappa} B_t$ and, for each $x \in \mathbb{R}$, let $V_t^x = g_t(x) - U_t$ and $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$. For each $0 < x < y$, let $g(x, y) = \mathbb{P}[\tau_x = \tau_y]$. Show that if $g(1, 1 + \epsilon/2) > 0$ for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$ then $g(x, y) > 0$ for all $0 < x < y$.

Problem 3. Fix $T > 0$ and let $D \subseteq \mathbb{H}$ be a simply connected domain. Suppose that $(A_t)_{t \in [0,T]}$ is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing with $A_0 = \emptyset$, $\mathrm{hcap}(A_t) = 2t$ for all $t \in [0, T]$, and $A_T \subseteq D$. Let $\psi \colon D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Show that the family of compact $\mathbb{H}$-hulls $\widetilde{A}_t = \psi(A_t)$ for $t \in [0, T]$ is locally growing with $\widetilde{A}_0 = \emptyset$ and with

$$\mathrm{hcap}(\widetilde{A}_t) = \int_0^t 2(\psi_s'(U_s))^2 ds \quad \text{where} \quad \psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1} \quad \text{for each} \quad t \in [0, T]$$

and $\widetilde{g}_t$ is the unique conformal transformation $\mathbb{H} \setminus \widetilde{A}_t \to \mathbb{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$.

Problem 4. In the setting of the previous problem, show that

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Problem 5. Suppose that (A_t) is a non-decreasing family of $\mathbb{H}$-hulls which are locally growing and with $A_0 = \emptyset$. For each $t \geq 0$, let $a(t) = \mathrm{hcap}(A_t)$ and assume that $a(t)$ is C^1. For each $t \geq 0$, let g_t be the unique conformal transformation which takes $\mathbb{H} \setminus A_t$ to $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Show that the conformal maps (g_t) satisfy the ODE:

$$\partial_t g_t(z) = \frac{\partial_t a(t)}{g_t(z) - U_t}, \quad g_0(z) = z$$

for some continuous, real-valued function U_t. (Hint: perform a time-change so that the hulls are parameterized by capacity, apply Loewner's theorem as proved in class, and then invert the time change.)

Problem 6. Suppose that B is a standard Brownian motion starting from $B_0 = x > 0$. For each $a \in \mathbb{R}$, let $\tau_a = \inf\{t \geq 0 : B_t = a\}$.

- For $a < x < b$, explain why $\mathbb{P}[\tau_a < \tau_b] = (b - x)/(b - a)$.
- Using the Girsanov theorem, explain why the law of B weighted by $B_{\tau_0 \wedge \tau_b}$ is equal to that of a BES^3 process stopped upon hitting b. That is, if $\mathbb{P}$ denotes the law of B and we define the law $\widetilde{\mathbb{P}}$ using the Radon-Nikodym derivative
$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{B_{\tau_0 \wedge \tau_b}}{\mathbb{E}[B_{\tau_0 \wedge \tau_b}]}$$
then the law of B under $\widetilde{\mathbb{P}}$ is that of a BES^3 process stopped upon hitting b.
- Explain why a standard Brownian motion conditioned to be non-negative is a BES^3 process.
- More generally, explain why a BES^d process with $d < 2$ conditioned to be non-negative is a BES^{4-d} process.

Problem 7. Suppose that (g_t) is the family of conformal maps associated with an SLE_κ with driving function U_t, i.e., $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion. Fix $z \in \mathbb{H}$ and let $z_t = x_t + iy_t = g_t(z)$. Assume that $\rho \in \mathbb{R}$ is fixed. Use Itô's formula to show that
$$M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} y_t^{\rho^2/8\kappa} |U_t - z_t|^{\rho/\kappa}$$
is a continuous local martingale. (Hint: let
$$Z_t = \frac{(8 - 2\kappa + \rho)\rho}{8\kappa} \log g_t'(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(U_t - z_t),$$
compute dZ_t using Itô's formula, take its real part, and exponentiate.)

Problem 8. Assume that we have the setup of Problem 7. Let $\Upsilon_t = y_t / |g_t'(z)|$.

- Explain why Υ_t is proportional to $\text{dist}(z, \gamma([0, t]) \cup \partial\mathbb{H})$. More precisely, explain why
$$\frac{1}{4} \leq \frac{\Upsilon_t}{\text{dist}(z, \gamma([0, t]) \cup \partial\mathbb{H})} \leq 4.$$
- Let $S_t = \sin(\arg(z_t - U_t))$. Explain why
$$M_t = |g_t'(z)|^{(8-\kappa+\rho)\rho/(4\kappa)} \Upsilon_t^{\rho(\rho+8)/(8\kappa)} S_t^{-\rho/\kappa}.$$
- By considering the above martingale with the special choice $\rho = \kappa - 8$, show that if $\kappa > 8$ then the SLE_κ curve γ almost surely hits z. Conclude that γ fills all of $\mathbb{H}$. (Hint: recall that we showed in class that γ fills $\partial\mathbb{H}$. Deduce from this and the conformal Markov property that γ cannot separate z from ∞ without hitting it. Consider the behavior of S_t when γ is hitting a point on $\partial\mathbb{H}$ with either very large positive or negative values.)

Problem 9. In the context of Problem 4, show that
$$\partial_t \psi_t'(U_t) = \lim_{z \to U_t} \partial_t \psi_t'(z) = \frac{\psi_t''(U_t)^2}{2\psi_t'(U_t)} - \frac{4}{3} \psi_t'''(U_t).$$

Problem 10. Prove that the Dirichlet inner product is conformally invariant. That is, show that if $f, g \in C_0^\infty(D)$ and $\varphi\colon D \to \widetilde{D}$ is a conformal transformation, then

$$(f, g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla.$$

(Hint: use the change of variables formula and the Cauchy-Riemann equations.)

Problem 11. Suppose that $f \in H_0^1(D)$ with $\Delta f = 0$ in U in the *distributional sense*: if $g \in C_0^\infty(U)$, then $(f, \Delta g) = 0$ where $(\cdot, \cdot)$ denotes the L^2 inner product. Show that $f|_U$ is C^∞ in U and $\Delta f = 0$ in U in (the usual sense) using the following steps.

- Let ϕ be a radially symmetric C_0^∞ bump function supported in $\mathbb{D}$. In other words, $\phi(x) \geq 0$ for all x, $\phi(x)$ depends only on $|x|$, $\phi(x) = 0$ for $|x| \geq 1$, and $\int \phi = 1$. For each $\epsilon > 0$, let
$$f_\epsilon(x) = \epsilon^{-2} \int f(y) \phi\left(\frac{x-y}{\epsilon}\right) dy.$$
Explain why f_ϵ is C^∞ in $U_\epsilon = \{z \in U : \mathrm{dist}(z, \partial U) > \epsilon\}$.
- Fix $\delta > 0$ and let $x \in U_\delta$. Explain why $f_\epsilon(x)$ does not depend on the value of ϵ for $\epsilon \in (0, \delta)$. (Hint: compute the derivative of $f_\epsilon(x)$ respect to ϵ, recall the form of Δ when expressed in polar coordinates, and consider the radially symmetric function $\psi(r) = \int r\phi(r)dr$.)
- Conclude that if $g \in C_0^\infty(U)$, then the value of (f_ϵ, g) does not depend on ϵ for sufficiently small values of ϵ.
- Explain why the previous parts imply that f is C^∞ in U and $\Delta f = 0$ in U (in the usual sense).

Bonus Problem. Fill in the missing details to the proof of Theorem 11.3 from the lecture notes by proving the following.

- Suppose that γ is an $\mathrm{SLE}_{8/3}$ in $\mathbb{H}$ from 0 to ∞. Suppose that for every $A \in \mathcal{Q}_\pm$ with the property that there exists a smooth, simple curve $\beta\colon (0,1) \to \mathbb{H}$ such that $\mathbb{H} \cap \partial A = \beta((0,1))$ we have that
$$\mathbb{P}[\gamma([0,\infty]) \cap A = \emptyset] = (\psi_A'(0))^{5/8}. \tag{0.1}$$
Show that (0.1) holds for all $A \in \mathcal{Q}_\pm$.
- Using the conformal invariance of Brownian motion, carefully justify (11.6) in the lecture notes.
- Carefully justify the last sentence in the proof of Theorem 11.3.