Part III — Ramsey Theory
Theorems

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What happens when we cut up a mathematical structure into many 'pieces' ? How big must the original structure be in order to guarantee that at least one of the pieces has a specific property of interest? These are the kinds of questions at the heart of Ramsey theory. A prototypical result in the area is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes, there is a class that contains arbitrarily long arithmetic progressions.

The course will cover both classical material and more recent developments in the subject. Some of the classical results that I shall cover include Ramsey's theorem, van der Waerden's theorem and the Hales–Jewett theorem; I shall discuss some applications of these results as well. More recent developments that I hope to cover include the properties of non-Ramsey graphs, topics in geometric Ramsey theory, and finally, connections between Ramsey theory and topological dynamics. I will also indicate a number of open problems.

Pre-requisites

There are almost no pre-requisites and the material presented in this course will, by and large, be self-contained. However, students familiar with the basic notions of graph theory and point-set topology are bound to find the course easier.

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0 Introduction

1 Graph Ramsey theory

1.1 Infinite graphs

Theorem (Ramsey's theorem)**.** Whenever we k-colour $\mathbb{N}^{(2)}$, there exists an infinite monochromatic set X, i.e. given any map $c : \mathbb{N}^{(2)} \to [k]$, there exists a subset $X \subseteq \mathbb{N}$ such that X is infinite and $c|_{X^{(2)}}$ is a constant function.

Corollary (Bolzano-Weierstrass theorem)**.** Let $(x_i)_{i\geq 0}$ be a bounded sequence of real numbers. Then it has a convergent subsequence.

Theorem (Ramsey's theorem for r sets)**.** Whenever $\mathbb{N}^{(r)}$ is k-coloured, there exists an infinite monochromatic set, i.e. for any $c : \mathbb{N}^{(r)} \to [k]$, there exists an infinite $X \subseteq \mathbb{N}$ such that $c|_{X^{(r)}}$ is constant.

Theorem (Canonical Ramsey theorem)**.** For any $c : \mathbb{N}^{(2)} \to \mathbb{N}$, there exists an infinite $X \subseteq \mathbb{N}$ such that one of the following hold:

(i) $c|_{X^{(2)}}$ is constant.

(ii) $c|_{X^{(2)}}$ is injective.

(iii) $c(ij) = c(k\ell)$ iff $i = k$ for all $i, j, k, \ell \in X$.

(iv) $c(ij) = c(k\ell)$ iff $j = \ell$ for all $i, j, k, \ell \in X$.

Theorem (Higher dimensional canonical Ramsey theorem)**.** Let $c : \mathbb{N}^{(r)} \to \mathbb{N}$ be a colouring. Then there exists $D \subseteq [r]$ and an infinite subset $X \subseteq \mathbb{N}$ such that for all $x, y \in X^{(r)}$, we have $c(x) = c(y)$ if $\{i : x_i = y_i\} \supseteq D$, where $x = \{x_1 < x_2 < \cdots < x_r\}$ (and similarly for y).

1.2 Finite graphs

Theorem (Finite Ramsey theorem)**.** For all n, we have $R(n) < \infty$.

Theorem. We have

$$R(n, m) \leq R(n - 1, m) + R(n, m - 1).$$

for all $n, m \in \mathbb{N}$. Consequently, we have

$$R(n, m) \leq \binom{n + m - 1}{n - 2}.$$

Theorem. We have $R(n) \geq \sqrt{2}^n$ for sufficiently large $n \in \mathbb{N}$.

Theorem. We have

$$R(3, n) \leq \frac{100n^2}{\log n}$$

for sufficiently large $n \in \mathbb{N}$.

2 Ramsey theory on the integers

Theorem (van der Waerden theorem)**.** Let $m, k \in N$. Then there is some $N = W(m, k)$ such that whenever $[N]$ is k-coloured, then there is a monochromatic arithmetic progression of length n.

Theorem (Gallai)**.** Whenever $\mathbb{N}^d$ is k-coloured, there exists a monochromatic (homothetic) copy of S for each finite $S \subseteq \mathbb{N}^d$.

Theorem (Hales–Jewett theorem)**.** For all $m, k \in \mathbb{N}$, there exists $N = HJ(m, k)$ such that whenever $[m]^N$ is k-coloured, there exists a monochromatic line.

Theorem (Gallai)**.** Whenever $\mathbb{N}^d$ is k-coloured, there exists a monochromatic (homothetic) copy of S for each finite $S \subseteq \mathbb{N}^d$.

Theorem (Szemerédi theorem)**.** Let $\delta > 0$ and $m \in \mathbb{N}$. Then there exists some $N = S(m, \delta) \in \mathbb{N}$ such that any subset $A \subseteq [N]$ with $|A| \geq \delta N$ contains an m-term arithmetic progression.

Theorem. For any $c : \mathbb{N} \to \mathbb{N}$, there exists a m-AP on which either

(i) c is constant; or

(ii) c is injective.

Theorem. For any $c : \mathbb{N}^{(2)} \to \{\text{red}, \text{blue}\}$, either

(i) There exists a blue m-AP for each $m \in \mathbb{N}$; or

(ii) There exists arbitrarily large red sets.

3 Partition Regularity

Theorem (Rado's theorem)**.** A matrix A is partition regular iff it has the column property.

Theorem. If $a_1, \cdots, a_n \in \mathbb{Q} \setminus \{0\}$, then $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is partition regular iff there exists a non-empty $I \subseteq [n]$ such that

$$\sum_{i \in I} a_i = 0.$$

Proposition. If $a_2, \cdots, a_n \in \mathbb{Q}\setminus\{0\}$ and $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ is partition regular, then

$$\sum_{i \in I} a_i = 0$$

for some non-empty I.

Proposition. The equation $\begin{pmatrix} 1 & \lambda & -1 \end{pmatrix}$ is partition regular for all $\lambda \in \mathbb{Q}$.

Theorem. If $a_1, \cdots, a_n \in \mathbb{Q}$, then $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is partition regular iff there exists a non-empty $I \subseteq [n]$ such that

$$\sum_{i \in I} a_i = 0.$$

Proposition. If A is an $m \times n$ matrix with rational entries which is partition regular, then A has the columns property.

Proposition. Let $m, p, c \in \mathbb{N}$. Then whenever $\mathbb{N}$ is finitely coloured, there exists a monochromatic (m, p, c)-set.

Corollary (Finite sum theorem)**.** For every fixed m, whenever we finitely-colour $\mathbb{N}$, there exists $x_1, \cdots, x_m$ such that $FS(x_1, \cdots, x_m)$ is monochromatic.

Proposition. If A is a rational matrix with the columns property, then there is some $m, p, c \in \mathbb{N}$ such that $Ax = 0$ has a solution inside any (m, p, c) set, i.e. all entries of the solution lie in the (m, p, c) set.

Theorem (Rado's theorem)**.** A matrix A is partition regular iff it has the column property.

Corollary (Consistency theorem)**.** If A, B are partition regular in independent variables, then

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is partition regular. In other words, we can solve $Ax = 0$ and $By = 0$ simultaneously in the same colour class.

Corollary. Whenever $\mathbb{N}$ is finitely-coloured, one colour class contains solutions to *all* partition regular systems!

4 Topological Dynamics in Ramsey Theory

Theorem (Topological van der Waerden)**.** Let (X, T) be a dynamical system. Then there exists an $\varepsilon > 0$ such that whenever $r \in \mathbb{N}$, then we can find $x \in X$ and $n \in \mathbb{N}$ such that $\rho(x, T^{in}x) < \varepsilon$ for all $i = 1, \cdots, r$.

Proposition. $(\bar{X}, T)$ is a dynamical system.

Corollary (van der Waerden theorem)**.** Let $r, k \in \mathbb{N}$. Then whenever $\mathbb{Z}$ is k-coloured, then there is a monochromatic arithmetic progression of length r.

Proposition. We have $c' \in \bar{c}$ iff $\mathrm{Seq}(c') \subseteq \mathrm{Seq}(c)$.

Proposition. Every dynamical system (X, T) has a minimal point.

Lemma. If (X, T) is a minimal system, then for all $\varepsilon > 0$, there is some $m \in \mathbb{N}$ such that for all $x, y \in X$, we have

$$\min_{|s| \leq m} \rho(T^s x, y) < \varepsilon.$$

Theorem (Topological van der Waerden)**.** Let (X, T) be a dynamical system. Then there exists an $\varepsilon > 0$ such that whenever $r \in \mathbb{N}$, then we can find $x \in X$ and $n \in \mathbb{N}$ such that $\rho(x, T^{in}x) < \varepsilon$ for all $i = 1, \cdots, r$.

Theorem (Topological Hindman's theorem)**.** Let (X, T) be a dynamical system, and suppose $X = \bar{x}$ for some $x \in X$. Then for any minimal subsystem $Y \subseteq X$, then there is some $y \in Y$ such that x and y are proximal.

Proposition. A colouring $c : \mathbb{Z} \to [k]$ is minimal iff c has the bounded gaps property.

Theorem (Hindman's theorem)**.** If $c : \mathbb{N} \to [k]$ is a k-colouring, then there exists an infinite $A \subseteq \mathbb{N}$ such that $FS(A)$ is monochromatic.

Theorem (Topological Hindman's theorem)**.** If (X, T) is a dynamical system, $X = \bar{x}$ and $Y \subseteq X$ is minimal, then there exists $y \in Y$ such that x and y are proximal.

Theorem (Idempotent theorem)**.** If $E \subseteq X^X$ is a compact semi-group, then there exists $g \in E$ such that $g^2 = g$.

5 Sums and products*

Theorem (Moreira, 2016)**.** If $\mathbb{N}$ is k-coloured, then there exists infinitely many x, y such that $\{x, x + y, xy\}$ is monochromatic.

Proposition. If A is piecewise syndetic and $A = X \cup Y$, then one of X and Y is piecewise syndetic.

Proposition. Let $A \subseteq \mathbb{N}$ be piecewise syndetic. Then for all $m \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that the set

$$A^* = \{x \in \mathbb{N} : x, x + d, \cdots, x + md \in A\}.$$

is piecewise syndetic.