Part III — Schramm–Loewner Evolutions
Definitions

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Schramm–Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

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0 Introduction

1 Conformal transformations

1.1 Conformal transformations

Definition (Conformal map)**.** Let U, V be domains in $\mathbb{C}$. We say a holomorphic function $f : U \to V$ is *conformal* if it is a bijection.

1.2 Brownian motion and harmonic functions

Definition (Harmonic function)**.** A function $f : \mathbb{R}^k \to \mathbb{R}$ is *harmonic* if it is C^2 and

$$\Delta f = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_k^2}\right) f = 0.$$

Definition (Complex Brownian motion)**.** We say a process $B = B^1 + iB^2$ is a *complex Brownian motion* if (B^1, B^2) is a standard Brownian motion in $\mathbb{R}^2$.

1.3 Distortion estimates for conformal maps

Definition (Compact hull)**.** A *compact hull* is a connected compact set $K \subseteq \mathbb{C}$ with more than one point such that $\mathbb{C} \setminus K$ is connected.

1.4 Half-plane capacity

Definition (Compact $\mathbb{H}$-hull)**.** A set $A \subseteq \mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if A is compact, $A = \mathbb{H} \cap \bar{A}$ and $\mathbb{H} \setminus A$ is simply connected. We write $\mathcal{Q}$ for the collection of compact $\mathbb{H}$-hulls.

Definition (Half-plane capacity)**.** Let $A \in \mathcal{Q}$. Then the *half-plane capacity* of A is defined to be

$$\mathrm{hcap}(A) = \lim_{z \to \infty} z(g_A(z) - z).$$

Thus, we have

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + \sum_{n=2}^{\infty} \frac{b_n}{z^n}.$$

2 Loewner's theorem

2.1 Key estimates

Definition. Suppose $(A_t)_{t\geq 0}$ is a family of compact $\mathbb{H}$-hulls. We say that (A_t) is

(i) *non-decreasing* if $s \leq t$ implies $A_s \subseteq A_t$.

(ii) *locally growing* if for all $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 \leq s \leq t \leq s + \delta \leq T$, we have

$$\operatorname{diam}(g_{A_s}(A_t \setminus A_s)) \leq \varepsilon.$$

This is a continuity condition.

(iii) *parametrized by half-plane capacity* if $\operatorname{hcap}(A_t) = 2t$ for all $t \geq 0$.

We write $\mathcal{A}$ be the set of families of compact $\mathbb{H}$-hulls which satisfy (i) to (iii). We write $\mathcal{A}_T$ for the set of such families defined on $[0, T]$.

2.2 Schramm–Loewner evolution

Definition (Conformal Markov property)**.** We say that (A_t) satisfy the *conformal Markov property* if

(i) Given $\mathcal{F}_t$, $(g_t(A_{t+s}) - U_t)_{s\geq 0} \overset{d}{=} (A_s)_{s\geq 0}$.

(ii) Scale-invariance: $(rA_{t/r^2})_{t\geq 0} \overset{d}{=} (A_t)$.

Definition (Schramm–Loewner evolution)**.** For $\kappa > 0$, SLE_κ is the random family of hulls encoded by $U_t = \sqrt{\kappa} B_t$, where B is a standard Brownian motion.

3 Review of stochastic calculus

4 Phases of SLE

Definition (Square Bessel process)**.** Let $X = (B^1, \ldots, B^d)$ be a d-dimensional standard Brownian motion. Then

$$Z_t = \|X_t\|^2 = (B_t^1)^2 + (B_t^2)^2 + \cdots + (B_t^d)^2$$

is a *square Bessel process* of dimension d.

Definition (Bessel process)**.** The *Bessel process* of dimension d, written BES^d, is

$$U_t = Z_t^{1/2}.$$

Definition (Equivalent events)**.** We say two events A, B are *equivalent* if

$$\mathbb{P}[A \setminus B] = \mathbb{P}[B \setminus A] = 0,$$

5 Scaling limit of critical percolation

6 Scaling limit of self-avoiding walks

Definition (Self-avoiding walk)**.** Let $G = (V, E)$ be a graph with uniformly bounded degree, and pick $x \in V$ and $n \in \mathbb{N}$. The self-avoiding in G starting from x of length n is the uniform measure on *simple* paths in G starting from x of length n.

Definition (Restriction property)**.** We say an SLE_κ satisfies the restriction property if whenever γ is an SLE_κ, for any $A \in \mathcal{Q}_\pm$, the law of $\psi_A(\gamma)$ conditional on V_A is that of an SLE_κ curve (for the same κ).

7 The Gaussian free field

Notation.

- C^∞ is the space of infinitely differentiable functions on $\mathbb{C}$
- C_0^∞ is the space of functions in C^∞ with compact support.
- If D is a domain, $C_0^\infty(D)$ is the functions in C_0^∞ supported in D.

Definition (Dirichlet inner product)**.** Let $f, g \in C_0^\infty$. The *Dirichlet inner product* of f, g is

$$(f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x)\,\mathrm{d}x.$$

Definition ($H_0^1(D)$)**.** We write $H_0^1(D)$ for the Hilbert space completion of $C_0^\infty(D)$ with respect to $(\,\cdot\,,\,\cdot\,)_\nabla$.