Part III — Ramsey Theory
Definitions

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

What happens when we cut up a mathematical structure into many 'pieces' ? How big must the original structure be in order to guarantee that at least one of the pieces has a specific property of interest? These are the kinds of questions at the heart of Ramsey theory. A prototypical result in the area is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes, there is a class that contains arbitrarily long arithmetic progressions.

The course will cover both classical material and more recent developments in the subject. Some of the classical results that I shall cover include Ramsey's theorem, van der Waerden's theorem and the Hales–Jewett theorem; I shall discuss some applications of these results as well. More recent developments that I hope to cover include the properties of non-Ramsey graphs, topics in geometric Ramsey theory, and finally, connections between Ramsey theory and topological dynamics. I will also indicate a number of open problems.

Pre-requisites

There are almost no pre-requisites and the material presented in this course will, by and large, be self-contained. However, students familiar with the basic notions of graph theory and point-set topology are bound to find the course easier.

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0 Introduction

1 Graph Ramsey theory

1.1 Infinite graphs

Definition ($\mathbb{N}$ and $[n]$)**.** We write

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\}.$$

We also write

$$[n] = \{1, 2, 3, \cdots, n\}.$$

Notation. For a set X, we write $X^{(r)}$ for the subsets of X of size r. The elements are known as *r-sets*.

Definition (Graph)**.** A graph G is a pair (V, E), where $E \subseteq V^{(2)}$.

Definition (k-colouring)**.** A k-colouring of a set X is a map $c : X \to [k]$.

Definition (Monochromatic set)**.** Let X be a set with a k-colouring. We say a subset $Y \subseteq X$ is *monochromatic* if the colouring restricted to Y is constant.

1.2 Finite graphs

Definition (Ramsey number)**.** We let $R(n) = R(K_n)$ to be the smallest $N \in \mathbb{N}$ whenever we red-blue colour the edges of K_N, then there is a monochromatic copy of K_n.

Definition (Off-diagonal Ramsey number)**.** We define $R(n, m) = R(K_n, K_m)$ to be the minimum $N \in \mathbb{N}$ such that whenever we red-blue colour the edges of K_N, we either get a red K_n or a blue K_m.

Definition ($R(G, H)$)**.** Let G, H be graphs. Then we define $R(G, H)$ to be the smallest N such that any red-blue colouring of K_N has either a red copy of G or a blue copy of H.

2 Ramsey theory on the integers

Definition (Focused progression)**.** We say a collection of arithmetic progressions $A_1, A_2, \cdots, A_r$ of length m with

$$A_i = \{a_i, a_i + d_i, \cdots, a_i + (m-1)d_i\}$$

are *focused* at f if $a_i + md_i = f$ for all $1 \leq i \leq r$.

Definition (Colour focused progression)**.** If $A_1, \cdots, A_r$ are focused at f, and each A_i is monochromatic and no two are the same colour, then we say they are *colour focused* at f.

Definition (Homothetic copy)**.** Given a finite $S \subseteq \mathbb{N}^d$, a *homothetic copy* of S is a set of the form

$$\ell S + M,$$

where $\ell, M \in \mathbb{N}^d$ and $\ell \neq 0$.

Definition (Combinatorial line)**.** A *combinatorial line* $L \subseteq [m]^n$ is a set of the form

$$\{(x_1, \cdots, x_n n) : x_i = x_j \text{ for all } i, j \in I; x_i = a_i \text{ for all } i \not\in I\}$$

for some fixed non-empty set of coordinates $I \subseteq [n]$ and $a_i \in [m]$.

I is called the set of *active coordinates*.

Definition (Focused lines)**.** We say lines $L_1, \cdots, L_r$ are *focused* at $f \in [m]^N$ if $L_i^+ = f$ for all $i = 1, \cdots, r$.

Definition (Colour focused lines)**.** If $L_1, \cdots, L_r$ are focused lines, and $L_i \setminus \{L_i^+\}$ is monochromatic for each $i = 1, \cdots, r$ and all these colours are distinct, then we say $L_1, \cdots, L_r$ are *colour focused* at f.

Definition (Density)**.** For $A \subseteq \mathbb{N}$, we let the *density* of A as

$$\bar{d}(A) = \limsup_{(b-a)\to\infty} \frac{A \cap [a, b]}{|b - a|}.$$

3 Partition Regularity

Definition (Partition regularity)**.** We say an $m \times n$ matrix A over $\mathbb{Q}$ is *partition regular* if whenever $\mathbb{N}$ is finitely coloured, there exists an $x \in \mathbb{N}^n$ such that $Ax = 0$ and x is monochromatic, i.e. all coordinates of x have the same colour.

Definition (Columns property)**.** Let

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ c^{(1)} & c^{(2)} & \cdots & c^{(n)} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}.$$

We say A has the *columns property* if there is a partition $[n] = B_1 \cup B_2 \cup \cdots \cup B_d$ such that

$$\sum_{i \in B_s} c^{(i)} \in \operatorname{span}\{c^{(i)} : c^{(i)} \in B_1 \cup \cdots \cup B_{s-1}\}$$

for $s = 1, \cdots, d$. In particular,

$$\sum_{i \in B_1} c^{(i)} = 0$$

Definition ((m, p, c)-set)**.** For $m, p, c \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is an (m, p, c)-set with generators $x_1, \cdots, x_m$ if S has the form

$$S = \left\{ \sum_{i=0}^{m} \lambda_i x_i : \begin{array}{c} \lambda_i = 0 \text{ for all } i < j \\ \lambda_j = c \\ \lambda_i \in [-p, p] \end{array} \right\}.$$

In other words, we have

$$S = \bigcup_{j=1}^{m} \{c x_j + \lambda_{j+1} x_{j+1} + \cdots + \lambda_m x_m : \lambda_i \in [-p, p]\}.$$

For each j, the set $\{c x_j + \lambda_{j+1} x_{j+1} + \cdots + \lambda_m x_m : \lambda_i \in [-p, p]\}$ is called a *row* of S.

4 Topological Dynamics in Ramsey Theory

Definition (Dynamical system)**.** A *dynamical system* is a pair (X, T) where X is the compact metric space and T is a homeomorphism on X.

Definition (Left shift)**.** The *left shift* operator $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ is defined by

$$(\mathcal{L}c)(i) = c(i + 1).$$

Definition (Orbital closure)**.** Let (X, T) be a dynamical system, and $x \in X$. The *orbital closure* $\bar{x}$ of x is the set $\mathrm{cl}\{T^s x : s \in \mathbb{Z}\}$.

Definition (Seq)**.** Let $c : \mathbb{Z} \to [k]$. We define

$$\mathrm{Seq}(c) = \{(c(i), \cdots, c(i + r)) : i \in \mathbb{Z}, r \in \mathbb{N}\}.$$

Definition (Minimal dynamical system)**.** We say (X, T) is *minimal* if $\bar{x} = X$ for all $x \in X$. We say $x \in X$ is *minimal* if $(\bar{x}, T)$ is a minimal dynamical system.

Definition (Proximal points)**.** We say $x, y \in X$ are *proximal* if

$$\inf_{s \in \mathbb{Z}} \rho(T^s x, T^s y) = 0.$$

Notation. For an interval $I \subseteq \mathbb{Z}$, we say $I = \{a, a + 1, \cdots, b\}$, we write $c(I)$ for the sequence $(c(a), c(a + 1), \cdots, c(b))$.

Definition (Bounded gaps property)**.** We say a colouring $c : \mathbb{Z} \to [k]$ has the *bounded gaps property* if for each interval $I \subseteq \mathbb{Z}$, there exists some $M > 0$ such that $c(I) \preccurlyeq c(\mathcal{U})$ for every interval $\mathcal{U} \subseteq \mathbb{Z}$ of length at least M.

5 Sums and products*

Definition (Syndetic set). We say $A = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}$ is *syndetic* if it has bounded gaps, i.e. $a_{i+1} - a_i < b$ for all i.

Definition (Piecewise syndetic set). We say $A \subseteq \mathbb{N}$ is *piecewise syndetic* if there exists $b > 0$ such that A has gaps bounded by b on arbitrarily long intervals.

More explicitly, if we again write $A = \{a_1 < a_2 < \cdots\}$, then there exists b such that for all $N > 0$, there is $i \in \mathbb{N}$ such that $a_{k+1} - a_k < b$ for $k = i, \cdots, i+N$.