Part III: Riemannian Geometry (Lent 2017)

Example Sheet 1

1. (i) Prove that any connection ∇ on M uniquely determines a covariant derivative on the cotangent bundle T^*M (still to be denoted by ∇), such that $\nabla_X : \Omega^1(M) \to \Omega^1(M)$ satisfies $X\langle \alpha, Y\rangle = \langle \nabla_X \alpha, Y\rangle + \langle \alpha, \nabla_X Y\rangle$. Here $\alpha \in \Omega^1(M)$, X, Y are vector fields on M, and $\langle \cdot, \cdot \rangle$ denotes the evaluation of a 1-form on a tangent vector. In particular, prove that if $\alpha = \sum_j \alpha_j dx^j$ in local coordinates and Γ^i_{jk} are the coefficients of ∇ on the tangent bundle then $(\nabla_X \alpha)_j = \sum_{ik}\big(\dfrac{\partial \alpha_j}{\partial x^k} - \Gamma^i_{jk}\alpha_i\big)X^k$.

   Show further that if ∇ is the Levi–Civita of some metric (g_{ij}) on M then the induced connection is compatible with the dual metric $g = (g^{ij})$ on T^*M in the sense that $X\big(g(\alpha, \beta)\big) = g(\nabla_X \alpha, \beta) + g(\alpha, \nabla_X \beta)$, for each $\alpha, \beta \in \Omega^1(M)$ and vector field X. (It is natural to call this induced connection the Levi–Civita on T^*M).

   (ii) Generalize the definition of the induced connection (still denoted by ∇) to the case of $(0, q)$-tensor bundle $T^*M^{\otimes q}$, $q > 1$, by writing out an appropriate version of 'Leibniz formula' for ∇. Give the expression for ∇ in local coordinates. Show that if ∇ is the Levi–Civita of a Riemannian metric g on M then $\nabla g = 0$. (Thus a Riemannian metric is covariant constant, or 'parallel', with respect to its Levi–Civita connection.)

2. (i) Let M be a Riemannian manifold. Show that the Levi–Civita covariant derivative of $R(X, Y) \in \Gamma(\operatorname{End} TM)$ is given by $\nabla_Z R(X, Y) = [\nabla_Z, R(X, Y)] - R(\nabla_Z X, Y) - R(X, \nabla_Z Y)$. Deduce from this a version of the *second Bianchi identity* for the Levi–Civita connection

   $$\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0. \qquad (*)$$

   (ii) When $\dim M \geq 3$, show, using $(*)$, that if $\mathrm{Ric} = fg$ for some smooth function f, then f is constant (M then is said to be an *Einstein manifold*).
   (You might like to consider a map $\delta : \Gamma(\mathrm{Sym}^2 T^*M) \to \Gamma(T^*M) = \Omega^1(M)$ defined by $(\delta h)(X) = -\sum_{i=1}^n (\nabla_{e_i} h)(e_i, X)$, where $\{e_i\}$ is any local orthonormal frame field on M, and put $h = \mathrm{Ric}$.)

3. For this question, recall that the Riemann curvature tensor $(R_{ij,kl})$ of (M, g) defines a symmetric bilinear form on the fibres of $\Lambda^2 T^*M$. Show that if $\dim M = 3$ then the Riemann curvature is determined at each point of M by the Ricci curvature $\mathrm{Ric}(g)$. [Hint: the assignment of $\mathrm{Ric}(g)$ to $R(g)$ is a linear map, at each point of M. A special feature of the dimension 3 is that the spaces of 1-forms and 2-forms on $\mathbb{R}^3$ have the same dimension.]

4. Prove that the scalar curvature $s(p)$, $p \in M$ is given by

   $$s(p) = \frac{n}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(x, x)\, dx$$

   where ω_{n-1} is the volume of the unit sphere S^{n-1} in T_pM.

5. Let G be a Lie group endowed with a Riemannian metric g which is left and right invariant and let X, Y, Z be left invariant vector fields of G.
   (*i*) Show that $g([X, Y], Z) + g(Y, [X, Z]) = 0$. (Consider the flow of X.)
   (*ii*) Show that $\nabla_X X = 0$. (Hint: consider $g(Y, \nabla_X X)$.)

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(*iii*) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

(*iv*) Prove that $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

(*v*) Suppose that X and Y are orthonormal, and let $K(\sigma)$ be the sectional curvature of the 2-plane σ spanned by X and Y. Prove that

$$K(\sigma) = \frac{1}{4}|[X, Y]|_g^2$$

6. Let M be a Riemannian manifold. M is said to be *locally symmetric* if $\nabla R = 0$, where $R = (R_{ij,kl})$ is the curvature tensor of M.

   (*i*) Let M be a locally symmetric space and let $\gamma : [0, \ell] \to M$ be a geodesic on M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel vector field along γ.

   (*ii*) Suppose that M is locally symmetric, connected and 2-dimensional. Prove that M has constant sectional curvature.

   (*iii*) Prove that if M has constant sectional curvature, then it is locally symmetric.

7. Let N be a connected Riemannian manifold and let $f : M \to N$ be a local diffeomorphism. Show that one can put a Riemannian metric on M such that f becomes a local isometry. Show that if M is complete then N is complete. Is the converse true? Is the converse true if f is a covering map?

8. A geodesic $\gamma : [0, \infty) \to M$ is called a *ray* if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in (0, \infty)$. Show that if M is complete and non-compact, there is a ray leaving from every point in M.

9. A Riemannian manifold M is said to be *homogeneous* if given p and q in M, there exists an isometry of M taking p to q. Show that a homogeneous Riemannian manifold is complete.

10. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and let $h : S^3 \to S^3$ be given by

$$h(z_1, z_2) = (e^{2\pi i/q} z_1, e^{2\pi ir/q} z_2),$$

where q and r are co-prime integers and $q > 1$.

(i) Show that $G = \{\text{id}, h, \ldots, h^{q-1}\}$ is a group of isometries of S^3 (equipped with the standard metric) that acts in such a way that S^3/G is a manifold and the projection $p : S^3 \to S^3/G$ is a local diffeomorphism. (The manifolds S^3/G are called *lens spaces*.)

(ii) Consider on S^3/G the metric induced by p. Show that all the geodesics of S^3/G are closed, but they could have different lengths.

11. Let M be a complete Riemannian manifold and let $N \subset M$ be a closed submanifold. Let $p \in M$, $p \notin N$, and let $d(p, N)$ be the distance from p to N. Show that there exists a point $q \in N$ such that $d(p, q) = d(p, N)$. Show that a minimizing geodesic between p and q must be orthogonal to N at q.

12. Let M be an orientable Riemannian manifold of even dimension and positive sectional curvature. Show that any closed geodesic in M is homotopic to a closed curve with length strictly smaller than that of γ.

13. Suppose that for every smooth Riemannian metric on a manifold M, M is complete. Show that M is compact (Hint: think about rays as in Problem 8.)

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Part III: Riemannian Geometry (Lent 2017)

Example Sheet 2

1. Give an example of a *non-compact* complete Riemannian manifold with Ricci curvature (strictly) positive-definite at each point.

2. Let G be a Lie group whose Lie algebra $\mathfrak{g}$ has trivial centre. Suppose that G admits a bi-invariant (i.e. left- and right-invariant) Riemannian metric. Show that G and its universal cover are compact. Deduce that $SL(2,\mathbb{R})$ admits no bi-invariant metric.

3. (i) Show that the Hodge star on $\Lambda^2(\mathbb{R}^4)^*$ determines an orthogonal decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda^+ \oplus \Lambda^-$ into the ± 1 eigenspaces and $\dim \Lambda^+ = \dim \Lambda^- = 3$. Deduce that on every oriented 4-dimensional Riemannian manifold M there is a decomposition of 2-forms $\Omega^2(M) = \Omega^+ \oplus \Omega^-$, so that $\alpha \wedge \alpha = \pm|\alpha|_g^2\, \omega_g$, for every $\alpha \in \Omega^\pm$, where ω_g is the volume form. (2-forms in the subspaces $\Omega^\pm$ are called, respectively, the *self-* and *anti-self-dual* forms on M.)

   (ii) Now assume that M is a *compact* 4-dimensional oriented Riemannian manifold. Show that the expression $\int_M \alpha \wedge \beta$, for closed $\alpha, \beta \in \Omega^2(M)$, induces a well-defined symmetric bilinear form on the de Rham cohomology $H^2_{\text{dR}}(M)$. Let $(b^+(M), b^-(M))$ denote the signature of this bilinear form. Show that $b^\pm(M) = \dim \mathcal{H}^\pm$, where $\mathcal{H}^\pm$ denotes the space of harmonic (anti-)self-dual forms on M.

4. (i) Derive explicit formulas for $*$, δ, and Laplace–Beltrami operator in Euclidean space. In particular, show that if
$$\alpha = \sum_{i_1<\ldots<i_p} \alpha_I dx_{i_1} \wedge \ldots \wedge dx_{i_p} \qquad (I = i_1, \ldots, i_p),$$
   then
$$\Delta\alpha = -\sum_{i_1<\ldots<i_p} \Big(\sum_{i=1}^{n} \frac{\partial^2 \alpha_I}{\partial {x_i}^2} \Big)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$
   (ii) For $u, v \in C^\infty(M)$, show that $\Delta(uv) = (\Delta u)v - 2\langle du, dv\rangle_g + u\Delta v$ (M is an oriented Riemannian manifold).

5. Calculate explicitly the expression of the Laplacian for functions:

   (a) on the hyperbolic plane $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, where the metric is $g(x,y) = \dfrac{dx^2 + dy^2}{y^2}$;

   (b) on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, in local coordinates given by stereographic projections. (The metric on S^n is the standard 'round' metric induced by the embedding.)

   *Express the Laplacian on the Euclidean $\mathbb{R}^{n+1} \setminus \{0\}$ in terms of the Laplacian on the unit sphere S^n (recall that the Euclidean metric can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^{n+1}$, and dS^2 is the 'round' metric on S^n). Deduce a formula for the Laplacian on spherically-symmetric functions $f(r)$.

6. Let α and β be n-forms on a compact oriented manifold M^n such that $\int_M \alpha = \int_M \beta$. Prove that α and β differ by an exact form. (Stokes' theorem may be assumed.)

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http://www.dpmms.cam.ac.uk/∼agk22/riemannian2.pdf

7. Show that the partial differential equation $\Delta f = \varphi$ for a function $f \in C^\infty(M)$ on a compact oriented Riemannian manifold (M, g), with a given $\varphi \in C^\infty(M)$, has a solution if and only if $\int_M \varphi\, \omega_g = 0$. ($\omega_g$ denotes the volume form.) Is the solution unique? *Discuss the solvability of $\Delta(\Delta f) = \varphi$ when $f, \varphi \in C^\infty(M)$ and more generally when f, φ are p-forms.

8. Let M be a compact oriented Riemannian manifold and F a diffeomorphism of M which preserves the volume form on M. We say that a form $\alpha \in \Omega^p(M)$ is *invariant* under F if $\alpha \circ F = \alpha$ and we say that the Laplacian Δ is *invariant* under F if $\Delta\alpha \circ F = \Delta(\alpha \circ F)$, for all $\alpha \in \Omega^p(M)$. Suppose that Δ and α are invariant under F and α is L^2-orthogonal to each harmonic form on M. Prove that there is an invariant solution η of $\Delta\eta = \alpha$.

9. (Holonomy transformations.) Show that the parallel transport defined by the Levi–Civita connection over any closed loop based at $x \in M$ defines an orthogonal linear transformation of T_xM which is in $SO(T_xM)$ when M is oriented.

   An *orthogonal almost complex structure* on a manifold (M, g) is an endomorphism J of its tangent bundle TM such that $J^2 = -1$ and $g(JX, JY) = g(X, Y)$, for all $X, Y \in \mathrm{Vect}(M)$. If M admits such J, show that M is orientable and even-dimensional. Show that $\omega = g(J\cdot, \cdot)$ defines a 2-form on M with $\omega^n \neq 0$ at each point ($\dim M = 2n$).

   Show further that the following statements are equivalent:
   (a) $\nabla J = 0$,
   (b) $\nabla\omega = 0$,
   (c) the parallel transport defined by ∇ along closed loops is represented by elements of $U(n) \subset SO(2n)$ (after some natural identifications).

   Here ∇ denotes the (induced) Levi-Civita connection on respective vector bundles. (Each of (a),(b),(c) is in fact equivalent to M being a *Kähler complex manifold* with Kähler form ω and J corresponding to multiplication by i in local complex coordinates.)

10. (i) For any two bilinear forms h, k on tangent spaces to M, define a $(0,4)$-tensor $(h\cdot k)(X,Y,Z,T) = h(X,Z)k(Y,T) + h(Y,T)k(X,Z) - h(X,T)k(Y,Z) - h(Y,Z)k(X,T)$, where $X, Y, Z, T \in T_xM$. Show that the curvature tensor $R = (R_{ij,kl})$ of a Riemannian n-dimensional manifold (M, g), $n \geq 4$, has an $SO(n)$-invariant, orthogonal decomposition $R = \dfrac{s}{2n(n-1)}\, g \cdot g + \dfrac{1}{n-2}\,(\mathrm{Ric} - \dfrac{s}{n} g) \cdot g + W$, where W satisfies $W(X,Y,Z,T) + W(Z,X,Y,T) + W(Y,Z,X,T) = 0$ (1st Bianchi identity) and $\sum_{i=1}^n W(X, e_i, Y, e_i) = 0$ for all $X, Y, Z, T \in T_xM$ and where e_i is an orthonormal basis of T_xM. (W is called the *Weyl tensor* of (M, g)).

    (ii) Suppose that $\dim M = 4$ and M is oriented. We consider R as a symmetric bilinear form on the fibres of $\Lambda^2 T^*M$. Let a bilinear form, $B : \Lambda^+ \times \Lambda^- \to \mathbb{R}$ be the restriction of R defined using the decomposition into self- and anti-self-dual forms as in Question 3. Show that B is equivalent to the trace-free part Ric_0 of the Ricci curvature (with respect to the metric g), that is, $g^{ik}(\mathrm{Ric}_0)_{kj} = g^{ik}\,\mathrm{Ric}_{kj} - \frac{1}{4} s\, \delta^i_j$ in local coordinates, where (g^{ij}) denotes the inverse matrix of $g = (g_{ij})$ (the summation convention is assumed).

    Show further that R, with respect to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, has the form

$$\begin{pmatrix} W^+ + \frac{s}{12} I & B \\ B^T & W^- + \frac{s}{12} I \end{pmatrix},$$

    where $W^\pm : \Lambda^\pm \times \Lambda^\pm \to \mathbb{R}$ are symmetric bilinear forms with $\mathrm{tr}\, W^- + \mathrm{tr}\, W^+ = 0$ and $W = W^+ \oplus W^-$ is an $SO(4)$-invariant orthogonal decomposition of the Weyl tensor.

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Part III: Riemannian Geometry (Lent 2017)

Example Sheet 3

1. Show that the volume form of a Riemannian manifold is parallel, $\nabla\omega_g = 0$, with respect to the Levi–Civita connection of g.

2. (normal frame fields) Show that for each point $p \in M$ of a Riemannian manifold there exists an orthonormal frame field $e_1, \ldots, e_n$ defined on some neighbourhood of p and such that ∇e_i vanishes at p for each i. [Hint: you might like to first verify that the coefficients Γ^i_{jk} of the Levi–Civita connection in the geodesic coordinates at $p \in M$, vanish at p.]

3. Show that for the Levi–Civita connection, the following diagram commutes

$$\begin{array}{ccc} \Gamma(\Lambda^p T^*M) & \xrightarrow{\ \nabla\ } & \Gamma(T^*M \otimes \Lambda^p T^*M) \\ & \searrow^{d} & \downarrow \text{alt} \\ & & \Gamma(\Lambda^{p+1} T^*M), \end{array}$$

where $\text{alt}(\xi \otimes \alpha) = \xi \wedge \alpha$ denotes projection to the subspace of anti-symmetric tensors ($p > 0$). Deduce the formula for the exterior derivative of one-forms

$$d\alpha(X, Y) = (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X)$$

as stated in the Lectures. * Show that these results hold for any torsion-free connection ∇ on M.

4. (*i*) The *divergence* of a $(1, 3)$-tensor R may be defined as a $(0, 3)$-tensor (i.e. a tri-linear function of tangent vectors X, Y, Z)

$$(\operatorname{div} R)(X, Y, Z) = \operatorname{tr}\big(V \to (\nabla_V R)(X, Y, Z)\big).$$

Show that if $R = (R^i_{j,kl})$ is the (1, 3)-curvature tensor, then

$$(\operatorname{div} R)(X, Y, Z) = (\nabla_X \operatorname{Ric})(Y, Z) - (\nabla_Y \operatorname{Ric})(X, Z).$$

Deduce that $\operatorname{div} R = 0$ if and only if the Ricci curvature satisfies:

$$(\nabla_X \operatorname{Ric})(Y, Z) = (\nabla_Y \operatorname{Ric})(X, Z) \quad \text{for all } X, Y, Z.$$

(*ii*) (M. Berger) On a closed Riemannian manifold (M, g) show that if $\operatorname{div} R = 0$ and the sectional curvature $K \geq 0$, then $\nabla \operatorname{Ric} = 0$. [Hint: use the relation $2 \operatorname{tr} \nabla \operatorname{Ric} = ds$ to conclude $\operatorname{tr} \nabla \operatorname{Ric} = 0$. Here $(\operatorname{tr} \nabla \operatorname{Ric})(X) = \sum_i \nabla_{e_i} \operatorname{Ric}(e_i, X))$, for orthonormal e_i.]

5. Show that the co-differential δ on p-forms may be equivalently defined by

$$(\delta\eta)(X_2, \ldots, X_p) = -\sum_{i=1}^{n} (\nabla_{e_i}\eta)(e_i, X_2, \ldots, X_p) = -\sum_{i=1}^{n} \big(i(e_i)(\nabla_{e_i}\eta)\big)(X_2, \ldots, X_p)$$

where e_i is some/any local orthonormal frame field. (In particular, δ and Δ, are independent of the choice of orientation and in fact may be defined on non-orientable manifolds too.)

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http://www.dpmms.cam.ac.uk/∼agk22/riemannian3.pdf

6. (i) Show that $\mathrm{Hol}^0(M)$ is a normal subgroup of $\mathrm{Hol}(M)$ and that there is a natural, surjective group homomorphism $\pi_1(M) \to \mathrm{Hol}(M)/\,\mathrm{Hol}^0(M)$.
   (ii) Show that $\mathrm{Hol}(\widetilde{M}) = \mathrm{Hol}^0(M)$, where $\widetilde{M}$ denotes the universal Riemannian cover of M.

7. Determine the holonomy of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ (with the round metric).

8. A theorem due to de Rham asserts that if we decompose the tangent bundle of a Riemannian manifold (M, g) into irreducible components according to the holonomy representation, $TM = \tau_1 \oplus \ldots \oplus \tau_k$, then around each point $p \in M$ there is a neighbourhood U that decomposes into a Riemannian product, $(U, g) = (U_1 \times \ldots \times U_k, g_1 + \ldots + g_k)$, so that $TU_i = \tau_i$ for each i.

   Deduce that if the holonomy group of (M, g) has no invariant subspaces (i.e. the holonomy representation is irreducible) and the Ricci tensor is parallel $\nabla \mathrm{Ric} = 0$, then M is Einstein ($\mathrm{Ric} = \lambda g$, for some $\lambda \in \mathbb{R}$). [Hint: eigenvalues.]

9.* (This question requires Frobenius theorem.) Suppose that (M, g) admits a parallel field of k-dimensional tangent subspaces ($k \leq n-1$), i.e. a rank k subbundle of TM invariant under parallel transport. Show that every such distribution is integrable (involutive).

10. Using the skew-symmetric linear maps

$$X \wedge Y : T_pM \to t_pM, \qquad X \wedge Y(V) = g(X, V)Y - g(Y, V)X,$$

show that $\Lambda^2 T_pM \cong \mathfrak{so}(T_pM)$. (Elements $X \wedge Y$ of $\Lambda^2 T_pM$ are sometimes called bi-vectors.) Now let $\mathfrak{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM$ be a linear endomorphism induced by the curvature (0,4)-tensor. Deduce that the image of $\mathfrak{R}$ is contained in the holonomy algebra $\mathfrak{R}(\Lambda^2 T_pM) \subset \mathfrak{hol}_p(M)$.

11. (i) Show that a compact Riemannian manifold with irreducible holonomy representation and $\mathrm{Ric} \geq 0$ has finite fundamental group.
   (ii) Let G be a compact Lie group endowed with a bi-invariant metric. Show that G admits a finite cover by $G' \times T^k$, where G' is compact simply connected and T^k is a torus.
   * Show that if G has finite fundamental group, then its Lie algebra has trivial centre.

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